Completely regular mappings

by

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The class of continuous mappings f of a topological space X onto a topological space Y can be ordered roughly by the amount of information about the local and in the large properties of X which can be learned from information about those properties of Y and the inverses under f of points of Y. Some of the groupings in this ordering are open mappings, homologically regular mappings [5], mappings which have the covering homotopy property, homotopically regular mappings, and projection mappings of direct products.

In this paper a new type of mapping is defined — *the completely regular mapping*. It will be shown that this type of mapping occupies a position in the ordering mentioned above just before that of the projection mapping, that under certain additional hypotheses such mappings become projection mappings and that if the inverses are certain low dimensional spaces, 0-regular maps are completely regular. Thus we will be able to show in some cases that spaces on which certain 0-regular maps are defined are direct products. Some of our results of this sort are related to results of B. J. Ball [2] and others nicely complement a result of R. H. Bing [3]. Part of Theorem 7 is a special case of a theorem of Whitney [15] proved by quite different methods.

In particular, we show that if f is a 0-regular mapping of a compact metric space onto an arc such that each inverse under f is a 2-cell then X is a 3-cell. This answers a question raised in [7], p. 84. We also show that if f is a closed mapping of E^3 onto a metric space X such that each inverse under f of a point of X is a compact continuum lying in a horizontal plane and not separating that plane, then X is homeomorphic to E^3. For information on related problems the reader is referred to [3] and [4]. The lemma of Alexander which proves so important in the argument for our principal theorem was called to our attention by

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(1) Part of this work was done while one of the authors held National Science Foundation grant NSF-G 2577 and the other held a National Science Foundation postdoctoral fellowship.

J. H. Roberts in a talk given at the Summer Institute on Set Theoretic Topology in 1955 [11].

The theorems in this paper depend on a selection theorem of E. A. Michael ([8] and [9]). We state here, in weaker form, the selection theorem we shall use.

THEOREM M. *If A and B are metric spaces such that A is complete and the covering dimension of $B \leqslant n+1$, Z is a closed subset of B, F is a mapping of A onto B such that the collection of inverses under F is lower semicontinuous* (see Lemma 4, below) *and* equi-LC^n (see Definition 4, below) *and f is a mapping of Z into A such that for $z \in Z$, $f(z) \in F^{-1}(z)$, then there is a neighborhood U of Z in B such that f can be extended to a mapping f^* of U into A such that for $b \in U$, $f^*(b) \in F^{-1}(b)$. If each inverse under F has the property that its homotopy groups of order $\leqslant n$ vanish then U may be taken to be the entire space B.*

Hereafter in this paper, if Y is a space then by the dimension of Y we mean the covering dimension.

DEFINITION 1. A mapping f of a metric space X onto a metric space Y is said to be *completely regular* provided that it is true that for each point y of Y and each positive number ε there is a positive number δ such that if x is a point of Y and $d(x, y) < \delta$, then there is a homeomorphism of $f^{-1}(y)$ onto $f^{-1}(x)$ which moves no point as much as ε.

It is to be noted that if the inverses have sufficiently strong local connectivity properties, then a completely regular mapping is n-regular in both the homology and homotopy senses and that the collection of inverses under such a mapping is continuous and equi-continuous.

In what follows we shall make use of the well known notion, the cone of a space K, and for completeness we include a definition.

DEFINITION 2. The *cone of a space K* is the decomposition space obtained from the direct product $K \times I$ of K with the unit interval I by identifying the set $K \times (1)$. Let $C(K)$ denote the cone of K.

If K is a compact metric space we may suppose that K is imbedded in Hilbert Space in such a way that the first coordinate of each point of K is 1. Then $C(K)$ may be considered as the point set consisting of the origin, P, plus the union of all line intervals PQ $(Q \in K)$. If Q is a point of K and t is a positive number we shall denote by (Q, t) the point R of the (directed) line PQ such that $PR/PQ = t$. Let K^* denote the set of all such points (Q, t) for Q in K and $t > 0$.

Let H denote the space of homeomorphisms of $C(K)$ onto itself which leave each point of K fixed. (Here, and elsewhere in this paper, if f and g are mappings of $C(K)$ onto a metric space the distance between f and g is defined as $\operatorname{lub}\big(d(f(x), g(x))\big)$ for x in $C(K)$ and is denoted by $d(f, g)$.) Let I denote the identity mapping of $C(K)$ onto itself. Let M denote a compact metric space imbedded in Hilbert Space in the manner described above for K and define M^* and (Q, t) for Q in M in the manner described above. The following is a trivial extension of a theorem of Alexander [1]. In this theorem K denotes an $(n-1)$-sphere and M consists of only one point. This theorem does not belong in the principal sequence of theorems in this paper but it is needed in their proofs.

THEOREM 1. *If f is a mapping of M into H such that for each point p in M, $d(f(p), I) \leqslant \varepsilon$, then f may be extended to a mapping F of $C(M)$ into H such that for each point p in $C(M)$, $d(F(p), I) \leqslant \varepsilon$ and $F(p) = f(p)$ for p in M.*

Proof. We shall use here the device already used by Alexander. The mapping F is defined as follows. First, $F(P) = I$. For each real number t such that $0 < t \leqslant 1$ let g_t be a homeomorphism of K^* onto itself which carries each point (q, r) of K^* onto the point $(q, r/t)$. If h is a homeomorphism in H we shall use the same letter, h, to indicate the mapping of K^* onto itself which is the identity outside $C(K)$ and is h on $C(K)$. For each point (p, t) of $C(M) - P$ let $F(p, t)$ denote $g_t^{-1} f(p) g_t$. When $t = 1$, $g_t = I$, so that $F(p, 1) = F(p) = f(p)$. That F is continuous follows from the construction of the g_t's and the continuity of f. (Clearly $F(p, t)$ converges to I as t approaches 0.) It also follows from the construction that $d(F(p, t), I) \leqslant \varepsilon$ for each point (p, t) of $C(M)$.

DEFINITION 3. A space Z is LC^m provided that for each point y of Z and each integer $k \leqslant m$ and each positive number ε there is a positive number δ such that every mapping of a k-sphere onto a subset of $S(y, \delta)$ (the set of points x of Z such that $d(x, y) < \delta$) is homotopic to a constant on a subset of $S(y, \varepsilon)$.

The following are obvious corollaries to Theorem 1.

COROLLARY 1. *The space H is LC^m for each integer m and all the homotopy groups of H vanish.*

COROLLARY 2. *If f is a homeomorphism of K onto a space L then for each integer m the space of homeomorphisms of $C(K)$ onto $C(L)$ which are extensions of f is LC^m and all its homotopy groups vanish.*

Note. Since these are topological properties this corollary remains true for any metric on this space of homeomorphisms which agrees with the topology defined by the original metric d.

We are now in a position to state and start to prove the principal theorem.

THEOREM 2. *Suppose K, X, and Y are metric spaces, K compact, X complete and Y finite dimensional, and f is a completely regular mapping of X onto Y such that* (1) *for each point p of Y there is a homeomorphism f_p of $C(K)$ onto $f^{-1}(p)$ and* (2) *there is a homeomorphism h of $\bigcup f_p(K)$ $(p \in Y)$ onto the direct product $Y \times K$ such that the diagram*

(1)
$$\begin{array}{ccc} \bigcup f_p(K) & \xrightarrow{h} & Y \times K \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \pi} \\ & & Y \end{array}$$

where π is the projection map, is commutative. Then there is a homeomorphism h^* of X onto the direct product $Y \times C(K)$ which is an extension of h and is such that the diagram

(2)
$$\begin{array}{ccc} X & \xrightarrow{h^*} & Y \times C(K) \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \pi} \\ & & Y \end{array}$$

is commutative.

Proof. Since diagram (1) is commutative the homeomorphism h carries $f_p(K)$ onto the set $(p) \times K$ for each point p of Y. Let h_p denote the homeomorphism of K onto $f_p(K)$ which carries each point x of K onto the inverse under h of the point (p, x) of $Y \times K$.

For each point p of Y let G_p denote the space of all homeomorphisms of $C(K)$ onto $f^{-1}(p)$ which coincide with h_p on K. Let G denote the collection of all G_p and let G^* denote the space of homeomorphisms which is the union of the elements of G. Let $\overline{G}_p$ denote the closure of G_p in the space of all mappings of $C(K)$ into $f^{-1}(p)$, let $\overline{G}$ denote the collection of all $\overline{G}_p$ and let $\overline{G}^*$ denote the union of the elements of $\overline{G}$.

Before proceeding with the proof of Theorem 2 we prove some lemmas.

LEMMA 1. *If ε is a positive number and y is a point of Y there is a positive number δ such that if x is a point of Y and $d(y, x) < \delta$, then there is a homeomorphism of $f_y(C(K))$ onto $f_x(C(K))$ which coincides with $h_x h_y^{-1}$ on $f_y(K)$ and moves no point as much as ε.*

Proof. There is a positive number $\delta' < \varepsilon$ such that each homeomorphism of $f_y(K)$ onto itself which moves no point as much as $\frac{1}{2}\delta'$ can be extended to a homeomorphism of $f_y(C(K))$ onto itself which moves no point as much as $\frac{1}{2}\varepsilon$. There is a positive number δ such that if $d(y, x) < \delta$ then (1) there is a homeomorphism t of $f_y(C(K))$ onto $f_x(C(K))$ which

moves no point as much as $\frac{1}{4}\delta'$ and (2) $h_x h_y^{-1}$ moves no point as much as $\frac{1}{4}\delta'$. The homeomorphism $t^{-1}h_x h_y^{-1}$ of $f_y(K)$ onto itself moves no point as much as $\frac{1}{2}\delta'$. Therefore $t^{-1}h_x h_y^{-1}$ can be extended to a homeomorphism q of $f_y(C(K))$ onto itself which moves no point as much as $\frac{1}{2}\varepsilon$. The homeomorphism tq moves no point as much as ε and coincides with $h_x h_y^{-1}$ on $f_y(K)$.

LEMMA 2. *The metric space G^* is complete.*

Proof. Since X is complete the space Z of all mappings of $C(K)$ into X is complete in the metric d. If $z_1, z_2, \ldots$ is a Cauchy sequence in $\overline{G}^*$, it converges to a mapping z, and since each z_i coincides with some h_{p_i} on K, z coincides with some h_p on K. Here z_i is an element of $\overline{G}_{p_i}$ and $p_1, p_2, \ldots$ converges to p in Y. By Lemma 1, if ε is a positive number there is an integer N such that if $n > N$ there is a homeomorphism f_n of $f_{p_n}(C(K))$ onto $f_p(C(K))$ which moves no point as much as $\frac{1}{3}\varepsilon$ and coincides with $h_p h_{p_n}^{-1}$ on $f_{p_n}(K)$ and $d(z, z_n) < \frac{1}{3}\varepsilon$. For each n there is a homeomorphism g_n in G_{p_n} such that $d(g_n, z_n) < \frac{1}{3}\varepsilon$. The homeomorphism $f_n g_n$ is an element of G_p and $d(f_n g_n, z) < d(f_n g_n, g_n) + d(g_n, z_n) + d(z_n, z) < \varepsilon$. Hence z is in $\overline{G}_p$ and $\overline{G}^*$ is complete in the metric d.

For each integer n let H_n denote the collection of all mappings g in $\overline{G}^*$ such that $\operatorname{lub} \operatorname{diam} g^{-1}(x) \geqslant 1/n$ for x in $g[C(K)]$. The subspace H_n of $\overline{G}^*$ is closed in $\overline{G}^*$. But $G^* = \overline{G}^* - \bigcup H_n$ and is therefore a G_δ. But any G_δ in a complete metric space is complete. Hence G^* is complete.

Let d^* denote a metric under which G^* is complete.

DEFINITION 4. A collection G of closed point sets filling a metric space is said to be equi-LC^m provided that it is true that if y is a point of an element g_0 of G and ε is a positive number there is a positive number δ such that if g is an element of G then any mapping of a k-sphere $(k \leqslant m)$ onto a subset of $g \cap S(y, \delta)$ is homotopic to a constant on a subset of $g \cap S(y, \varepsilon)$.

It is clear that this is a topological property and hence that the following lemma will be true in the metric d^* if it is proved for the metric d.

LEMMA 3. *The collection G is* equi-LC^m *for each integer m.*

Proof. Suppose that s is an element of G_p and ε is a positive number. By Theorem 1 and its corollaries there is a positive number $\delta' < \frac{1}{2}\varepsilon$ such that every mapping of the k-sphere S^k $(k \leqslant m)$ onto a subset of $G_p \cap S(s, \delta')$ can be extended to a mapping of R^{k+1} (S^k plus its interior) onto a subset of $G_p \cap S(s, \frac{1}{2}\varepsilon)$. By Lemma 1 there is a positive number d such that if $d(p, q) < d$, $q \in Y$, then there is a homeomorphism h of $f_p(C(K))$ onto

$f_q(C(K))$ which moves no point as much as $\frac{1}{2}\delta'$ and coincides with $h_q h_p^{-1}$ on $f_p(K)$. Let δ be a positive number such that $\delta < \frac{1}{2}\delta'$ and if r is a homeomorphism in $S(s, \delta)$ there is a point q of Y such that r is in G_q and $d(p, q) < d$. This is the required δ. To see this let t be a mapping of S^k onto a subset of $G_q \cap S(s, \delta)$. Since $\delta < \frac{1}{2}\delta'$, $h^{-1}t$ carries S^k onto a subset of $G_p \cap S(s, \delta')$. Therefore $h^{-1}t$ can be extended to a mapping T of R^{k+1} onto a subset of $G_p \cap S(s, \frac{1}{2}\varepsilon)$. The mapping hT coincides with t on S^k and maps R^{k+1} onto a subset of $G_q \cap S(s, \varepsilon)$. Thus G is equi-LC^m.

Lemma 4. *The collection G is lower semicontinuous in the sense that if the sequence of points $p_1, p_2, \ldots$ of Y converges to a point p then G_p is in the closure of $\bigcup G_{p_i}$.*

Proof. This is a topological property so that the lemma will be true for the metric d^* if it is true for d. Suppose g belongs to G_p and ε is a positive number. By Lemma 1 there is an integer N such that if $n > N$ there is a homeomorphism g_n of $f_{p_n}(C(K))$ onto $f_p(C(K))$ which coincides with $h_p h_{p_n}^{-1}$ on $f_{p_n}(K)$ and moves no point as much as ε. The homeomorphism $g_n^{-1}g$ is in G_{p_n} and $d(g, g_n^{-1}g) < \varepsilon$. Thus g is in the closure of $\bigcup G_{p_i}$.

Completion of the proof of Theorem 2. It follows from Theorem 1 that each G_p has the property that all its homotopy groups vanish. We have shown that the function which carries each element G_p of G onto the point p of Y is a mapping of G^* onto Y satisfying the hypothesis of Theorem M. Thus there is a homeomorphism g of Y into G^* such that for each point y of Y, $g(y) \in G_y$. For each point x of X let $h^*(x)$ denote the point $(f(x), [g(f(x))]^{-1}(x))$ of $Y \times C(K)$. The mapping h^* is the one required by Theorem 2. The mapping h^* is clearly continuous. To see that it is 1-1, suppose $h^*(x_1) = h^*(x_2)$. Then $f(x_1) = f(x_2)$. Therefore $g(f(x_2)) = g(f(x_2))$ and since this is a homeomorphism and $[g(f(x_1))]^{-1}(x_1) = [g(f(x_2))]^{-1}(x_2)$, $x_1 = x_2$. To see that h^* is onto, suppose that (y, z) is a point of $Y \times C(K)$. The symbol $[g(y)](z)$ denotes a point r of $f^{-1}(y)$, $f(r) = y$, and since $g(y) = g(f(r))$, $[g(f(r))]^{-1}(r) = z$. Thus $h^*(r) = (y, z)$. Since $\pi(y, z) = y$ and $f(r) = y$, the diagram (2) is commutative. Finally, h^* is an extension of h, for if x is a point of $f_y(K)$, $h(x) = (y, h_y^{-1}(x))$. (Recall that if z is a point of K then $h_y(z)$ is the point $h^{-1}(y, z)$ of X and that diagram (1) is commutative.) But each mapping in G_y coincides with h_y on K. Therefore $h(x) = (f(x), [g(f(x))]^{-1}(x))$, h^* extends h and the proof of Theorem 2 is complete.

We shall now apply Theorem 2 to the proofs of the theorems mentioned in the introduction. But first we show in Theorems 3, 4, and 5 that with certain added conditions on K and Y condition (2) in the hypothesis of Theorem 2 may be omitted.

Theorem 3. *Suppose that f is a completely regular mapping of the complete metric space X onto the finite dimensional space Y. Suppose, further, that there is a point set K which is either* (1) *discrete or* (2) *a simple closed curve such that for each point y of Y, $f^{-1}(y)$ is homeomorphic to K. Then (X, f, Y) is a locally trivial fibre space.*

Proof. If K is discrete, then, since X is complete and $f^{-1}(y)$ is closed in X, there is, for each $y \in Y$, a positive number ε such that if $p, q \in f^{-1}(y)$ $(p \neq q)$, then $d(p, q) > 2\varepsilon$. Since f is completely regular there is a positive number δ such that if $d(y, z) < \delta$, $(y, z \in Y)$, then there is a homeomorphism of $f^{-1}(y)$ onto $f^{-1}(z)$ which moves no point as much as ε. Let U be a δ-neighborhood of y. Then $f^{-1}(U) = \bigcup V_p$ $(p \in f^{-1}(y))$ where $V_p = f^{-1}(U) \cap S(p, \varepsilon)$. Clearly f is a homeomorphism on each V_p. The mapping h of $f^{-1}(U)$ onto $K \times U$ which carries the point q of V_p into the point $(g(p), f(q))$ (g is a homeomorphism of $f^{-1}(y)$ onto K) is a homeomorphism. This shows that (X, f, Y) is a locally trivial fibre space.

In case K is a simple closed curve, for each point y of Y, let $S(y)$ denote the space of all homeomorphisms of K onto $f^{-1}(y)$ and let $S = \bigcup S(y)$ $(y \in Y)$. The following statements are established as before: S can be remetrized so as to be complete, under this metric the collection of all $S(y)$ is lower semicontinuous and equi-LC^m for each integer m and each of the two components of $S(y)$ is arcwise connected. To see that these statements are true, repeat the arguments for Lemmas 2, 3 and 4, recalling that the properties of being LC^m and lower semi-continuous are topological properties and hence hold for the new metric if they hold for the original metric.

Let F be the mapping of S onto Y which carries each $S(y)$ onto y. By the above remarks, F satisfies the conditions for Theorem M. Hence if y is a point of Y there is a neighborhood U_y of y and a mapping g of U_y into S such that for u in U_y, $g(u) \in S(u)$. Let h be the mapping of $U_y \times K$ onto $f^{-1}(U_y)$ defined by $h(u, x) = g(u)(x)$. Since g is continuous, h is continuous. Since $g(u)(x) \in f^{-1}(u)$ and $g(u)$ is a homeomorphism onto, h is a homeomorphism onto. Hence (X, f, Y) is a locally trivial fibre space.

Notes. If it were known that the space of homeomorphisms of S^n onto itself were LC^k for every k, a slight variation of the above argument would establish this theorem in case K were an n-sphere.

If Y is contractible, locally compact and separable then since (X, f, Y) is a locally trivial fibre space X is the direct product $K \times Y$. (See [13], p. 53.)

THEOREM 4. *Suppose that f is a completely regular mapping of the complete metric space X onto the 1-dimensional, contractible, locally compact and separable metric space Y. Suppose, further, that for each point y of Y, $f^{-1}(y)$ is a homeomorphic to point set K where K is a 2-sphere or a 3-sphere. Then there is a homeomorphism h of X onto $Y\times K$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y\times K \\ & \searrow^{f} & \downarrow{\pi} \\ & & Y \end{array}$$

is commutative.

Proof. For each point y of Y let $S(y)$ denote the space of homeomorphisms of K onto $f^{-1}(y)$. It is known ([6], [11], and [12]), that each $S(y)$ is locally connected, (*i. e.* LC^0). Duplicating the proofs of Lemmas 2, 3, and 4, we see that $S=\bigcup S(y)$ can be remetrized so as to be complete and that under this new metric the collection of all $S(y)$ is lower semi-continuous and equi-LC^0, these last properties being topological. The proof of Theorem 3 can now be repeated word for word, using Theorem M in the case $n=0$, to show that (X, f, Y) is a locally trivial fibre space. Since Y is contractible, locally compact and separable, X is, as above, the required direct product.

THEOREM 5. *If f is a completely regular mapping of a complete metric space X onto an n-cell, Y, such that each inverse under f is homeomorphic to a point set K, where K is a 2-sphere or a 3-sphere, then there is a homeomorphism h of X onto $Y\times K$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y\times K \\ & \searrow^{f} & \downarrow{\pi} \\ & & Y \end{array}$$

is commutative.

Proof. By theorem 4, if $\dim Y=1$, Theorem 5 holds. We shall use induction on the dimension of Y, assuming that Theorem 5 holds if $\dim Y\leqslant k$ and proving it if $\dim Y=k+1$. But first we prove some lemmas.

LEMMA A. *If Y' is a contractible metric space and K' is a compact metric space for which the space $H(K')$ of all homeomorphisms of K' onto itself is locally connected then the space $H(Y', K')$ of all homeomorphisms of $Y'\times K'$ onto itself which leave the set (y', K') invariant for each $y'\in Y'$ is locally connected.*

Proof. Since each of the spaces $H(K')$ and $H(Y'\times K')$ is a topological group we have only to prove that $H(Y'\times K')$ is locally arcwise connected at its identity, I^*. Let I denote the identity of $H(K')$ and $S(I,\varepsilon)$ and $S^*(I^*,\varepsilon)$ the elements of $H(K')$ and $H(Y'\times K')$ respectively which move no point as much as ε. Suppose $\varepsilon>0$. Since $H(K')$ is locally arcwise connected there is a $\delta>0$ such that if $g\in S(I,\delta)$ then there is an arc from I to g in $S(I,\varepsilon)$. Let t be a homeomorphism in $S^*(I^*,\delta)$. Note that t leaves each (y', K') invariant. For each $y\in Y'$, let t_y denote the homeomorphism of K' onto itself with the property that if $t(y,x)=(y,x')$, $t_y(x)=x'$. Let z be the mapping of Y' into $H(K')$ such that $z(y)=t_y$. Clearly $z(Y')$ lies in $S(I,\delta)$. Since Y' is contractible, there is a mapping Z of $Y'\times(0,1)$ into $S(I,\varepsilon)$ such that $Z(y,0)=I$ and $Z(y,1)=z(y)$ for each $y\in Y'$. Let T be a mapping of $(0,1)$ into $H(Y'\times K')$ such that $T(s)(y,x)=\big(y, Z(y,s)(x)\big)$. Clearly $T(0,1)$ lies in $S^*(I^*,\varepsilon)$, $T(0)(y,x)=(y,x)$ and $T(1)(y,x)=\big(y,z(y)(x)\big)=\big(y,t_y(x)\big)$ $=t(y,x)$. Hence $H(Y'\times K')$ is locally connected.

LEMMA B. *If Y' and K' are as in Lemma A and ε is a positive number then there is a positive number δ such that, in $Y'\times K'\times(0,1)$, if g is a homeomorphism of $Y'\times K'\times(1)$ onto itself moving no point as much as δ and leaving each set $(y', K', 1)$ invariant then there is a homeomorphism G of $Y'\times K'\times(0,1)$ onto itself which extends g, leaves each set (y', K', t) invariant $(0\leqslant t\leqslant 1)$, moves no point as much as ε and is the identity on $Y'\times K'\times(0)$.*

Proof. Suppose $\varepsilon>0$. By Lemma A there is a $\delta>0$ such that if g is a homeomorphism in the space H^* of homeomorphisms of $Y'\times K'\times(1)$ onto itself which leave each set $(y', K', 1)$ invariant and g moves no point as much as δ then there is a mapping Z of $(0,1)$ into H^* such that $Z(1)=g$ and $Z(0)=I$, the identity mapping, and $Z(t)$ moves no point as much as ε for t in $(0,1)$. If $Z(t)(y,x,1)=(y,x',1)$ let $G(y,x,t)=(y,x',t)$. Then $G(y,x,1)=g(y,x,1)$ and $G(y,x,0)=(y,x,0)$ since $Z(0)(y,x,1)$ $=(y,x,1)$. The mapping G satisfies the conditions required by the Lemma.

Note. It is clear that if, in the statement of Lemma B, $(0,1)$ is replaced by (s,s'), $s<s'$, the Lemma still holds.

Continuation of the proof of Theorem 5. Let Y be a $(k+1)$-cell and suppose that $Y=Y'\times(0,1)$ where Y' is a k-cell and $Y'=Y''\times$ $\times(0,1)$ where Y'' is a $(k-1)$-cell. Let F be the mapping of X onto $(0,1)$ which carries $f^{-1}(Y',t)$ into t. By the induction hypothesis there is, for each $t\in(0,1)$, a homeomorphism h_t of $f^{-1}(Y',t)$ onto $Y'\times K$ such that the diagram

$$\begin{array}{ccc} f^{-1}(Y',t) & \xrightarrow{h_t} & Y'\times K \\ & \searrow^{f} & \downarrow{\scriptstyle \pi_1} \\ & & Y' \end{array}$$

where π_1 is the projection mapping, is commutative. Also, if $s \in (0,1)$ there is a homeomorphism f_s of $f^{-1}(Y''\times s\times(0,1))$ onto $Y''\times(0,1)\times K$ such that the diagram

$$\begin{array}{ccc} f^{-1}(Y''\times s\times(0,1)) & \xrightarrow{f_t} & Y''\times(0,1)\times K \\ & \searrow^{f} & \downarrow{\scriptstyle \pi_2} \\ & & Y''\times(0,1), \end{array}$$

where π_2 is the projection mapping, is commutative.

Lemma C. *The mapping F is completely regular. Moreover, if $\varepsilon > 0$, there is a $\delta > 0$ such that if $|t-t'| < \delta$, there is a mapping z of $F^{-1}(t) = f^{-1}(Y',t)$ onto $F^{-1}(t') = f^{-1}(Y',t')$ which moves no point as much as ε and carries each set $h_t^{-1}(y,K)$ onto $h_{t'}^{-1}(y,K)$ (i. e. $f^{-1}(y'',s,t)$ onto $f^{-1}(y'',s,t')$).*

Proof. Suppose $\varepsilon > 0$. Let $\delta' < \varepsilon/100$ be a positive number such that if $t, s, s' \in (0,1)$, $s < s'$, and g is a homeomorphism of $f^{-1}(Y'',s',t)$ onto itself moving no point as much as δ' and leaving each set $f^{-1}(y'',s',t)$ invariant, then there is a homeomorphism G of $f^{-1}(Y'',(s,s'),t)$ onto itself which moves no point as much as $\varepsilon/100$, leaves each set $f^{-1}(y'',s'',t)$, $s'' \in (s,s')$, invariant, extends g and is the identity on $f^{-1}(Y'',s,t)$. This follows from Lemma B and the facts that Y and K are compact and the space of homeomorphisms of K onto itself is locally connected. Fix t. Let $\delta < \delta'/100$ be such that if $|t-t'| < \delta$ and $|s-s'| < \delta$, then the natural homeomorphisms of $f^{-1}(Y'',s,t)$ onto $f^{-1}(Y'',s,t')$ and $f^{-1}(Y'',s,t')$ onto $f^{-1}(Y'',s',t')$ induced by f_s and $h_{t'}$, respectively, move no point as much as $\delta'/100$. These homeomorphisms carry each set $f^{-1}(y'',s,t)$ onto $f^{-1}(y'',s,t')$ and $f^{-1}(y'',s,t')$ onto $f^{-1}(y'',s',t')$, respectively. (The natural homeomorphism of $f^{-1}(Y'',s,t)$ onto $f^{-1}(Y'',s,t')$ induced by f_s is that homeomorphism h such that $f_s h(x)$ is the point of $Y''\times t'\times K$ into which $f_s(x)$ projects in the projection mapping of $Y''\times(0,1)\times K$ onto $Y''\times t'\times K$.)

Now suppose $|t-t'| < \delta$ and $0 = s_0 < \ldots < s_n = 1$, $(s_{i+1}-s_i) < \delta$. Let g_i be the natural homeomorphism of $f^{-1}(Y'',s_i,t)$ onto $f^{-1}(Y'',s_i,t')$ induced by f_{s_i} and h_{i+1} and h'_{i+1} the natural homeomorphisms of $f^{-1}(Y'',s_i,t)$ and $f^{-1}(Y'',s_i,t')$ onto $f^{-1}(Y'',s_{i+1},t)$ and $f^{-1}(Y'',s_{i+1},t')$, respectively, induced by h_t and $h_{t'}$, respectively. Let $F_{i+1} = g_{i+1}^{-1}h'_{i+1}g_i h_{i+1}^{-1}$ and $F_0 = I$.

The mapping F_{i+1} is a homeomorphism of $f^{-1}(Y'',s_{i+1},t)$ onto itself which leaves each set $f^{-1}(y'',s_{i+1},t)$ invariant and moves no point as much as δ'. Thus there is a homeomorphism G_{i+1} of $f^{-1}(Y'',(s_i,s_{i+1}),t)$ onto itself which leaves each $f^{-1}(y'',s,t)$ invariant, leaves $f^{-1}(Y'',s_i,t)$ pointwise fixed, is identical with F_{i+1} on $f^{-1}(Y'',s_{i+1},t)$ and moves no point as much as $\varepsilon/100$.

Let z_{i+1} be the homeomorphism of $f^{-1}(Y'',(s_i,s_{i+1}),t')$ onto $f^{-1}(Y'',(s_i,s_{i+1}),t)$ defined in the following manner. Suppose $x \in (y'',s,t')$. Let $\pi(x)$ denote the image of x in $f^{-1}(y'',s_i,t')$ under the natural homeomorphism of $f^{-1}(Y'',s,t')$ onto $f^{-1}(Y'',s_i,t')$ induced by $h_{t'}$. Let π' denote the natural homeomorphism of $f^{-1}(Y'',s_i,t)$ onto $f^{-1}(Y'',s,t)$ induced by h_t. Let $z_{i+1}(x) = G_{i+1}\pi' g_i^{-1}\pi(x)$. This moves no point as much as ε. If $x \in f^{-1}(Y'',s_i,t')$, $z_{i+1}(x) = g_i^{-1}(x)$ and $z_i(x) = g_i^{-1}h'_i g_{i-1}h_i^{-1}h_i g_{i-1}^{-1}h_i'^{-1}(x) = g_i^{-1}(x)$. Therefore the z_i can be fitted together to give a homeomorphism z of $f^{-1}(Y''\times(0,1)\times t')$ onto $f^{-1}(Y''\times(0,1)\times t)$ which is z_{i+1} on $f^{-1}(Y'',(s_i,s_{i+1}),t')$ and has the required properties.

For each t in $(0,1)$, let $S(t)$ denote the space of homeomorphisms of $Y'\times K$ onto $F^{-1}(t)$ which carry each $y'\times K$ onto $f^{-1}(y',t)$, $y' \in Y'$. By Lemma A, $S(t)$ is LC^0. The mapping F is completely regular in the sense of Lemma C, (*i. e.* if $\varepsilon > 0$, there is a $\delta > 0$ such that if $|t-t'| < \delta$, there is a homeomorphism of $F^{-1}(t)$ onto $F^{-1}(t')$ carrying each $f^{-1}(y,t)$ onto $f^{-1}(y,t')$, $y \in Y'$, and moving no point as much as ε) and $(0,1)$ is one dimensional, locally compact, separable and contractible. A repetition of the argument for Lemma 2 shows that $S = \bigcup S(t)$ can be remetrized so as to be complete and repetitions of the arguments for Lemmas 3 and 4 show that the collection of all $S(t)$ is equi-LC^0 and lower semicontinuous. (In this argument, Lemma C is a counterpart of Lemma 1.) Thus the mapping g of S into $(0,1)$ which carries each $S(t)$ onto t satisfies the conditions of Theorem M for the case $n = 0$. Therefore, for each t in $(0,1)$ there is a neighborhood U of t and a mapping q of U into S such that for each u in U, $q(u) \in S(u)$. The mapping h of $Y'\times U\times K$ onto $F^{-1}(U)$ defined by $h(y,u,x) = q(u)(y,x)$ is a homeomorphism. Hence $(X, F, (0,1))$ is a locally trivial fibre space. However, $Y'\times U$ is an open subset of $Y'\times(0,1) = Y$ and the homeomorphism h carries each $y\times t\times K$ onto $q(t)(y,K) = f^{-1}(y,t)$. Thus (X, f, Y) is a locally trivial fibre space. Since Y is locally compact, separable and contractible, it follows as before (see [13], p. 53) that X is the product $Y\times K$.

Remark. Theorem 5 remains true if K is any compact metric space such that the space of homeomorphisms of K onto itself is locally connected. Also, Theorems 3-5 show that if K, in Theorem 2, is a 2-sphere or a 3-sphere and Y is an n-cell or a contractible one-dimensional, locally compact, separable metric space then condition (2) of Theorem 2 may be omitted. This condition may also be omitted if K is a 0-sphere or 1-sphere and Y is contractible and finite dimensional.

DEFINITION 5. The mapping f of a metric space X onto a metric space Y is said to be 0-*regular* provided that it is true that f is open and if ε is a positive number, y is a point of Y, and p is a point of $f^{-1}(y)$ then there is a positive number δ such that if x is a point of Y and a and b are points of $f^{-1}(x)$ in $S(p,\delta)$ then there is an arc from a to b in the common part of $f^{-1}(x)$ and $S(p,\varepsilon)$.

THEOREM 6. *Suppose that f is a 0-regular mapping of a metric space X onto a metric space Y. Suppose, further, that there is a space K which is either an i-cell or an i-sphere ($i \leqslant 2$) such that for each point y of Y, $f^{-1}(y)$ is homeomorphic to K. Then the mapping f is completely regular.*

Proof. We leave to the reader the essentially elementary details of the proof for the cases $i \leqslant 1$. We present the details of the proof for the case where K is a 2-cell. If K is a 2-sphere the proof is essentially unaltered.

We show, first, that if K is a 2-cell then f is 1-regular in the sense that if ε is a positive number, y is a point of Y and p is a point of $f^{-1}(y)$ then there is a positive number δ such that if x is a point of Y and J is a 1-sphere in $f^{-1}(x)\cap S(p,\delta)$ then J bounds a 2-cell in $f^{-1}(x)\cap S(p,\varepsilon)$. Suppose that f is not 1-regular. Then there are a positive number δ, a sequence of points $y_1, y_2, \ldots,$ of Y converging to a point y of Y and a sequence $J_1, J_2, \ldots$ of simple closed curves such that this sequence converges to a point P of $f^{-1}(y)$ and for each i, J_i is a subset of $f^{-1}(y_i)$ and the interior U_i of J_i relative to $f^{-1}(y_i)$ has diameter greater than ε. There is a point Q of $f^{-1}(y)$ which is different from P and is in the limiting set of the U_i. There is a simple closed curve M in $f^{-1}(y)$ whose interior V relative to $f^{-1}(y)$ contains P and whose exterior contains Q. (If P is on the boundary of $f^{-1}(y)$ then P is on M.) It follows from a result of Whyburn [14] that there is a sequence $M_1, M_2, \ldots$ of simple closed curves which converges to M 0-regularly and is such that for each i, M_i is a subset of $f^{-1}(y_i)$. (*I. e.* the mapping f defined on $M\cup(\bigcup M_i)$ is 0-regular.) Let V_i denote the interior of M_i relative to $f^{-1}(y_i)$. It is easy to show that each convergent subsequence of the sequence $M_1\cup V_1$, $M_2\cup V_2, \ldots$ converges 0-regularly. We assume, without loss of generality, that the sequence itself has this property. It then follows from the results in [14] that this sequence converges to $M\cup V$. But for sufficiently large i, V_i contains U_i. Since Q is a limit point of $\bigcup U_i$, we have a contradiction. Thus f is 1-regular.

It follows from results in that same paper of Whyburn that the mapping f defined on $\bigcup Bd(f^{-1}(y))$, $y \in Y$, is 0-regular and hence completely regular.

Since f is 1-regular and each $f^{-1}(y)$ is compact it follows that if y is a point of Y and ε is a positive number there are positive numbers δ and d such that if $d(y,x)<d$ and J is a 1-sphere in $f^{-1}(x)$ of diameter less than δ, then J bounds a 2-cell in $f^{-1}(x)$ of diameter less than $\frac{1}{3}\varepsilon$. Let $s_0, s_1, \ldots, s_n$ and $t_0, t_1, \ldots, t_n$ be arcs in $f^{-1}(y)$ such that (1) the union of $s_0, s_n, t_0,$ and t_n is $\mathrm{Bd}(f^{-1}(y))$ and these arcs are non-overlapping, (2) for each i, t_i has one endpoint on s_0 and the other on s_n and separates t_{i-1} from t_{i+1} in $f^{-1}(y)$ and s_i has one endpoint on t_0 and the other on t_n and separates s_{i-1} from s_{i+1} in $f^{-1}(y)$, (3) $s_i\cap t_j$ is a point P_{ij} and (4) the diameter of each component of $f^{-1}(y)-\bigcup(s_i\cup t_i)$ is less than $\frac{1}{3}\delta$. It follows from Whyburn's results that if t is an arc in $f^{-1}(y)$ and $y_1, y_2, \ldots$ converges to y, then there is a sequence $t'_1, t'_2, \ldots$ of arcs converging 0-regularly to t such that for each i, t'_i is a subset of $f^{-1}(y_i)$. Using this result, it is seen that there is a positive number $d'<d$ such that if $d(y,x)<d'$, then in $f^{-1}(x)$ there are arcs $s'_0, s'_1, \ldots, s'_n$ and $t'_0, t'_1, \ldots, t'_n$ such that (1) the union of $s'_0, s'_n, t'_0,$ and t'_n is $\mathrm{Bd}(f^{-1}(x))$ and these arcs are non-overlapping, (2) for each i, t'_i has one endpoint on s'_0 and the other on s'_n and separates t'_{i-1} from t'_{i+1} in $f^{-1}(x)$ and s'_i has one endpoint on t'_0 and the other on t'_n and separates s'_{i-1} from s'_{i+1} in $f^{-1}(x)$, (3) $s'_i\cap t'_j$ is a point P'_{ij} and (4) there is a homeomorphism h carrying $\bigcup(t_i\cup s_i)$ onto $\bigcup(t'_i\cup s'_i)$ such that $h(s_i)=s'_i$, $h(t_i)=t'_i$ and h moves no point as much as $\frac{1}{3}\delta$.

If U is a component of $f^{-1}(x)-\bigcup(t'_i\cup s'_i)$, its boundary clearly has diameter less than δ. Therefore U has diameter less that $\frac{1}{3}\varepsilon$. A homeomorphism H of $f^{-1}(y)$ onto $f^{-1}(x)$ which is an extension of h moves no point as much as ε. Thus we have shown that f is completely regular.

THEOREM 7. *If f is a 0-regular mapping of a complete metric space X onto a finite dimensional, contractible, locally compact, separable metric space Y such that for each point y of Y, $f^{-1}(y)$ is homeomorphic to the point set K, where K is a 1-cell, a 2-cell or a 1-sphere, then is a homeomorphism h of X onto the direct product $Y\times K$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y\times K \\ & \searrow^{f} & \downarrow \pi \\ & & Y \end{array}$$

is commutative. If K is 2-sphere and Y is one-dimensional and contractible or Y is an n-cell, then the theorem remains true.

Proof. Since f is 0-regular, f is completely regular. If K is a 1-sphere or a 2-sphere, Theorem 7 follows immediately from Theorems 3-5. If K is a 1-cell or a 2-cell, the mapping f defined on $\bigcup \operatorname{Bd}(f^{-1}(y))$ $(y \in Y)$ is completely regular. Thus, by Theorems 3-5, condition (2) of Theorem 2 is satisfied and Theorem 7 now follows.

The next theorem complements results of Bing [3]. The reader is particularly referred to the final section of that paper.

THEOREM 8. *If q is a closed mapping of E^3 onto a metric space Y such that for each point y of Y, $q^{-1}(y)$ is a compact continuum which lies in a horizontal plane and does not separate that plane, then Y is homeomorphic to E^3.*

Proof. Let C_1 and C_2 be concentric spheres in E^3, let K denote the set of points between C_1 and C_2 and let L denote the common part of K and a ray terminating at the common center of C_1 and C_2. If $y \in L$ let S_y denote the sphere concentric with C_1 which contains y. Let f denote a closed mapping of K onto a space X such that if x is a point of X then either $f^{-1}(x)$ is a point of L or there is a point y of L such that $f^{-1}(x)$ is a compact continuum which lies in $S_y - y$ and does not separate S_y. Since S_y is a 2-sphere and $f(S_y)$ is homeomorphic to the decomposition space whose "points" are the sets $f^{-1}(x)$, $x \in f(S_y)$, and none of these sets separates S_y it follows from the paper [10] that $f(S_y)$ is homeomorphic to a 2-sphere, S. Let g denote the mapping of X onto L such that $gf(S_y) = y$ for each y in L. The space X is imbeddable as an open set in the space X' which is the image of $K \cup C_1 \cup C_2$ under a mapping f' whose inverses are C_1, C_2, and the inverses under f. Since X' is compact, X can be metrized so as to be complete. To see that g is 0-regular, suppose p is a point of $f(S_y)$, an inverse under g, and ε is a positive number. Let d be a positive number such that if x is in the d-neighborhood, V_d, of $f^{-1}(p)$ then $d(p, f(x)) < \varepsilon$. Since $f^{-1}(p)$ is connected, $V_d \cap S_y$ is arcwise connected. Thus there is a positive number $d' < d$ such that if $a, b \in S_x \cap V_{d'}$, where x is any point of L, then there is an arc from a to b in $S_x \cap V_d$. Let δ be a positive number such that if $d(p, q) < \delta$ then $f^{-1}(q)$ lies in $V_{d'}$. Thus if $q, q' \in f(S_x)$ and $d(p, q) < \delta$ and $d(p, q') < \delta$ then there is an arc ab in $S_x \cap V_d$ from a point a of $f^{-1}(q)$ in $V_{d'}$ to a point b of $f^{-1}(q')$ in $V_{d'}$. By the definition of d, $f(ab)$ is in the common part of $f(S_x)$ and the ε-neighborhood of p and since f is continuous, $f(ab)$ contains an arc with endpoints q and q'. Thus g is 0-regular.

Clearly X can be metrized so as to be complete and g is 0-regular. Thus, by Theorem 7, there is a homeomorphism h of X onto $S \times L$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & S \times L \\ & \searrow^{g} & \downarrow \pi \\ & & L \end{array}$$

is commutative. Also, h can be chosen so that $h(f(L)) = (p) \times L$ for some point p of S. Thus the diagram

$$\begin{array}{ccc} X - f(L) & \xrightarrow{h} & (S - p) \times L \\ & \searrow^{g} & \downarrow \pi \\ & & L \end{array}$$

is also commutative.

There is a homeomorphism t of E^3 onto $K - L$ which carries each horizontal plane onto some $S_y - y$ $(y \in L)$. Let f be a mapping of K onto a space X such that if x is a point of X then either $f^{-1}(x)$ is a point of L or there is a point y of Y such that $f^{-1}(x) = tq^{-1}(y)$. The mapping f satisfies the conditions required of f in the above paragraph. The space Y is thus homeomorphic to $X - f(L)$ through the homeomorphism which maps y onto $ftq^{-1}(y)$. But $X - f(L)$ is homeomorphic to $(S - p) \times L$ which, in turn, is homeomorphic to E^3. Thus Theorem 8 is proved.

References

[1] J. W. Alexander, *On the deformation of an n-cell*, Proc. Nat. Acad. Sci. U.S.A. 9 (1923), p. 406-407.

[2] B. J. Ball, *Continuous and equicontinuous collections of arcs*, Duke Math. J. 19 (1952), p. 423-433.

[3] R. H. Bing, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Annals of Mathematics 65 (1957), p. 484-500.

[4] — *Decomposition of E^3 into points and tame arcs*, Summary of lectures, Summer Institute on Set Theoretic Topology, Madison, Wisconsin, 1955, p. 40.

[5] Eldon Dyer, *Open mappings and regular convergence*, to appear in Annales of Mathematics.

[6] M. K. Fort, Jr., *A proof that the group of homeomorphisms of the plane ono itself is locally arcwise connected*, Proc. Amer. Math. Soc. 1 (1955), p. 59-62.

[7] M.-E. Hamstrom, *Continuous collections of continuous curves*, Summary of lectures, Summer Institute on Set Theoretic Topology, Madison, Wisconsin, 1955, p. 82.

[8] E. A. Michael, *Continuous selections, I, II and III*, Annals of Mathematics 63 (1956), p. 361-382; 64 (1956), p. 562-580; 65 (1957), p. 375-390.

[9] — *Selection theorems for continuous functions*, Proceedings of the International Congress of Mathematicians, Amsterdam 1955, vol. II, p. 241.

[10] R. L. Moore, *Collections of continua*, Trans. Amer. Math. Soc. 27 (1925), p. 416-428.

[11] J. H. Roberts, *Local arcwise connectivity in the space H^n of homeomorphisms of S^n onto itself*, Summary of lectures, Summer Institute on Set Theoretic Topology, Madison, Wisconsin, 1955, p. 100.

[12] D. E. Sanderson, *On isotopy of homeomorphisms of E^3 and S^3*, Bull. Amer. Math. Soc., Abstract 60-4-545.

[13] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, N. J., U.S.A. 1951.

[14] G. T. Whyburn, *On sequences and limiting sets*, Fund. Math. 25 (1935), p. 408-426.

[15] H. Whitney, *Regular families of curves*, Ann. of Math. 34 (1933), p. 244-270.

THE INSTITUTE FOR ADVANCED STUDY

Presented to the American Mathematical Society, April 20, 1956

Reçu par la Rédaction le 15. 12. 1956

On simple regular mappings

by

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1. Introduction. K. Borsuk and R. Molski studied in [3] a class of continuous mappings, called the mappings of finite order. A continuous mapping f of the space X onto the space Y is of order $\leqslant k$ if for every point $y \in Y$ the set $f^{-1}(y)$ contains at most k points (see also [9], p. 52). A mapping of order $\leqslant 2$ is called by K. Borsuk and R. Molski a *simple mapping*. The authors have given many examples which show that simple mappings may have many singular properties, *e. g.*, they can raise the dimension. Moreover, the authors have distinguished a certain class of mappings of finite order, called elementary mappings, and have proved that, in particular, every elementary mapping may be obtained by finite superpositions of simple mappings.

In this paper we shall study some properties of a certain class of simple mappings, called by us *regular mappings*, which contains, in particular, all simple elementary mappings. For a simple regular mapping f of a space X we introduce the notion of so-called doubling of X by f, which enables us to reduce the study of regular mappings to the study of simple interior mappings (¹).

2. Let f be a simple mapping of a space X into Y. The union X_0 of all sets $f^{-1}(y)$, with $y \in Y$, containing two different points is called in [3] the *seam* of f. Let us denote, for every $x \in X$, by $\Phi_f(x)$ a point of X such that $f^{-1}(f(x)) = \{x, \Phi_f(x)\}$ (²). Then Φ_f is an involution of X; it is called in [3] "the involution assigned to f" and is denoted there by ι_f (see [3], Nr 3). We shall call it the *involution induced by f*. The set $X - X_0$ is the set of fixed points of Φ_f.

3. Let $\mathfrak{F}$ be an upper semi-continuous decomposition of a compact space X and let Y be the hyperspace of this decomposition. Then, by

(¹) A continuous mapping $f\colon X \to Y$ is called interior if it carries open sets onto open ones.

(²) We denote by $\{a_1, a_2, \ldots, a_n\}$ the set composed of the elements $a_1, a_2, \ldots, a_n$ and by $(a_1, a_2, \ldots, a_n)$ the ordered sequence of these elements.