On simple regular mappings

by

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1. Introduction. K. Borsuk and R. Molski studied in [3] a class of continuous mappings, called the mappings of finite order. A continuous mapping f of the space X onto the space Y is of order $\leqslant k$ if for every point $y \in Y$ the set $f^{-1}(y)$ contains at most k points (see also [9], p. 52). A mapping of order $\leqslant 2$ is called by K. Borsuk and R. Molski a *simple mapping*. The authors have given many examples which show that simple mappings may have many singular properties, *e. g.*, they can raise the dimension. Moreover, the authors have distinguished a certain class of mappings of finite order, called elementary mappings, and have proved that, in particular, every elementary mapping may be obtained by finite superpositions of simple mappings.

In this paper we shall study some properties of a certain class of simple mappings, called by us *regular mappings*, which contains, in particular, all simple elementary mappings. For a simple regular mapping f of a space X we introduce the notion of so-called doubling of X by f, which enables us to reduce the study of regular mappings to the study of simple interior mappings (¹).

2. Let f be a simple mapping of a space X into Y. The union X_0 of all sets $f^{-1}(y)$, with $y \in Y$, containing two different points is called in [3] the *seam* of f. Let us denote, for every $x \in X$, by $\Phi_f(x)$ a point of X such that $f^{-1}(f(x)) = \{x, \Phi_f(x)\}$ (²). Then Φ_f is an involution of X; it is called in [3] "the involution assigned to f" and is denoted there by i_f (see [3], Nr 3). We shall call it the *involution induced by* f. The set $X - X_0$ is the set of fixed points of Φ_f.

3. Let $\mathfrak{F}$ be an upper semi-continuous decomposition of a compact space X and let Y be the hyperspace of this decomposition. Then, by

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(¹) A continuous mapping $f\colon X \to Y$ is called interior if it carries open sets onto open ones.

(²) We denote by $\{a_1, a_2, \ldots, a_n\}$ the set composed of the elements $a_1, a_2, \ldots, a_n$ and by $(a_1, a_2, \ldots, a_n)$ the ordered sequence of these elements.

Alexandroff's theorem (see [9], p. 42), the decomposition $\mathfrak{F}$ induces a continuous mapping $f\colon X\to Y$ such that the sets of $\mathfrak{F}$ are the same as the sets $f^{-1}(y)$ with $y\in Y$. Moreover, in order that the decomposition $\mathfrak{F}$ be continuous it is necessary and sufficient that the induced mapping f be an interior mapping (see [4]).

4. Let $\mathfrak{F}$ be a continuous decomposition of a compact space X on sets containing at most two points. Hence the mapping f induced by $\mathfrak{F}$ is a simple interior mapping. It is evident that the involution Φ_f induced by f is continuous.

Conversely, let $\varphi\colon X\to X$ be a continuous involution of a compact space X and let $\mathfrak{F}_\varphi$ be the decomposition of X on the sets of the form $\{x,\varphi(x)\}$. Now, let E be an arbitrary subset of X. Then the union S of all sets of $\mathfrak{F}_\varphi$ intersecting E is equal to $E\cup\varphi(E)$. Hence, if E is closed (or open), then S is closed (or, respectively, open). It follows that the decomposition $\mathfrak{F}_\varphi$ is continuous and the induced mapping f_φ defined by $f_\varphi(x)=\{x,\varphi(x)\}$, for every $x\in X$, is a simple interior mapping of X onto the hyperspace Y of the decomposition $\mathfrak{F}_\varphi$. The simple mapping $f_\varphi\colon X\to Y$ is said by us to be *induced by the continuous involution* φ.

Let us observe that if, for some $p\in X$, we have $\varphi(p)\neq p$, then there exists a neighbourhood U of p such that $\overline{U}\cap\varphi(\overline{U})=0$. Hence f_φ maps U homeomorphically onto a neighbourhood $f_\varphi(U)$ of $f_\varphi(p)$ in Y. In particular, if the involution φ has no fixed points, then the induced mapping f_φ is a local homeomorphism. Therefore, we have obtained

THEOREM 1. *In order that a simple mapping f of a compact space X be an interior mapping it is necessary and sufficient that the involution Φ_f induced by f be continuous. Moreover, every continuous involution φ of X induces a simple interior mapping f_φ; if φ has no fixed points, then the induced mapping f_φ is a local homeomorphism.*

5. A simple mapping f of the space X with the seam X_0 is said by us to be regular provided that the partial mapping $f|\overline{X}_0$ is interior. Theorem 1 yields

THEOREM 2. *Let f be a simple mapping of a compact space X and let X_0 denote the seam of f. Then the following conditions are equivalent:*

(i) *f is regular.*

(ii) *The involution $\varphi=\Phi_f|\overline{X}_0$ is continuous.*

6. A mapping f of a space X is called in [3] an *elementary mapping* if there exists an $\varepsilon>0$ such that, for every two different points $x',x''\in X$, from $f(x')=f(x'')$ follows $\varrho(x',x'')\geqslant\varepsilon$.

THEOREM 3. *Let f be a simple mapping of a compact space X and let X_0 be the seam of f. Then the following conditions are equivalent:*

(i) *f is elementary.*

(ii) *f is regular and the seam X_0 is closed.*

(iii) *The involution $\varphi=\Phi_f|\overline{X}_0$ is continuous and has no fixed points.*

Proof. The implication (i)→(iii) is proved in [3]. Nr 3, Corollary 1, (iii)→(ii). Since the involution φ of $\overline{X}_0$ is continuous, then, by Theorem 2, f is regular. Let $\{x_n\}\subset X_0$ be a sequence which is convergent to a point $x\in\overline{X}_0$. Since $x_n\neq\varphi(x_n)$ and since the involution φ has not fixed points, then $\varphi(x)\neq x$. Therefore, $x\in X_0$. Hence $\overline{X}_0=X_0$.

(ii)→(i). Since X_0 is closed, then it is compact. By Theorem 2, the involution φ of X_0 is continuous. Moreover, it has no fixed points (since X_0 is the seam of f). It follows that there exists a number $\varepsilon>0$ such that $\varrho(\varphi(x'),\varphi(x''))\geqslant\varepsilon$ for every two points x',x'' belonging to X_0. This means that f is elementary.

This theorem shows that an elementary mapping may be defined as a regular mapping with a closed seam.

7. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two groups and let $f\colon\mathfrak{A}\to\mathfrak{B}$ be a homomorphism. Then f is called an *r-homomorphism*, if it posesses a right inverse, *i. e.* if there exists a homomorphism $h\colon\mathfrak{B}\to\mathfrak{A}$ such that fh is the identity on $\mathfrak{B}$ (see [2]). If f is an r-homomorphism, then $f(\mathfrak{A})=\mathfrak{B}$.

Let $\mathfrak{G}$ be an abelian group which is a D-module over a domain of integrity D, such that *every element of $\mathfrak{G}$ is divisible by* 2. Let $H_k(X)$ denotes the kth homology group of a compact space X over $\mathfrak{G}$, in the sense of Vietoris. The homomorphism of $H_k(X)$ into $H_k(Y)$ induced by a continuous mapping $f\colon X\to Y$ is denoted by f_*. The rank $r(H_k(X))$, *i. e.*, the kth Betti number of X over $\mathfrak{G}$, will be denoted by $p_k(X)$.

8. Simple interior mappings.

THEOREM 4. *Let f be a simple interior mapping of the compact space X. Then the homomorphism $f_*\colon H_k(X)\to H_k(f(X))$ induced by f is an r-homomorphism.*

By theorem 1, this theorem can be formulated as follows:

THEOREM 4′. *Let $\varphi\colon X\to X$ be a continuous involution of the compact space X and let $f=f_\varphi$ be the simple mapping induced by φ. Then f induces an r-homomorphism $f_*\colon H_k(X)\to H_k(f(H))$.*

Proof of theorem 1′. Let $\bar\varrho$ be a metric in X. Setting

$$\varrho(x_1,x_2)=\bar\varrho(x_1,x_2)+\bar\varrho(\varphi(x_1),\varphi(x_2)),$$

we obtain a metric ϱ which is topologically equivalent to $\bar\varrho$ and which satisfies the condition

$$\varrho(x_1,x_2)=\varrho(\varphi(x_1),\varphi(x_2)), \tag{1}$$

for every $x_1,x_2\in X$.

The space $Y = f(X)$ is the hyperspace of the decomposition $\mathfrak{F}$ of X on the sets $\{x, \varphi(x)\}$. Thus Y is metrized by the Hausdorff metric (see [8], p. 106). Therefore, if $y_1, y_2 \in Y$ and $y_1 = \{x_1, \varphi(x_1)\}$, $y_2 = \{x_2, \varphi(x_2)\}$, then, by (1),

$$\varrho(y_1, y_2) = \min\{\varrho(x_1, x_2), \varrho(x_1, \varphi(x_2))\}. \tag{2}$$

Since the decomposition $\mathfrak{F}$ is continuous, it follows that the diameter $\delta(f^{-1}(y))$ is a continuous function in Y. Let us set for $\varepsilon > 0$

$$Y(\varepsilon) = \mathop{E}_{y \in Y}[\delta(f(y)) < \varepsilon].$$

Hence the set $Y(\varepsilon)$ is open. If X_0 is the set of fixed points of the involution φ (*i. e.*, the complement of the seam of f), then

$$Y_0 = f(X_0) \subset Y(\varepsilon).$$

Let

$$Z(\varepsilon) = Y - Y(\varepsilon) = \mathop{E}_{y \in Y}[\delta(f^{-1}(y)) \geqslant \varepsilon].$$

Hence the set $Z(\varepsilon)$ is closed.

Now let E be an arbitrary subset of $Z(\varepsilon)$ of the diameter $\delta(E) < \frac{1}{2}\varepsilon$. If $x_1, x_2 \in f^{-1}(E)$, then $\varrho(f(x_1), f(x_2)) < \frac{1}{2}\varepsilon$, and $\delta(f^{-1}(x_m)) \geqslant \varepsilon$, for $m = 1, 2$. Hence, by (2), either $\varrho(x_1, x_2) < \frac{1}{2}\varepsilon$ or $\varrho(x_1, \varphi(x_2)) < \frac{1}{2}\varepsilon$. In the first case, by (1),

$$\varrho(x_1, \varphi(x_2)) \geqslant \varrho(x_1, \varphi(x_1)) - \varrho(\varphi(x_1), \varphi(x_2)) = \\ = \varrho(x_1, \varphi(x_1)) - \varrho(x_1, x_2) \geqslant \varepsilon - \tfrac{1}{2}\varepsilon = \tfrac{1}{2}\varepsilon$$

and, similarly, in the second case

$$\varrho(x_1, x_2) \geqslant \varrho(x_1, \varphi(x_1)) - \varrho(\varphi(x_1), x_2) = \\ = \varrho(x_1, \varphi(x_1)) - \varrho(x_1, \varphi(x_2)) \geqslant \varepsilon - \tfrac{1}{2}\varepsilon = \tfrac{1}{2}\varepsilon.$$

Therefore, either $\varrho(x_1, x_2) < \frac{1}{2}\varepsilon$ and $\varrho(x_1, \varphi(x_2)) \geqslant \frac{1}{2}\varepsilon$, or $\varrho(x_1, x_2) \geqslant \frac{1}{2}\varepsilon$ and $\varrho(x_1, \varphi(x_2)) < \frac{1}{2}\varepsilon$. Hence the set $f^{-1}(E)$ splits uniquely onto the sum of two disjoint sets:

$$f^{-1}(E) = E^{(-1)} \cup E^{(+1)} \tag{3}$$

such that if $x_1, x_2 \in E^{(i)}$, $i = \mp 1$, then

$$\varrho(x_1, x_2) = \varrho(f(x_1), f(x_2)) < \tfrac{1}{2}\varepsilon, \tag{4}$$

and if $x_1 \in E^{(i)}$ and $x_2 \in E^{(-i)}$, then $\varrho(x_1, x_2) \geqslant \frac{1}{2}\varepsilon$. Moreover,

$$f(E^{(-1)}) = f(E^{(+1)}) = E. \tag{5}$$

It follows from (1) and (2) that if $y_0 \in Y_0$, $y \in Y$, $y_0 = f(x_0)$ and $y = f(x)$, then

$$\varrho(x_0, x) = \varrho(y_0, y). \tag{6}$$

Let us denote by ϑ_ε the function which assigns to every point $y \in Y(\varepsilon)$ a certain point from among the points of Y_0 nearest to the point y.

Let $\lambda = \{\lambda_n\}$ be a sequence of k-dimensional ε_n-chains with coefficients of the space Y with coefficients in the group $\mathfrak{G}$, such that $\varepsilon_n > 0$ and

$$\lim_{n=\infty} \varepsilon_n = 0. \tag{7}$$

Let $\tau = (q_0, q_1, \ldots, q_k)$ be a k-dimensional ε_n-simplex which appears in λ_n with the coefficient a. We may assume that $q_\nu \in Y(2\varepsilon_n)$ for $0 \leqslant \nu < j$ and $q_\nu \in Z(2\varepsilon_n)$ for $j \leqslant \nu \leqslant k$ where $0 \leqslant j \leqslant k$. Let $q'_\nu = \vartheta_{2\varepsilon_n}(q_\nu)$ for $0 \leqslant \nu < j$. By (7)

$$\prod_{n=1}^{\infty} Y(2\varepsilon_n) = Y_0.$$

Hence there exists a sequence of positive numbers η_n such that

$$\lim_{n=\infty} \eta_n = 0 \tag{8}$$

and

$$\varrho(y, \vartheta_{2\varepsilon_n}(y)) < \eta_n, \quad \text{for every} \quad y \in Y(2\varepsilon_n) \quad \text{and} \quad n = 1, 2, \ldots \tag{9}$$

By (9) the simplex $\tau' = (q'_0, \ldots, q'_{j-1}, q_j, \ldots, q_k)$ is an η_n-simplex of Y and the set $E = \{q_j, \ldots, q_k\}$ is of the diameter $\delta(E) < \varepsilon_n$. Let

$$f^{-1}(E) = E^{(-1)} \cup E^{(+1)}$$

be the decomposition of $f^{-1}(E)$ of the form (3), where $\varepsilon = 2\varepsilon_n$. Since $q'_\nu \in Y_0$, for $0 \leqslant \nu < j$, let us set

$$f^{-1}(q'_\nu) = p_\nu, \quad \text{for} \quad 0 \leqslant \nu < j,$$

and

$$f^{-1}(q'_\nu) = \{p_\nu^{(-1)}, p_\nu^{(+1)}\}, \quad \text{for} \quad j \leqslant \nu \leqslant k,$$

where $p^{(i)} \in E^{(i)}$ for $i = \mp 1$.

Let us assign to the chain $a\tau$ the chain

$$\tfrac{1}{2}a\sigma_{(-1)} + \tfrac{1}{2}a\sigma_{(+1)}$$

where $\sigma_{(i)} = (p_0, \ldots, p_{j-1}, p_j^{(i)}, \ldots, p_k^{(i)})$ for $i = \mp 1$.

By (4) and (6) this correspondence carries the ε_n-chain λ_n onto an η_n-chain $\varkappa_n$ of X and, by (8), it induces a homomorphism $\bar{h}$ of the group of true chains $\lambda = \{\lambda_n\}$ of Y into the group of true chains $\varkappa = \{\varkappa_n\}$ of X. Moreover, $\bar{h}$ satisfies the condition

$$\partial\bar{h}(y) = \bar{h}(\partial\lambda) \tag{10}$$

for every true chain λ of Y. It follows from (9) and (5) that if γ is a true cycle in Y, then $f\bar{h}(\gamma)\sim\gamma$, and therefore, by (10), $\bar{h}$ induces a homomorphism

$$h\colon H_k(Y)\to H_k(X)$$

such that f_*h is the identity in $H_k(X)$. Then h is a right inverse of f_*. Hence the proof of theorem 1' is finished.

THEOREM 5. *If f is a simple interior mapping of the compact space X, then* $\dim f(X) = \dim X$.

This theorem may also be formulated otherwise:

THEOREM 5'. *Let $\varphi\colon X\to X$ be a continuous involution of a compact space X and let $f = f_\varphi$ be the simple mapping induced by φ. Then* $\dim f(X) = \dim X$.

Proof of theorem 2' (³). Let $Y = f(X)$. It suffices to show that $\dim Y \leqslant \dim X$ (see [3], Nr 4).

Let $\dim X \leqslant n$. Theorem 2' is true if $n = -1$. Let us suppose that it is true for $n = k-1$.

Let $p\in X$. If $\varphi(p)\neq p$, then there exists a neighbourhood U of p in X such that f maps U homeomorphically onto a neighbourhood of $f(p)$ in Y (see Nr 4). Hence $\dim_p X = \dim_{f(p)} Y$.

Now let $\varphi(p) = p$ and let V be a neighbourhood of $f(p)$. Let U be a neighbourhood of p such that $f(U)\subset V$. Since $\dim X\leqslant k$, there exists a neighbourhood $U_1\subset U$ of p such that $\dim\mathrm{Fr}(U_1)\leqslant k-1$. Let $U_0 = U_1\cap\varphi(U_1)$. Then U_0 is a neighbourhood of p such that

$$\varphi(U_0) = U_0 \tag{11}$$

and since $\mathrm{Fr}(U_0)\subset\mathrm{Fr}(U_1)\cup\mathrm{Fr}\big(\varphi(U_1)\big)$, then

$$\dim\mathrm{Fr}(U_0)\leqslant k-1\,. \tag{12}$$

By (11), the partial mapping $f|\mathrm{Fr}(U_0)$ is a simple interior mapping of the compact set $\mathrm{Fr}(U_0)$, whence from the hypothesis of induction and from (12) follows

$$\dim f\big(\mathrm{Fr}(U_0)\big)\leqslant k-1\,. \tag{13}$$

(³) Theorem 2' also follows from a more general theorem proved by E. E. Floyd; see [6], Nr 4, p. 35.

Let us observe that, by (11), $f(x) = \{x, \varphi(x)\}\in\overline{f(U_0)}$ if and only if $x\in\overline{U_0}$. Therefore $f(\overline{U_0}) = \overline{f(U_0)}$. Similarly, since $\varphi(X-U_0) = X-U_0$, we have $f(\overline{X-U_0}) = \overline{f(X-U_0)} = \overline{Y-f(U_0)}$. It follows that $f\big(\mathrm{Fr}(U_0)\big) = f(\overline{U_0}\cap\overline{X-U_0}) = f(\overline{U_0})\cap f(\overline{X-U_0}) = \overline{f(U_0)}\cap\overline{Y-f(U_0)}$, that is

$$f\big(\mathrm{Fr}(U_0)\big) = \mathrm{Fr}\big(f(U_0)\big)\,. \tag{14}$$

Thus, by (11), $V_0 = f(U_0)$ is a neighbourhood of the point $f(p)$ in Y such that $V_0\subset V$ and by (13) and (14), $\dim\mathrm{Fr}(V_0)\leqslant k-1$. It follows that $\dim_{f(p)}Y\leqslant k$. Therefore theorem 2' is proved.

9. The notion of the doubling of the space by a simple regular mapping. Let f be a simple regular mapping of the compact space X. Let X_0 denote the seam of f and Φ_f — the involution of X induced by f. Therefore $\varphi = \Phi_f|\bar{X}_0$ is a continuous involution of the compact set $\bar{X}_0$.

Let $Z = X\times\{-1, +1\}$ be the Cartesian product of X by the two-point set $\{-1, +1\}$. The points of Z are of the form (x, i), where $x\in X$ and $i = \mp 1$. Let $\mathfrak{F}$ be the decomposition of Z on the two-point sets of the form $\big\{(x, i), (\varphi(x), -i)\big\}$, where $x\in\bar{X}_0$, $i = \mp 1$, and on the one-point sets of the form $\{(x, i)\}$, where $x\in X-\bar{X}_0$, $i = \mp 1$. We shall show that the decomposition is upper semi-continuous.

Let A be a closed subset of Z. It should be shown that the set

$$S = \bigcup_{C\in\mathfrak{F}}[C\cap A\neq 0]$$

is closed.

Let us put $Z^{(-)} = X\times\{-1\}$, $Z^{(+)} = X\times\{+1\}$ and $Z_0 = \bar{X}_0\times\{-1, +1\}$. Moreover, if E is a subset of Z, let us put $E^{(-)} = E\cap Z^{(-)}$, and $E^{(+)} = E\cap Z^{(+)}$.

Since the sets $A^{(-)}$ and $A^{(+)}$ are closed, it suffices to prove that the set

$$S_1 = \bigcup_{C\in\mathfrak{F}}[C\cap A^{(-)}\neq 0]$$

is closed. Let us consider the sets $S_1^{(-)}$ and $S_1^{(+)}$. We have

$$S_1^{(-)} = A^{(-)}$$

and

$$S_1^{(+)} = \underset{(\varphi(x),+1)}{E}[(x, -1)\in A^{(-)}\cap Z_0^{(-)}]. \tag{15}$$

Let us observe that from the continuity of $\varphi\colon\bar{X}_0\to\bar{X}_0$ it follows that the involution $\alpha\colon Z_0\to Z_0$ defined by

$$\alpha(x, i) = \big(\varphi(x), -1\big)$$

is continuous. Moreover, by (15), $a(A^{(-)}\cap Z_0^{(-)})=S_1^{(+)}$, and since $A^{(-)}\cap Z_0^{(-)}$ is closed, it follows that $S_1^{(+)}$ is closed. Therefore S is a closed set and hence $\mathfrak{F}$ is upper semi-continuous.

Let us denote by $D(X,f)$ the hyperspace of the decomposition $\mathfrak{F}$, and by $g\colon Z\to D(X,f)$ the mapping induced by $\mathfrak{F}$. Then g is continuous. We define in $D(X,f)$ an involution putting

$$\psi\big(\{(x,i),(\varphi(x),-i)\}\big)=\{(\varphi(x),i),(x,-i)\},\quad \text{for}\quad x\in\overline{X}_0,$$

and

$$\psi(\{(x,i)\})=\{(x,-i)\},\quad \text{for}\quad x\in X-\overline{X}_0.$$

We shall show that the involution ψ is continuous. It should be proved that if the set $A\subset D(X,f)$ is closed, then $\psi(A)$ is closed. Set $P=g^{-1}(A)$ and $Q=g^{-1}(\psi(A))$.

Putting $\beta(x,i)=(x,-i)$ we obtain a continuous involution β of Z such that $\beta(P)=Q$. Therefore, since P is closed (from the continuity of g), it follows that Q is closed and hence compact. Consequently, the set $\psi(A)=g(Q)$ is closed. Thus ψ is continuous.

Let us put $D_1=g(Z^{(-)})$ and $D_2=g(Z^{(+)})$. Then

$$D(X,f)=D_1\cup D_2.$$

Moreover, the mappings h_1 and h_2 defined by

$$h_1(x)=g(x,-1)\quad\text{and}\quad h_2(x)=g(x,1)$$

map topologically X onto D_1 and D_2, respectively, and $\overline{X}_0$ onto

$$D_0=D_1\cap D_2$$

such that

(16) $$h_1\varphi(x)=\psi h_1(x),\quad \text{for every}\quad x\in\overline{X}_0,$$

and, similarly, $h_2\varphi(x)=\psi h_2(x)$.

Let F be the simple mapping induced by the involution ψ. Thus $F(D(X,f))=F(D_1)=F(D_2)$. It follows by (16) that $F(D_1)$ (and $F(D_2)$) is homeomorphic with $f(X)$. Hence $F(D(X,f))$ is homeomorphic with $Y=f(X)$.

The space $D(X,f)$ will be called by us the *doubling of X by the mapping f* and the involution ψ — the *natural involution of $D(X,f)$*. We have proved the following

THEOREM 6. *Let f be a simple regular mapping of the compact space X with the seam X_0. Then the doubling $D=D(X,f)$ of X by f is of the form*

$$D=D_1\cup D_2,$$

such that D_1 and D_2 are homeomorphic with X, and

$$D_0=D_1\cap D_2$$

is homeomorphic with $\overline{X}_0$. Moreover, the simple mapping F induced by the natural involution of D maps D onto a set homeomorphic with $Y=f(X)$.

By theorem 1, this yields the following

COROLLARY 1. *If f is a simple regular mapping of a compact space X, then $f(X)$ is an image of the doubling $D(X,f)$ by a simple interior mapping.*

Now let us suppose that f is a simple elementary mapping. Then the seam X_0 of f is closed and the involution $\varphi=\Phi_f|X_0$ has no fixed points. Hence the natural involution $\psi\colon D(X,f)\to D(X,f)$ has no fixed points, either. It follows by theorem 1 that the simple mapping F induced by ψ is a local homeomorphism. Therefore we have

COROLLARY 2. *If f is a simple elementary mapping of a compact space X, then $f(X)$ is an image of the doubling $D(X,f)$ by a simple mapping which is a local homeomorphism.*

Corollary 1 enables us to reduce the study of certain invariants of simple regular mappings to the study of invariants of simple interior mappings. Similarly, corollary 2 sometimes reduces the study of simple elementary mappings to the study of local homeomorphisms.

10. Simple regular mappings.

THEOREM 7. *Let f be a simple regular mapping of a compact space X and let X_0 denote the seam of f. If $p_k(X)=0$ and $p_{k-1}(\overline{X}_0)=0$, then $p_k(f(X))=0$.*

Proof. Let $D=D_2\cup D_2$ be the decomposition of the doubling $D=D(X,f)$ as in theorem 6. Applying the Mayer-Vietoris formula (see [5], p. 53), we have

(17) $$p_k(D)=2p_k(D_1)-p_k(D_0)+r(N_k)+r(N_{k-1}),$$

where N_m is the kernel of the homomorphism

$$i_{*m}\colon H_m(D_0)\to H_m(D_1)$$

induced by the inclusion mapping $i\colon D_0\to D_1$. By theorem 3 we have $p_k(D_1)=p_k(X)$ and $p_k(D_0)=p_k(\overline{X}_0)$. Since $p_k(X)=0$, hence $H_k(D_1)=0$ and then $N_k=H_k(D_0)$. Therefore $r(N_k)=p_k(D_0)=p_k(\overline{X}_0)$. Since $p_{k-1}(\overline{X}_0)=p_{k-1}(D_0)=0$, hence $N_{k-1}=0$ and then $r(N_{k-1})=0$. It follows from (17) that

(18) $$p_k(D)=0.$$

By corollary 1, and theorem 4, the group $H_k(f(X))$ is a homomorphic image of $H_k(D)$. Hence, by (18), $p_k(f(X))=0$.

Now, let $\mathfrak{G}$ be the field of rational numbers. If X is locally connected, then the condition $p_1(X) = 0$ means that X is unicoherent (see [1], p. 230). Therefore, theorem 7 yields

COROLLARY 3. *Let f be a simple regular mapping of a compact locally connected space X and let X_0 denote the seam of f. If X is unicoherent and the set $\bar{X}_0$ is connected, then $f(X)$ is unicoherent.*

Remark. Corollary 3 is an answer to a question posed by K. Borsuk. An example given by K. Borsuk and R. Molski (see [3], Example 6) shows that the hypothesis that f is regular is essential in theorem 7 and in corollary 3. We shall show that the hypothesis of local connectedness of X in corollary 3 is also essential.

EXAMPLE. Let (ϱ, ϑ) denote the polar coordinates on the plane and ψ — the antipodal mapping, *i. e.*, $\psi(\varrho, \vartheta) = (\varrho, \vartheta + \pi)$. Denote by K the circle $\varrho = 2$, by D the ring $1 \leqslant \varrho \leqslant 2$ and by S the spire defined by the equation $\varrho = (1+2\vartheta)/(1+\vartheta)$. Then the set $S \cup \psi(S)$ cuts the ring $1 \leqslant \varrho < 2$ in two parts A and B such that $B = \psi(A)$. Let $X = \bar{A}$.

Let F be the simple mapping induced by the involution ψ on D and let $f = F|X$. Hence f is a simple regular (even elementary) mapping of X with the seam $X_0 = S \cup \psi(S) \cup K$. Thus X is unicoherent, the seam X_0 is a continuum. However, the set $f(X)$ is homeomorphic with $F(D)$, whence with D, and thus it is not unicoherent.

THEOREM 8. *If f is a simple regular mapping of a compact space X, then* $\dim f(X) = \dim X$.

Proof. It follows from theorem 6 that $\dim D(X, f) = \dim X$. By corollary 1 and theorem 5, we have $\dim f(X) = \dim D(X, f)$. Hence $\dim f(X) = \dim X$.

11. Simple elementary mappings.

THEOREM 9. *Let f be a simple elementary mapping of the compact space X and let X_0 denote the seam of f. If X and X_0 are* ANR*-sets, then $f(X)$ is also an* ANR*-set.*

Proof. In this case, the sets D_1 and D_2 in theorem 6 are ANR-sets. Since X_0 is closed, it is homeomorphic with D_0. Hence $D_0 = D_1 \cap D_2$ is also an ANR-set. Consequently, the doubling $D(X, f) = D_1 \cup D_2$ is an ANR-set (see, for example, [9], p. 260, Nr 1). According to corollary 2, $f(X)$ is an image of $D(X, f)$ by a local homeomorphism. From a local characterization of ANR-sets (see, for example, [7]) it follows that $f(X)$ is an ANR-set.

Remark. Theorem 9 constitutes a slight extension of a theorem proved in an another way by K. Borsuk and R. Molski (see [3], Nr 6).

If $\dim X < \infty$, then, by using a theorem of E. E. Floyd (see [6], p. 38, (4.6)) it my be shown that theorem 9 holds for any regular mapping f. The question is open if it is so in the general case.

References

[1] K. Borsuk, *Über die Abbildungen der metrischen kompakten Räume auf die Kreislinie*, Fund. Math. 20 (1933), p. 224-231.

[2] K. Borsuk and A. Kosiński, *On Connection Between the Homology Properties of a Set and of Its Frontier*, Bull. Acad. Polon. Sci., Cl. III, 4 (1956), p. 323-325.

[3] K. Borsuk and R. Molski, *On a class of continuous mappings*, this issue.

[4] S. Eilenberg, *Sur les transformations d'espaces métriques en circonférence*, Fund. Math. 24 (1935), p. 174.

[5] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton 1952.

[6] E. E. Floyd, *Orbit spaces of finite transformation groups*, Duke Math. Journal 22 (1955), p. 33-38.

[7] O. Hanner, *Some theorems on absolute neighborhood retracts*, Arkiv för Math. 1 (1952), p. 389-408.

[8] C. Kuratowski, *Topologie I*, Monografie Matematyczne vol. XX, Warszawa-Wrocław 1948.

[9] — *Topologie II*, Monografie Matematyczne vol. XXI, Warszawa-Wrocław 1950.

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