On the factorizability of maps of S^n into S^n

by

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It is well known that a continuous map of S^1 into S^1 is inessential (*i. e.* homotopic to a constant) if and only if it can be represented as a superposition $\varphi = \varphi_2\varphi_1$, $\varphi_1: S^1 \to R^1$, $\varphi_2: R^1 \to S^1$, where R^1 is the real line. This is an immediate consequence of the fact that R^1 is a contractible covering space of S^1. K. Borsuk [1] has shown that the corresponding statement is in no way true for inessential maps of S^2 into S^2: he has exhibited an example of a map which cannot be factored through R^2 (the Euclidian 2-space).

In his paper, K. Borsuk raises the question whether inessential maps of S^n into S^n, which are not factorizable through R^n, do exist for all $n \geqslant 2$. He also asks whether there exists a space X such that a map $\varphi: S^2 \to S^2$ is inessential if and only if it is of the form $\varphi_2\varphi_1$, where $\varphi_1: S^2 \to X$, $\varphi_2: X \to S^2$. It is obvious, though not explicitly stated in Borsuk's problem, that X must be no more than 2-dimensional; for, without this restriction, a closed 3-cell X with S^2 as boundary behaves as required: φ is inessential if and only if it can be extended to a map $\psi: X \to S^2$ and then $\varphi = \psi i$, where $i: S^2 \to X$ is the inclusion.

The following theorem answers completely all these questions:

THEOREM. *There exists no n-dimensional paracompact space X, $n \geqslant 2$, such that a continuous map φ of S^n into S^n is inessential if and only if it is representable as $\varphi = \varphi_2\varphi_1$, where φ_1 and φ_2 are maps of S^n and X respectively, into X and S^n respectively.*

The proof of this theorem will be achieved by producing a certain inessential map $\Phi: S^n \to S^n$, which is somehow related to that used by K. Borsuk.

Let the sets on the left be defined by the condition on the right:

n-sphere	S^n:	$x_1^2 + \ldots + x_{n+1}^2 = 1$,
upper cap	E_+^n:	$\{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} > 0\}$,
lower cap	E_-^n:	$\{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} < 0\}$,
equator	S^{n-1}:	$\{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} = 0\}$.

The suspension $E\alpha: S^n \to S^n$ over α (α — any map $S^{n-1} \to S^{n-1}$) is defined by

$$E\alpha(x_1, \ldots, x_{n+1}) = \left(y_1\sqrt{x_1^2 + \ldots + x_n^2}, \ldots, y_n\sqrt{x_1^2 + \ldots + x_n^2}, x_{n+1}\right)$$

where

$$(x_1, \ldots, x_{n+1}) \in S^n, \qquad \left(\frac{x_1}{\sqrt{x_1^2 + \ldots + x_n^2}}, \ldots, \frac{x_n}{\sqrt{x_1^2 + \ldots + x_n^2}}, 0\right) \in S^{n-1}$$

and

$$\alpha\left(\frac{x_1}{\sqrt{x_1^2 + \ldots + x_n^2}}, \ldots, \frac{x_n}{\sqrt{x_1^2 + \ldots + x_n^2}}, 0\right) = (y_1, \ldots, y_n, 0)$$

$$\left(E\alpha(0, \ldots, 0, \pm 1) = (0, \ldots, 0, \pm 1)\right).$$

For any α, $E\alpha|_{S^{n-1}} = \alpha$; further the suspension over the identity map $\vartheta: S^{n-1} \to S^{n-1}$ is the identity map $\theta: S^n \to S^n$. It is also easy to check that if α_t is a homotopy joining α_0 to α_1, $E\alpha_t$ is a homotopy joining $E\alpha_0$ to $E\alpha_1$ and $E\alpha_t(S^{n-1}) \subset S^{n-1}$ for all $t \in [0, 1]$.

Now, for $n \geqslant 2$, let τ be a topologically transitive homeomorphism of S^{n-1} onto S^{n-1}, *i. e.*, a homeomorphism for which the sequence of iterated images $\tau^k(y_0)$ is everywhere dense in S^{n-1} for at least a point $y_0 \in S^{n-1}$. We suppose also that τ is homotopic to the identity (¹). Let $\sigma = E\tau: S^n \to S^n$ be the suspension over τ and let r be the reflexion across the hyperplane $x_{n+1} = 0$; $r(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1})$.

Now construct a map φ of the closed cell $\overline{E_+^n} \subset S^n$ onto S^n, sending S^{n-1} into $x_0 = (0, \ldots, 0, -1)$ and sending E_+^n homeomorphically onto $S^n - x_0$. For instance choose the φ given by

$$\varphi(x_1, \ldots, x_{n+1}) = (2x_1x_{n+1}, 2x_2x_{n+1}, \ldots, 2x_nx_{n+1}, 2x_{n+1}^2 - 1)$$

for $(x_1, \ldots, x_{n+1}) \in E_+^n$.

Set

$$\Phi(x) = \begin{cases} \varphi(x), & x \in \overline{E_+^n}, \\ \varphi\left(r\left(\sigma(x)\right)\right), & x \in \overline{E_-^n}. \end{cases}$$

From $\varphi(S^{n-1}) = \varphi\left(r\left(\sigma(S^{n-1})\right)\right) = x_0$ we infer that Φ is continuous. Next, consider the following maps

$$(*) \qquad \Phi_t(x) = \begin{cases} \varphi(x), & x \in \overline{E_+}, \\ \varphi\left(r\left(E\tau_t(x)\right)\right), & x \in \overline{E_-^n}, \end{cases}$$

(¹) Such homeomorphisms are known to exist for $n - 1 \geqslant 2$ and we can choose it sufficiently near to the identity map (see [4]). For $n = 1$ we simply take a rotation of S^1 with $2\pi\alpha$, the arbitrary α being irrational.

$t \in [0,1]$, where $\tau_0 = \tau$, $E\tau = \sigma$ and $\tau_1 = 0$ (the identity map). The incontinuity is ensured since $E\tau_t(S^{n-1}) \subset S^{n-1}$ for all t, and $\varphi(S^{n-1}) = x_0$. Thus, Φ is homotopic to the map $\Phi_1\colon S^n \to S^n$, given by

$$\Phi_1(x) = \begin{cases} \varphi(x), & x \in \overline{E^n_+}, \\ \varphi(r(x)), & x \in \overline{E^n_-}. \end{cases}$$

Clearly,

$$\Phi_1(x) = \Phi(r(x)), \qquad x \in S^n$$

for $r^2 =$ identity. Since r is sense reversing, its degree is -1 and we have: degree $\Phi_1 =$ degree $\Phi_1 \cdot (-1)$, hence: degree $\Phi_1 = 0$ and Φ_1 is inessential. Therefore Φ, being homotopic to Φ_1, is also inessential.

Suppose now that X is a paracompact space of dimension $\leqslant n$, such that a map of S^n into S^n is inessential if and only if it is factorizable through X. The inessentiality of Φ implies then

$$\Phi = f\psi, \quad \psi\colon S^n \to X, \quad f\colon X \to S^n.$$

Both the restrictions of ψ to E^n_+ and E^n_- must be one-one since so are $\Phi|_{E^n_+}$ and $\Phi|_{E^n_-}$. The sets $\psi(E^n_+)$ and $\psi(E^n_-)$ are then homeomorphic to open cells, and

$$\overline{\psi(E^n_+)} = \psi(\overline{E^n_+}) = \psi(E^n_+) \cup \psi(S^{n-1}),$$
$$\overline{\psi(E^n_-)} = \psi(\overline{E^n_-}) = \psi(E^n_-) \cup \psi(S^{n-1}),$$
$$\psi(E^n_+ \cup E^n_-) \cap \psi(S^{n-1}) = \emptyset$$

(since $\Phi(E^n_+) \cap \Phi(E^n_-) = \emptyset$).

Two cases are possible:

I. $\psi(E^n_+) \neq \psi(E^n_-)$. Then, at least one of the sets $X_1 = \psi(E^n_+) - \psi(E^n_-)$ or $X_2 = \psi(E^n_-) - \psi(E^n_+)$ is non-void. Suppose $X_1 \neq \emptyset$ (the other case is similar). $X_1 = \psi(E^n_+) - \overline{\psi(E^n_-)}$ is open relatively to $\psi(E^n_+)$. Let $U \subset \psi^{-1}(X_1)$ be an open cell and $V = \psi(U)$. Since ψ is clearly a homeomorphism on $\psi^{-1}(X_1)$, and in particular on U, the map of pairs

$$\psi\colon (S^n, S^n - U) \to (\psi(S^n), \psi(S^n) - V)$$

is a relative homeomorphism [3] $(\psi(S^n - U) \subset \psi(S^n) - V)$. Consider the commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 \to & H^n(S^n, S^n - U) & \xrightarrow{j_1^*} & H^n(S^n) & \to 0 \\ & \uparrow \psi^* & & \uparrow \psi^* & \\ & H^n(\psi(S^n), \psi(S^n) - V) & \xrightarrow{j_2^*} & H^n(\psi(S^n)) & \end{array}$$

where H^n are the Čech cohomology groups with integral coefficients and ψ^*, j_i^* are the homomorphisms induced by ψ and the inclusion maps j_1, j_2. Since $S^n - U$ is a closed cell, $H^{n-1}(S^n - U) = H^n(S^n - U) = 0$. The first vertical map is an *isomorphism onto* due to the invariance of the Čech cohomology groups with respect to relative homeomorphisms [3]; so is also j_1^*. We infer that the second vertical map

$$\psi^*\colon H^n(\psi(S^n)) \to H^n(S^n)$$

is *onto*.

II. $\psi(E^n_+) = \psi(E^n_-)$. We claim that for all $x \in E^n_-$, $\psi(x) = \psi(r(\sigma(x)))$. Indeed, there must be a point $x' \in E^n_+$ such that $\psi(x') = \psi(x)$; it follows that $\Phi(x') = \Phi(x)$, whence $x' = r(\sigma(x))$. The continuity of ψ yields $\psi(x) = \psi(r(\sigma(x)))$ for $x \in \overline{E^n_-}$, whence also for $x \in S^{n-1}$. On S^{n-1}, $r\sigma = \tau$, so that $\psi(x) = \psi(r(x))$ for $x \in S^{n-1}$. The point $y_0 \in S^{n-1}$, for which $\{\tau^k(y_0)\}$ is everywhere dense in S^{n-1}, satisfies $\psi(y_0) = \psi(\tau(y_0)) = \ldots = \psi(\tau^k(y_0)) = \ldots$ The set $\psi^{-1}(\psi(y_0)) \supset \{\tau^k(y_0)\}$ is closed, whence $\psi^{-1}(\psi(y_0)) \supset \overline{\{\tau^k(y_0)\}} = S^{n-1}$ and $\psi(S^{n-1}) = \psi(y_0)$. Since $\psi(S^{n-1}) = f^{-1}(\Phi(S^{n-1}))$, it follows that $f^{-1}(x_0) = \psi(y_0)$. Together with the obvious fact that $f|_{\psi(E^n_+)}$ is one-one, this yields that

$$f^{-1}\colon S^n \to \psi(S^n)$$

is a homeomophism. Therefore

$$(f^{-1})^*\colon H^n(S^n) \to H^n(\psi(S^n))$$

is an isomorphism *onto*:

Since $\dim X \leqslant n$, $H^{n+1}(X, \psi(S^n)) = 0$ and the exact sequence

$$H^n(X) \xrightarrow{i^*} H^n(\psi(S^n)) \to 0$$

implies that i^* is *onto* ($i\colon \psi(S^n) \to X$ is the inclusion). Let $g = \psi$ in the first case and $g = f^{-1}$ in the second. The composition gi, which will also be denoted by g,

$$g\colon S^n \to X$$

induces a homomorphism

$$g^*\colon H^n(X) \to H^n(S^n),$$

which is *onto* (whence $H^n(X)$ and g^* are $\neq 0$).

Let $a \in H^n(X)$ satisfy $g^*(a) \neq 0$. From Dowker's extension to paracompact spaces of the Hopf classification theorem [2], it follows that a map $h\colon X \to S^n$ with cohomology degree a, *i. e.*, such that $h^*(a) = a$, exists, where a is a generator of the group $H^n(S^n) \approx Z$. It follows that $g^*h^*(a) \neq 0$, whence the homomorphism

$$g^*h^* = (hg)^*\colon H^n(S^n) \to H^n(S^n)$$

is nontrivial. This proves that hg is an essential map, contradicting the assumption that each composition hg with $g\colon S^n \to X$, $h\colon X \to S^n$ is inessential.

Remark 1. In particular, the map Φ given above yields an example of an inessential map of S^n into S^n which is not factorizable through R^n. Indeed, setting $X = R^n$ and assuming $\Phi = f\psi$, $\psi\colon S^n \to R^n$, $f\colon R^n \to S^n$, we would get as above $H^n(R^n) \neq 0$, which is absurd.

Remark 2. If we choose τ sufficiently near to the identity, we get $|\Phi(x) - \Phi_1(x)| < \varepsilon$; but it is easy to check that the map Φ_1 (see (*)) can be factored through R^n. This shows that the set of maps which are factorizable through R^n *is not open* in the set of all inessential maps of S^n into S^n.

References

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After this paper had been submitted to the Fundamenta, professor K. Borsuk kindly informed me that H. Hopf had already communicated to him the solution of problem 1 of [1].

Note on dimension theory for metric spaces *

by

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Recently, a dimension theory for general metric spaces has been established by M. Katětov and by K. Morita (see [4] and [5]) independently. They have extended the sum, decomposition and product theorems to non-separable metric spaces and have shown the equivalence of the Lebesgue dimension and the inductive dimension (1). On the other hand, the following theorem of P. Alexandroff and P. Urysohn is well known:

In order that a T_1-topological space R be metrizable it is necessary and sufficient that there exists a sequence $\mathfrak{V}_1 > \mathfrak{V}_2^* > \mathfrak{V}_2 > \mathfrak{V}_3^* > \ldots$ of open coverings such that $\{S(p, \mathfrak{V}_m) | m = 1, 2, \ldots\}$ (2) is a nbd (neighbourhood) basis for each point p of R.

The purpose of the present note is to refine this theorem to a theorem concerning n-dimensionality of metric spaces and to develop the dimension theory for general metric spaces. In § 1 we shall prove that Alexandroff-Urysohn's theorem turns into a theorem asserting a necessary and sufficient condition for n-dimensionality if we add the condition order $\mathfrak{V}_m \leqslant n+1$ $(m = 1, 2, \ldots)$ to the original condition. Furthermore, concerning that theorem it will be shown that we may replace order $\mathfrak{V}_m \leqslant n+1$ by Alexandroff-Kolmogoroff's length of $\mathfrak{V}_m \leqslant n+1$ (see [1]). In § 2 we shall apply the result of § 1 to the study of the connections between dimension and metric function. § 3 contains applications of the result of § 1 to the embedding of n-dimensional metric spaces into products of 1-dimensional spaces. The final section is devoted to

* The content of this paper is a development in detail of our brief notes published in Proc. of Japan Acad. 32 (1956).

(1) $\operatorname{ind}\dim \emptyset = -1$ for a vacuous set $\emptyset$, and $\operatorname{ind}\dim R \leqslant n$ if and only if for any pair of a closed set F and an open set G with $F \subseteq G$ there exists an open set U such that $F \subseteq U \subseteq G$, $\dim B(U) \leqslant n-1$, where we denote by $B(U)$ the boundary of U.

(2) $S(p, \mathfrak{V}) = \bigcup \{V | p \in V \in \mathfrak{V}\}$ for a covering $\mathfrak{V}$ of R, $S(A, \mathfrak{V}) = \bigcup \{V | V \cap A \neq \emptyset, V \in \mathfrak{V}\}$ for a subset A of R, $\mathfrak{V}^* = \{S(V, \mathfrak{V}) | V \in \mathfrak{V}\}$. $\mathfrak{V}$ is called a star-refinement of $\mathfrak{U}$ if $\mathfrak{V}^* < \mathfrak{U}$. The notation of this paper is chiefly due to [8]. See also [2] with respect to the notions.