Rings with Hausdorff structure space *

by

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This paper is motivated by the possibility of extending to more general settings some of the results obtained by Henriksen and the author in [3] and [5] for rings $C(X)$ of continuous real-valued functions. As an illustration, we cite the following: it is shown in [3], Theorem 3.3 ff. that every ideal N^p in $C(X)$ is contained in exactly one maximal ideal, namely, M^p (notation as in [3]); for an arbitrary ring, the corresponding statement turns out to be *equivalent* to the statement that the structure space of the ring is a Hausdorff space (Theorem 3.1). (The structure space of $C(X)$ is the Stone-Čech compactification βX.)

Section 1 contains preliminary remarks. Section 2 introduces the ideals $N(S)$, which play an important role in the sequel. Section 3 deals with Hausdorff structure spaces. The results here include generalizations of some theorems about $C(X)$ (*e. g.*, Theorem 3.16), the theorem that every proper prime ideal in a biregular ring is primitive (Theorem 3.13), and a characterization of strongly regular rings (Theorem 3.14), as well as some other purely algebraic results (*e. g.*, Theorem 3.10). Finally, section 4 contains some simple results about ordered rings.

1. Preliminary results. Let A be an arbitrary ring, and $\mathfrak{S}=\mathfrak{S}(A)$ a set of (two-sided) proper ideals in A. The elements of $\mathfrak{S}$ will be called $\mathfrak{S}$-*ideals*. We say that $\mathfrak{S}$ is a *structure set of* A if whenever I, J are intersections of $\mathfrak{S}$-ideals, and S is an $\mathfrak{S}$-ideal, then

(1) $I \cap J \subset S$ implies that $I \subset S$ or $J \subset S$.

The structure sets of chief interest are

$\mathfrak{P}=\mathfrak{P}(A)=$ set of all primitive ideals in A,

$\mathfrak{Q}=\mathfrak{Q}(A)=$ set of all proper prime ideals in A.

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We recall that an ideal S in A is prime if and only if for arbitrary ideals I and J in A, $IJ\subset S$ implies that $I\subset S$ or $J\subset S$ (see [14]); obviously, then $\mathfrak{Q}$ is a structure set. Since $\mathfrak{P}\subset\mathfrak{Q}$ ([9], Lemma 4), $\mathfrak{P}$ is also a structure set [10].

Let $\mathfrak{S}$ be a set of ideals in A. For $\mathfrak{A}\subset\mathfrak{S}$, we define

$$\Delta\mathfrak{A}=\bigcap_{S\in\mathfrak{A}} S\,.$$

Jacobson [10] showed that the Stone topology can be defined on $\mathfrak{P}$; he called the resulting space the structure space of A. The construction and many of the results are valid for $\mathfrak{Q}$ as well [14], and, in fact, for any structure set $\mathfrak{S}$ [15]. The Stone topology is defined as follows: for any $\mathfrak{A}\subset\mathfrak{S}$, the closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ is given by

$$\overline{\mathfrak{A}}=\{S\in\mathfrak{S}:\ S\supset\Delta\mathfrak{A}\}\,.$$

Additivity of the closure operation follows from (1). Any structure set $\mathfrak{S}$, equipped with the Stone topology, will be called a *structure space of A*. Every such $\mathfrak{S}$ is a T_0-space, but need not be T_1 ([10], [11]).

Let $\mathfrak{M}$ denote the space of all prime maximal ideals in A; then $\mathfrak{P}\cap\mathfrak{M}$ is a T_1-space [10]. If A has an identity element, then every maximal ideal is primitive, and $\mathfrak{P}$ is compact [10]. If A is commutative, then $\mathfrak{P}=\mathfrak{M}$ [11], so that $\mathfrak{M}$ is a T_1-space; hence if A is a commutative ring with identity, then $\mathfrak{P}=\mathfrak{M}$, and $\mathfrak{M}$ is a compact T_1-space. (When A is commutative, we shall always write $\mathfrak{M}$ in place of $\mathfrak{P}$.)

THEOREM 1.1. *Let B be an ideal in A, and let φ denote the natural homomorphism from A onto A/B. Given any structure space $\mathfrak{S}$ of A, let $\mathfrak{S}/B$ denote the set of all $\mathfrak{S}$-ideals that contain B, and define*

$$\Phi(\mathfrak{S}/B)=\{\varphi(S):\ S\in\mathfrak{S}/B\}\,.$$

Then $\Phi(\mathfrak{S}/B)$ is a structure space of the ring A/B, and is homeomorphic with $\mathfrak{S}/B$. Moreover, the mapping Φ is one-one from the set of all structure spaces $\mathfrak{S}/B$ onto the set of all structure spaces of A/B.

The proof follows readily from the fact that the mapping φ is one-one and inclusion-preserving from the set of all ideals in A that contain B onto the set of all ideals in A/B.

In particular, we have $\Phi(\mathfrak{P}/B)=\mathfrak{P}(A/B)$ [10]; a similar proof shows that $\Phi(\mathfrak{Q}/B)=\mathfrak{Q}(A/B)$.

COROLLARY 1.2. *Let $\mathfrak{B}$ be a subset of $\mathfrak{S}$, where $\mathfrak{S}\supset\mathfrak{P}$. If the ring $A/\Delta\mathfrak{B}$ has an identity element, then $\overline{\mathfrak{B}}$ (= closure in $\mathfrak{S}$) is compact.*

Proof. By Theorem 1.1, $\Phi(\mathfrak{S}/\Delta\mathfrak{B})$ is a structure space of $A/\Delta\mathfrak{B}$. Since $\mathfrak{S}\supset\mathfrak{P}$, we have $\Phi(\mathfrak{S}/\Delta\mathfrak{B})\supset\Phi(\mathfrak{P}/\Delta\mathfrak{B})=\mathfrak{P}(A/\Delta\mathfrak{B})$. Since $A/\Delta\mathfrak{B}$ has an identity element, it follows that $\Phi(\mathfrak{S}/\Delta\mathfrak{B})$ is compact ([11], Corollary 3.4). Therefore, by Theorem 1.1, $\mathfrak{S}/\Delta\mathfrak{B}$ is compact, *i. e.*, $\overline{\mathfrak{B}}$ is compact. (Cf. [1], p. 460.)

LEMMA 1.3. *A one-element subset $\{S\}$ of $\mathfrak{S}$ is closed if and only if for all $T\in\mathfrak{S}$, $T\supset S$ implies $T=S$. Hence $\mathfrak{S}$ is a T_1-space if and only if no $\mathfrak{S}$-ideal contains another.*

The proof is immediate from the definition of closure. (Cf. [10].) As an example, in the ring of integers, $\mathfrak{M}$ is a T_1-space, while $\mathfrak{Q}$ is not.

THEOREM 1.4. *If A has an identity element, and if $\mathfrak{S}\supset\mathfrak{P}$, then $\mathfrak{S}$ is a T_1-space if and only if every $\mathfrak{S}$-ideal is maximal.*

Proof. If every $\mathfrak{S}$-ideal is maximal, then no $\mathfrak{S}$-ideal contains another, whence, by Lemma 1.3, $\mathfrak{S}$ is a T_1-space. Conversely, suppose that there exists an $\mathfrak{S}$-ideal S that is not maximal. Since A has an identity element, S can be imbedded in a maximal ideal $M\neq S$; also, every maximal ideal is primitive. Therefore, $M\in\mathfrak{P}\subset\mathfrak{S}$. Hence, by Lemma 1.3, $\mathfrak{S}$ is not a T_1-space.

For all $a\in A$, we define

$$\mathfrak{S}(a)=\{S\in\mathfrak{S}:\ a\in S\}\,,$$

which we call the $\mathfrak{S}$-set of a. Evidently, every $\mathfrak{S}$-set is closed. For $\mathfrak{A}\subset\mathfrak{S}$, we write

$$\Sigma\mathfrak{A}=\bigcup_{S\in\mathfrak{A}} S\qquad\text{(set union)},$$

$$\mathfrak{C}\mathfrak{A}=\text{complement of } \mathfrak{A} \text{ in } \mathfrak{S}\,.$$

LEMMA 1.5. *If $a\notin\Sigma\mathfrak{A}$, then there exists a neighborhood $\mathfrak{U}$ of $\mathfrak{A}$ such that $a\notin\Sigma\mathfrak{U}$.*

Proof. $\Sigma\mathfrak{A}\subset\mathfrak{C}\mathfrak{S}(a)$, and this latter is open.

LEMMA 1.6. *If $\mathfrak{A}$ is closed and $S\notin\mathfrak{A}$, then there exists $b\in\Delta\mathfrak{A}$ such that $b\notin S$; in other words, the $\mathfrak{S}$-sets are a base for the closed sets.*

Proof. $S\not\supset\Delta\mathfrak{A}$.

COROLLARY 1.7. *If $\mathfrak{A}$ is closed, $S\notin\mathfrak{A}$, and A/S is a division ring, then*
(2) *for all $u,v\in A$, there exists $c\in A$ such that $c\equiv u(\Delta\mathfrak{A})$ and $c\equiv v(S)$.*

Proof. Put $c=xb+u$, where b is as in Lemma 1.6, and $xb\equiv v-u(S)$.

In connection with this corollary, we may introduce the following terminology: $\mathfrak{S}$ is *completely regular in A* if whenever $\mathfrak{A}$ is closed and $S\notin\mathfrak{A}$, then (2) holds. In particular, if A is commutative, then $\mathfrak{P}=\mathfrak{M}$, whence $\mathfrak{M}$ is completely regular in A. For a ring with identity, the defining condition may be expressed in terms of the specific elements $u=0$, $v=1$ (or $u=1$, $v=0$).

2. The ideals $N(S)$. DEFINITION 2.1. For $S\in\mathfrak{S}$, we define

$N(S)=\{a\in A$: there exists a neighborhood $\mathfrak{U}$ of S such that $a\in\Delta\mathfrak{U}\}$.

A more complete notation is $N_{\mathfrak{S}}(S)$, but the simpler one will suffice here.

THEOREM 2.2. *For every $S\in\mathfrak{S}$, $N(S)$ is an ideal in A, and $N(S)\subset S$.*

The proof is obvious.

More generally, one could introduce the notion of "$\mathfrak{Z}$-ideal" as in [12]. Many of our results concerning $N(S)$ will be seen to hold for this more general type of ideal.

THEOREM 2.3. *For every $S\in\mathfrak{S}$, we have $N(S)\supset\Delta\mathfrak{S}$.*

Proof. Since $\mathfrak{S}$ itself is a neighborhood of S, $a\in\Delta\mathfrak{S}$ implies $a\in N(S)$.

We recall that an ideal Q in A is prime if and only if $axb\in Q$ for all x implies that $a\in Q$ or $b\in Q$ ([14], Theorem 1).

THEOREM 2.4. *If $S\supset Q\supset\Delta\mathfrak{S}$, where $S\in\mathfrak{S}$ and Q is prime, then $Q\supset N(S)$.*

Proof. Let $a\in N(S)$, and let $\mathfrak{U}$ be a neighborhood of S such that $a\in\Delta\mathfrak{U}$. By Lemma 1.6, there exists b such that $b\in\Delta\mathfrak{C}\mathfrak{U}$, $b\notin S$. Then for all x, we have

$$axb\in\Delta\mathfrak{U}\cap\Delta\mathfrak{C}\mathfrak{U}=\Delta\mathfrak{S}\subset Q\,,$$

i. e., $axb\in Q$. But $b\notin Q$, since $Q\subset S$ and $b\notin S$. Therefore $a\in Q$, since Q is prime: Thus, $N(S)\subset Q$. (Cf. [3], Lemma 3.2.)

LEMMA 2.5. *Let N be an ideal in a commutative ring, and let K be a subset closed under multiplication. If N does not meet K, then there exists a prime ideal containing N that does not meet K.*

This is a well-known consequence of Zorn's lemma (see, *e. g.*, [13], ch. 5).

THEOREM 2.6. *If A is commutative and $\mathfrak{S}\subset\mathfrak{Q}$, then for every $S\in\mathfrak{S}$, $N(S)$ is an intersection of prime ideals.*

Proof. Evidently, all powers of an element a belong to the same prime ideals as a. Hence if $a\notin N(S)$, then $N(S)$ does not meet the multiplicative system $\{a^n\colon n=1,2,\ldots\}$. By Lemma 2.5, there is a prime ideal containing N but not a. The result follows.

I do not know whether the corresponding result holds in a general (non-commutative) ring.

THEOREM 2.7. *Every $\mathfrak{S}$-set is open if and only if $N(S)=S$ for all $S\in\mathfrak{S}$.*

Proof. $a\in N(S)$ if and only if there is a neighborhood $\mathfrak{U}$ of S such that $a\in\Delta\mathfrak{U}$, *i. e.*, $S\in\mathfrak{U}\subset\mathfrak{S}(a)$.

We close this section with two lemmas that will be used later.

LEMMA 2.8. *Let $\mathfrak{A}$, $\mathfrak{B}$ be disjoint subsets of $\mathfrak{S}$, with $\mathfrak{A}$ closed and $\mathfrak{B}$ compact, such that $A/\Delta\mathfrak{A}$ has an identity element, and such that A/S is a division ring and $N(S)=S$ for every $S\in\mathfrak{B}$. Then*

(3) *for all $u,v\in A$, there exists $c\in A$ such that $c\equiv u(\Delta\mathfrak{A})$ and $c\equiv v(\Delta\mathfrak{B})$.*

Proof. Let e be an identity element modulo $\Delta\mathfrak{A}$ (that is, $ex\equiv xe\equiv x(\Delta\mathfrak{A})$ for every $x\in A$). By Corollary 1.7, for every $S_\lambda\in\mathfrak{B}$, there exists $d_\lambda\in A$ such that $d_\lambda\equiv e(\Delta\mathfrak{A})$ and $d_\lambda\equiv 0(S_\lambda)$. Then $d_\lambda\equiv 0\big(N(S_\lambda)\big)$, so there is a neighborhood $\mathfrak{U}_\lambda$ of S_λ such that $d_\lambda\in\Delta\mathfrak{U}_\lambda$. Since $\mathfrak{B}$ is compact, there exist indices $\lambda_1,\ldots,\lambda_n$ such that $\bigcup_{k=1}^{n}\mathfrak{U}_{\lambda_k}\supset\mathfrak{B}$. Define $d=d_{\lambda_1}\ldots d_{\lambda_n}$. Then $d\equiv e(\Delta\mathfrak{A})$ and $d\equiv 0(\Delta\mathfrak{B})$. The element $c=d(u-v)+v$ is then as required.

In connection with this lemma, we may introduce the following terminology: $\mathfrak{A}$, $\mathfrak{B}$ are *completely separated by A* if (3) holds. Clearly, two arbitrary subsets of $\mathfrak{S}$ are completely separated by A if and only if their closures are. For a ring with identity, the condition may be expressed in terms of the specific elements $u=0$, $v=1$. We may call $\mathfrak{S}$ *normal in A* if every pair of disjoint closed subsets of $\mathfrak{S}$ are completely separated by A. Clearly, if $\mathfrak{S}$ is a T_1-space and normal in A, then $\mathfrak{S}$ is completely regular in A (Corollary 1.7 ff.). In particular, if A is a commutative ring with identity such that every $\mathfrak{M}$-set is open, then $\mathfrak{M}$ is normal in A.

LEMMA 2.9. *Let $\mathfrak{A}$ be a compact subset of $\mathfrak{S}$ such that $A/\Delta\mathfrak{A}$ has an identity element, and such that A/S is a division ring and $N(S)=S$ for all $S\in\mathfrak{A}$. Then every element $a\notin\Sigma\mathfrak{A}$ has an inverse modulo $\Delta\mathfrak{A}$.*

Proof. Let e be an identity element modulo $\Delta\mathfrak{A}$. If $a\notin\Sigma\mathfrak{A}$, then for every $S_\lambda\in\mathfrak{A}$, there exists $x_\lambda\in A$ such that $ax_\lambda\equiv e(S_\lambda)$. Since $S_\lambda=N(S_\lambda)$, there is a neighborhood $\mathfrak{U}_\lambda$ of S_λ such that $ax_\lambda-e\in\Delta\mathfrak{U}_\lambda$. Since $\mathfrak{A}$ is compact, there exist indices $\lambda_1,\ldots,\lambda_n$ such that $\bigcup_{k=1}^{n}\mathfrak{U}_{\lambda_k}\supset\mathfrak{A}$. Then

$$\prod_{k=1}^{n}(ax_{\lambda_k}-e)\in\Delta\bigcup_{k=1}^{n}\mathfrak{U}_{\lambda_k}\subset\Delta\mathfrak{A}\,,$$

which reduces to the form $ax\equiv e(\Delta\mathfrak{A})$.

3. Hausdorff structure spaces. THEOREM 3.1. *The following statements are equivalent.*

(a) *$\mathfrak{S}(A)$ is a Hausdorff space.*

(b) *For any two distinct elements S, T of $\mathfrak{S}$, there exists an element $a\in A$ such that $a\in N(S)$, $a\notin T$.*

(c) *For every $S\in\mathfrak{S}$, $N(S)$ is contained in exactly one $\mathfrak{S}$-ideal (namely, S).*

Proof. The equivalence of (a) with (b) is an immediate consequence of Lemma 1.6. Since $N(S) \subset S$ (Theorem 2.2), it is evident that (b) is equivalent to (c).

If every A/S ($S \in \mathfrak{S}$) is a division ring, then, clearly, (b) is equivalent to

(d) *For any two distinct elements S, T of $\mathfrak{S}$, and arbitrary $v \in A$, there exists $c \in A$ such that $c \in N(S)$, $c \equiv v(T)$.*

If (b) holds, then, clearly, $S \neq T$ implies $N(S) \neq N(T)$. If every proper ideal in A can be imbedded in an $\mathfrak{S}$-ideal, then the conditions of the theorem are also equivalent to

(e) *For any two distinct elements S, T of $\mathfrak{S}$, we have $\big(N(S), N(T)\big) = A$.*

For if $\big(N(S), N(T)\big) \neq A$, then there is an $\mathfrak{S}$-ideal that contains both $N(S)$ and $N(T)$, contradicting (c). Conversely, if $N(S) \subset T \neq S$, then $\big(N(S), N(T)\big) \subset T$, contradicting (e).

In particular, if A is commutative and $\mathfrak{S} \subset \mathfrak{M}$, then the properties (a), (b), (c), (d) are equivalent; if A has an identity element and $\mathfrak{S} \supset \mathfrak{P}$, then (a), (b), (c), (e) are equivalent.

As an example, consider the ring of integers. As pointed out by Jacobson [10], $\mathfrak{M}$ is not Hausdorff: the nonempty open sets are the complements of finite sets. Note that $N(M) = (0)$ for every $M \in \mathfrak{M}$. (As for $\mathfrak{Q}$, we have already observed in section 1 that it is not even a T_1-space.)

If $\mathfrak{S}(A)$ is a Hausdorff space, we call A an *$H_{\mathfrak{S}}$-ring*. Since every subspace of a Hausdorff space is Hausdorff, Theorem 1.1 ff. yields

THEOREM 3.2. *Every homomorphic image of an $H_{\mathfrak{P}}$-ring [$H_{\mathfrak{Q}}$-ring] is an $H_{\mathfrak{P}}$-ring [$H_{\mathfrak{Q}}$-ring].*

The corresponding theorem for T_1-spaces is also valid.

In particular, the ring of integers is not a homomorphic image of any $H_{\mathfrak{P}}$-ring.

THEOREM 3.3. *If every $\mathfrak{S}$-set is open, then $\mathfrak{S}$ is a Hausdorff space.*

Proof. If $S \neq T$, then there exists an element $a \in A$ such that, say, $a \in S$, $a \notin T$. Then $\mathfrak{S}(a)$, $\mathfrak{C}\mathfrak{S}(a)$ are disjoint neighborhoods of S, T, respectively.

The following theorem generalizes [3], Lemma 3.2 and Theorem 3.3.

THEOREM 3.4. *If $\mathfrak{S}$ is a Hausdorff space, then every prime ideal Q that contains $\Delta\mathfrak{S}$ is contained in at most one $\mathfrak{S}$-ideal; if, in addition, Q can be imbedded in an $\mathfrak{S}$-ideal, then $Q \subset S$ ($\in \mathfrak{S}$) if and only if $Q \supset N(S)$.*

Proof. Theorems 2.4 and 3.1 (a, c).

If $\Delta\mathfrak{S} = (0)$, we say that A is $\mathfrak{S}$-semi-simple. For $\mathfrak{S} = \mathfrak{P}$, this coincides with Jacobson's semi-simplicity.

COROLLARY 3.5. *If A is a semi-simple integral domain containing at least two maximal ideals, then $\mathfrak{M}$ is not a Hausdorff space.*

This also follows from Kohls' characterization of Hausdorff structure spaces ([11], Theorem 4.1 ff.). Examples include the ring of integers, other rings of algebraic integers, the ring of entire functions, and various other rings of analytic functions.

COROLLARY 3.6. *If A is a commutative $H_{\mathfrak{S}}$-ring, with $\mathfrak{S} \subset \mathfrak{Q}$, and if every $\mathfrak{Q}$-ideal containing $\Delta\mathfrak{S}$ can be imbedded in an $\mathfrak{S}$-ideal, then for every $S \in \mathfrak{S}$, $N(S)$ coincides with the intersection of all the prime ideals Q such that $S \supset Q \supset \Delta\mathfrak{S}$.*

Proof. Theorems 2.6 and 3.4.

In particular, the hypotheses of this corollary are satisfied if A is a commutative $\mathfrak{S}$-semi-simple $H_{\mathfrak{S}}$-ring with identity, and $\mathfrak{M} \subset \mathfrak{S} \subset \mathfrak{Q}$. (Cf. [5], Theorem 1.4.)

THEOREM 3.7. *If $\mathfrak{S}$ is a Hausdorff space contained in $\mathfrak{Q}$, then the following are equivalent.*

(a) *Every $\mathfrak{Q}$-ideal containing $\Delta\mathfrak{S}$ is an $\mathfrak{S}$-ideal.*

(b) *Every $\mathfrak{Q}$-ideal containing $\Delta\mathfrak{S}$ is an intersection of $\mathfrak{S}$-ideals.*

(c) (i) *Every $\mathfrak{Q}$-ideal containing $\Delta\mathfrak{S}$ is imbeddable in an $\mathfrak{S}$-ideal; and*

(ii) *for all $S \in \mathfrak{S}$, S is the unique proper prime ideal containing $N(S)$.*

If in addition, every $N(S)$ ($S \in \mathfrak{S}$) is an intersection of prime ideals, then (ii) *may be replaced by*

(ii′) *every $\mathfrak{S}$-set is open, that is, $N(S) = S$ for all $S \in \mathfrak{S}$.*

Remarks. In particular, this additional hypothesis will be satisfied in case the ring is commutative (Theorem 2.6). The equivalence of the two statements in (ii′) was pointed out in Theorem 2.7.

Proof. Trivially, (a) implies both (b) and (i). Now let S be any $\mathfrak{S}$-ideal. If Q is a proper prime ideal containing $N(S)$, then $Q \supset \Delta\mathfrak{S}$ (2.3). By (a), Q is an $\mathfrak{S}$-ideal, and Theorem 3.1 now shows that $S = Q$. This establishes (ii). Thus, (a) implies (c).

By Theorem 3.4, (b) implies (a). Next, we deduce (a) from either form of (c). Let Q be any $\mathfrak{Q}$-ideal containing $\Delta\mathfrak{S}$. By (i), Q is contained in an $\mathfrak{S}$-ideal S; hence $Q \supset N(S)$, by Theorem 3.4. Each of (ii), (ii′) now implies that $Q = S$, and this yields (a).

Finally, if $N(S)$ is an intersection of prime ideals, then obviously (ii) implies (ii′).

If $\mathfrak{S} = \mathfrak{Q}$, then (a) (or (b)) is trivially satisfied. Hence from (c) and Theorem 3.3, we obtain the following rather interesting result.

COROLLARY 3.8. *If A is commutative, then $\mathfrak{Q}$ is a Hausdorff space if and only if every $\mathfrak{Q}$-set is open.*

We next prove two general theorems that do not deal with structure spaces.

THEOREM 3.9. *For any family $\mathfrak{A}$ of prime ideals in a ring A, the following are equivalent.*

(a) *For all $\mathfrak{B}\subset\mathfrak{A}$, the ideal $\bigcap_{Q\in\mathfrak{B}} Q$ is prime.*

(b) *For any $P,Q\in\mathfrak{A}$, the ideal $P\cap Q$ is prime.*

(c) *$\mathfrak{A}$ is totally ordered under set inclusion.*

Proof. (a) *implies* (b). Trivial.

(b) *implies* (c). Consider any $P,Q\in\mathfrak{A}$. Since $PQ\subset P\cap Q$, and $P\cap Q$ is prime, by hypothesis, we must have either $P\subset P\cap Q$ or $Q\subset P\cap Q$. Hence either $P\subset Q$ or $Q\subset P$.

(c) *implies* (a). Suppose that (a) fails. Then there exists a subset $\mathfrak{B}$ of $\mathfrak{A}$ such that the ideal $B=\bigcap_{Q\in\mathfrak{B}} Q$ is not prime. Hence there exist elements $a,b\in A$ such that $a\notin B$, $b\notin B$, but $axb\in B$ for all $x\in A$. Since $a\notin B$, there exists $P\in\mathfrak{B}$ such that $a\notin P$. Likewise, there exists $Q\in\mathfrak{B}$ such that $b\notin Q$. By hypothesis, $P\supset Q$, say. Then $a\notin Q$, $b\notin Q$. But $axb\in Q$ for all x, contradicting the fact that Q is prime. (Cf. [7], Theorem 4.)

THEOREM 3.10. *Let A be a commutative ring with identity, let $\mathfrak{A}$ denote the family of all prime ideals in A that are contained in a given proper ideal I, and suppose that every finitely generated ideal in A that is contained in I is a principal ideal. Then $\mathfrak{A}$ is totally ordered under set inclusion, and the intersection of any subfamily of $\mathfrak{A}$ is a prime ideal.*

Proof. By Theorem 3.9, the two conclusions are equivalent. Suppose that they fail. Then there exist $P,Q\in\mathfrak{A}$ such that neither contains the other. Hence there exist elements $p,q\in A$ satisfying $p\in P$, $p\notin Q$, $q\in Q$, $q\notin P$. Since $(p,q)\subset(P,Q)\subset I$, the ideal (p,q) is principal, by hypothesis. Let $(p,q)=(a)$. Then there exist elements b, c, s, t such that $p=ba$, $q=ca$, and $a=sp+tq$. Then $a\notin P$, $a\notin Q$. Since $sp=sba\equiv 0(P)$, and P is prime, we have $sb\equiv 0(P)$. Also, $a=sp+tq\equiv sp\equiv sba(Q)$, whence $sb\equiv 1(Q)$, since Q is prime. Thus, $sb\in P\subset(P,Q)$, and $sb-1\in Q\subset(P,Q)$. But then $1\in(P,Q)\subset I$, contradicting the hypothesis that I is proper.

A particular consequence of this theorem is [7], Corollary to Theorem 4.

THEOREM 3.11. *If A is a commutative $H_{\mathfrak{S}}$-ring with identity, where $\mathfrak{M}\subset\mathfrak{S}\subset\mathfrak{Q}$, and if*

(4) *every finitely generated ideal in A is principal,*

then for every $S\in\mathfrak{S}$, the ideal $N(S)$ is prime.

Proof. Since A has an identity element and $\mathfrak{S}\supset\mathfrak{M}$, every proper ideal is imbeddable in an $\mathfrak{S}$-ideal. The result is now an immediate consequence of Corollary 3.6 and Theorem 3.10.

The hypothesis that $\mathfrak{S}$ be a Hausdorff space (or even T_1) is not necessary, as shown by the example of the ring of integers.

In [5], Theorem 2.5, it is shown, using an entirely different argument, that if A is a ring $C(X)$ (of continuous real-valued functions), then the condition (4) is *equivalent* to the condition that every $N(M)$ $(M\in\mathfrak{M})$ be prime. In this connection, we have:

EXAMPLE 3.12. Let A denote the ring of all formal power series in two indeterminates x and y, with coefficients in a field, and consider the space $\mathfrak{M}(A)$. It is not hard to see that $M=(x,y)$ is the only maximal ideal in A. Hence $N(M)=M$, so that $N(M)$ is prime. Thus, A is a commutative $H_{\mathfrak{M}}$-ring with identity in which every $N(M)$ $(M\in\mathfrak{M})$ is prime. But the finitely generated ideal M is not principal.

A ring is *regular* [16] if for every a, there exists x such that $axa=a$; *biregular* [1] if every principal ideal is generated by an idempotent in the center; *strongly regular* [1] if for every a, there exists x such that $a^2x=a$. Strong regularity implies both regularity and biregularity, and for commutative rings, the three concepts coincide ([1], Theorem 3.2).

Every regular ring is semi-simple [9]. A commutative ring is regular if and only if every ideal is an intersection of prime ideals ([13], Theorem 49).

Every biregular ring A is semi-simple; its primitive ideals coincide with its maximal ideals, and every ideal is an intersection of primitive ideals; $\mathfrak{P}$ is locally compact and zero-dimensional, hence Hausdorff, and $\mathfrak{P}$ is compact if and only if A has an identity element ([1], § 2).

If A is strongly regular, then every A/P $(P\in\mathfrak{P})$ is a division ring ([1], Theorem 3.2).

THEOREM 3.13. *In a biregular ring, every proper prime ideal is primitive, and every $\mathfrak{P}$-set is open.*

Proof. As just pointed out, $\Delta\mathfrak{P}=(0)$, and all the hypotheses and (b) of Theorem 3.7 (with $\mathfrak{S}=\mathfrak{P}$) are satisfied; (a) and (ii′) of that theorem yield our present conclusions.

THEOREM 3.14. *A necessary and sufficient condition that a ring A be strongly regular is that* (i) *A be semi-simple,* (ii) *every A/P $(P\in\mathfrak{P})$ be a division ring,* (iii) *every $\mathfrak{P}$-set be open, and* (iv) *every $A/\Delta\mathfrak{C}\mathfrak{P}(a)$ $(a\in A)$ have an identity element.*

Proof. Necessity. (i) and (ii) have been noted above, and (iii) is given in Theorem 3.13. Finally, given $a\in A$, let x satisfy $a^2x=a$; it follows easily from (i) and (ii) that ax is an identity element modulo $\Delta\mathfrak{C}\mathfrak{P}(a)$ (see [1], p. 463).

Sufficiency. Consider any $a \in A$. By (iii) and (iv), $\mathfrak{C}\mathfrak{P}(a)$ is compact (Corollary 1.2). Now by (iii), $N(P)=P$ for every P (Theorem 2.7). Thus (iv), (ii) and (iii) imply that the hypotheses of Lemma 2.9, with $\mathfrak{A}=\mathfrak{C}\mathfrak{P}(a)$, and $\mathfrak{S}=\mathfrak{P}$, are satisfied. We conclude from that lemma that there exists x such that ax is an identity modulo $\Delta\mathfrak{C}\mathfrak{P}(a)$. It follows easily, using (i), that $a^2x=a$.

From Lemma 2.8, we get.

Corollary 3.15. *If A is a strongly regular ring with identity, then $\mathfrak{P}$ is normal in A.*

A commutative ring A with identity is *adequate* if it satisfies (4) and

(5) given a, b, with $a\neq 0$, there exist r, s such that $a=rs$, $(r,s)=A$, and for every non-unit divisor s' of s, we have $(s',b)\neq A$.

Every commutative regular ring with identity is adequate ([4], Theorem 11). (Further discussion and references are also given in [4].)

The following theorem is a generalization of parts of [3], Theorem 5.3 ff., and [5], Theorems 6.2 and 6.5.

Theorem 3.16. *If A is a semi-simple commutative $H_{\mathfrak{M}}$-ring with identity, then the following are equivalent.*

(a) *A is regular.*

(b) *Every ideal is an intersection of maximal ideals.*

(c) *Every prime ideal is an intersection of maximal ideals.*

(d) *Every proper prime ideal is maximal.*

(e) *A is adequate.*

(f) *A satisfies the condition* (5).

(g) *Every $\mathfrak{M}$-set is open, i. e., $N(M)=M$ for all $M \in \mathfrak{M}$.*

Proof. In view of the preceding discussion, Theorem 3.7 (and remarks), and Theorem 3.14, all that remains to be shown is that (f) implies (g). The proof of this fact is an exact analogue of the proof of [5], Theorem 6.5 (reference being made, at the appropriate places, to the hypothesis that $\mathfrak{M}$ is a Hausdorff space, and to Theorem 2.7, Lemmas 1.6 and 1.5, and Corollary 1.7, above).

In particular, under the given hypotheses on A, each of the above properties implies (4).

The example of the ring of integers shows that these properties need not be equivalent in case $\mathfrak{M}$ is not a Hausdorff space.

4. Ordered rings. All rings considered in this section are commutative. We recall that a partially ordered group is a commutative group A on which is defined a partial ordering relation $\geqslant$ that is invariant under translation — that is, A is a commutative group, and for all $a,b,c \in A$, we have

Axiom **O**. (i) $a\geqslant a$; (ii) *if $a\geqslant b$ and $b\geqslant a$, then $a=b$*; (iii) *if $a\geqslant b$ and $b\geqslant c$, then $a\geqslant c$*;

and

Axiom **G**. *If $a\geqslant b$, then $a+c\geqslant b+c$.*

Alternate forms of **G**, evidently equivalent to it, are:

$\mathbf{G}_1$. *$a\geqslant b$ if and only if $a-b\geqslant 0$.*

$\mathbf{G}_2$. *If $a\geqslant b$ and $c\geqslant d$, then $a+c\geqslant b+d$.*

By a partially ordered ring is meant a commutative ring A whose additive group is a partially ordered group, and that satisfies

Axiom **R**. *If $a\geqslant 0$ and $b\geqslant 0$, then $ab\geqslant 0$.*

The discussion that follows is stated for rings, but the considerations up to and including Theorem 4.5 apply equally well to groups ("subgroup" replacing "ideal").

As usual, the symbol $\sup(a,b)$ denotes an element c such that $c\geqslant a$ and $c\geqslant b$, and such that, whenever $x\geqslant a$ and $x\geqslant b$, then $x\geqslant c$. The symbol $\inf(a,b)$ is defined correspondingly. The symbol $|a|$ denotes $\sup(a,-a)$. By **O** (ii), any such elements are necessarily unique. Clearly,

(6) *if $|a|$ exists, then* $2|a|\geqslant 0$;

(7) *if $2x\geqslant 0$, then $|x|$ exists and $|x|=x$.*

It follows from **G** that $\sup(a+c,b+c)=\sup(a,b)+c$, if either side exists. Hence we have:

(8) $\sup(2a,2b)=a+b+|a-b|$, *if either side exists.*

Furthermore, we have $a\geqslant b$ if and only if $-b\geqslant -a$. It follows that $\inf(a,b)=-\sup(-a,-b)$, if either exists.

We now consider some additional axioms for partially ordered rings.

Axiom **I**. *If $|a|$ exists, then $|a|\geqslant 0$.*

Alternatively:

$\mathbf{I}_1$. *If $2x\geqslant 0$, then $x\geqslant 0$.*

$\mathbf{I}_2$. *If $2a\geqslant 2b$, then $a\geqslant b$.*

The equivalence of **I** with $\mathbf{I}_1$ follows from (6) and (7); that of $\mathbf{I}_1$ with $\mathbf{I}_2$ is evident.

Axiom **II**. *$|a|$ exists for every a.*

Alternatively:

$\mathbf{II}_1$. $\sup(2a,0)$ *exists for every a.*

$\mathbf{II}_2$. $\sup(2a,2b)$ *exists for all a and b.*

The equivalence of these with **II** is immediate from (8).

Theorem 4.1. *If the partially ordered ring A satisfies* **I** *and* **II**, *then the relation $\geqslant$ is maximal in A.*

Proof. By $\mathbf{G}_1$, any relation $x \geqslant y$ is equivalent to one of the form $a \geqslant 0$. Suppose that the relation $a \geqslant 0$ is consistent with the given ones. Then $|a|$ must exist and satisfy $|a| = a$. But by **II**, $|a|$ exists in any case, and by **I**, $|a| \geqslant 0$. Therefore the relation $a \geqslant 0$ is already known to hold.

If the partially ordered ring A is a lattice (*i. e.*, $\sup(a,b)$ exists for all a and b), then A satisfies both **I** and **II** (see, *e. g.*, [2], p. 15, Prop. 9). The following simple example to show that the converse need not hold was supplied by M. Henriksen.

EXAMPLE 4.2. Let B denote the direct sum of the ring of integers with itself, define $(a,b) \geqslant 0$ if and only if both a and b are non-negative, and let A denote the subset of B consisting of all pairs of integers whose sum is even. Then A is a partially ordered ring with identity, and satisfies both **I** and **II**. But A is not a lattice: *e. g.*, $\sup((2,0),(1,1))$ does not exist.

THEOREM 4.3. *Let A be a partially ordered ring satisfying* **I** *and* **II**. *If for every $a \in A$, there exists $x \in A$ such that $a = 2x$, then A is a lattice.*

Proof. Let $a, b \in A$ be given. By $\mathbf{II}_2$, $\sup(2a, 2b)$ exists. By hypothesis, we may write $\sup(2a,2b) = 2x$. It follows easily from $\mathbf{I}_2$ that $x = \sup(a,b)$.

Let S be any ideal in A. For $a \in A$, the image of a in the residue class ring A/S (under the natural homomorphism) will be denoted by a_S.

AXIOM $\mathbf{III}_S$. *If $a \geqslant b \geqslant 0$, and $a \equiv 0(S)$, then $b \equiv 0(S)$.*

It is well known (see, for example, [2], p. 23, Ex. 4) that the partially ordered ring A satisfies $\mathbf{III}_S$ if and only if A/S is a partially ordered ring under the following definition: $a_S \geqslant 0$ if and only if there exists $x \in A$ with $x \geqslant 0$ and $x_S = a_S$. We shall refer to this as the natural order on A/S (induced by the given order on A).

Let $\mathfrak{S}$ be a set of ideals in A, with $\Delta\mathfrak{S} = (0)$, and suppose that A/S is a partially ordered ring for every $S \in \mathfrak{S}$. Then, evidently, A becomes a partially ordered ring under the following definition: $a \geqslant 0$ if and only if $a_S \geqslant 0$ for all $S \in \mathfrak{S}$. (The condition $\Delta\mathfrak{S} = (0)$ is used to obtain axiom **O** (ii).) We shall call this the natural order on A (induced by the given orders on the rings A/S).

AXIOM $\mathbf{IV}_S$. *For each $a \in A$, if $|a|$ exists, then either $a \equiv |a|(S)$ or $a \equiv -|a|(S)$.*

THEOREM 4.4. *If the partially ordered ring A satisfies* **I**, **II**, $\mathbf{III}_S$ *and* $\mathbf{IV}_S$, *then*

(i) *the residue class ring A/S is totally ordered;*

(ii) $a_S \geqslant 0$ *if and only if* $a \equiv |a|(S)$, *and if and only if* $\sup(2a, 0) \equiv 2|a|(S)$;

(iii) $a_S \leqslant 0$ *if and only if* $a \equiv -|a|(S)$, *and if and only if* $\sup(2a,0) \equiv 0(S)$.

Proof. (i) is evident; (ii) and (iii) follow from (8).

If the partially ordered ring A satisfies $\mathbf{III}_S$ [$\mathbf{IV}_S$] for all $S \in \mathfrak{S}$, we say that A satisfies $\mathbf{III}_{\mathfrak{S}}$ [$\mathbf{IV}_{\mathfrak{S}}$]. If A satisfies **I**, **II**, $\mathbf{III}_{\mathfrak{S}}$ and $\mathbf{IV}_{\mathfrak{S}}$, we call A an $L_{\mathfrak{S}}$-ring.

THEOREM 4.5. *A necessary and sufficient condition that an $\mathfrak{S}$-semi-simple commutative ring A be an $L_{\mathfrak{S}}$-ring is that*

(i) *for every $S \in \mathfrak{S}$, A/S be totally ordered, and*

(ii) *for every $a \in A$, there exist an element $a' \in A$ such that $a'_S = |a|_S$ for every $S \in \mathfrak{S}$ (in fact, $a' = |a|$), the induced order(s), in each case, being the natural one(s). Furthermore, the composition in either direction of the two operations of inducing order(s) yields the order(s) originally given.*

Proof. Necessity. ($\mathfrak{S}$-semi-simplicity is not needed here.) We have already observed that (i) holds. To prove (ii), define $a' = |a|$. Consider any $S \in \mathfrak{S}$. If $a \equiv |a|(S)$, then $a_S \geqslant 0$, so that $a_S = |a_S|$. But if $a \equiv |a|(S)$, then $a_S = |a|_S$, *i. e.*, $a_S = a'_S$. Therefore $a'_S = |a_S|$. The case $a \equiv -|a|(S)$ is similar.

Sufficiency. We have already observed (using $\mathfrak{S}$-semi-simplicity) that A is a partially ordered ring. Verification of axiom $\mathbf{III}_{\mathfrak{S}}$ is trivial in view of the fact that every A/S is totally ordered. Likewise, for each $a \in A$, it is clear that $a' = \sup(a, -a)$; consequently, we have $|a| = a'$, and axioms **I** and **II** are verified. Since for each S, we have $|a|_S = |a_S| = a_S$ or $-a_S$, axiom $\mathbf{IV}_{\mathfrak{S}}$ also holds.

Next, suppose that the rings A/S are totally ordered. Let us refer to their ordering relations collectively as $\geqslant_0$. Let $\geqslant_1$ denote the partial ordering relation that these induce on A, and let $\geqslant_2$ denote the order relations induced by $\geqslant_1$ on the rings A/S. We wish to show that $\geqslant_2$ coincides with $\geqslant_0$. Since both are *total* ordering relations, it suffices to show that one contains the other, *e. g.*, that $a_S \geqslant_2 0$ implies $a_S \geqslant_0 0$ ($S \in \mathfrak{S}$). Let a_S be given, with $a_S \geqslant_2 0$, and let a denote any pre-image in A of a_S. By definition of $\geqslant_2$, there exists an element x of A such that $x \geqslant_1 0$ and $x_S = a_S$. By definition of $\geqslant_1$, we have $x_S \geqslant_0 0$. Thus, $a_S \geqslant_0 0$.

Finally, let A be partially ordered by $\geqslant_1$, let $\geqslant_2$ denote the order relations induced by $\geqslant_1$ on the rings A/S, and let $\geqslant_3$ denote the order relation induced by $\geqslant_2$ on A. We are to show that $\geqslant_3$ coincides with $\geqslant_1$. Let $a \in A$ be given, with $a \geqslant_1 0$. Then $a_S \geqslant_2 0$ for every S. Therefore $a \geqslant_3 0$. Thus $a \geqslant_1 0$ implies $a \geqslant_3 0$. This shows that $\geqslant_1$ is contained in $\geqslant_3$. It now follows from the maximality of $\geqslant_1$ (Theorem 4.1) that $\geqslant_1$ coincides with $\geqslant_3$. This completes the proof of the theorem.

Remark 4.6. If A is $\mathfrak{S}$-semi-simple and satisfies **II** and $\mathbf{IV}_{\mathfrak{S}}$, then $|a|^2 = a^2$ for every $a \in A$. In the opposite direction, if $|a|^2 = a^2$ whenever $|a|$ exists, then A satisfies $\mathbf{IV}_S$ for every prime ideal S.

LEMMA 4.7. *Let A satisfy* **I** *and* **II** *and have an identity element. If* (i): *A is $\mathfrak{S}$-semi-simple and satisfies* $\mathbf{IV}_{\mathfrak{S}}$, *then* (ii): $1 \geqslant 0$. *If* (iii): $\mathfrak{S} \supset \mathfrak{M}$, *and A satisfies both* $\mathbf{III}_{\mathfrak{S}}$ *and* (ii), *then* (iv): *$a \geqslant 1$ implies that a is a unit (whence A contains the rationals). Finally, if 2 is a unit, then A is a lattice.*

Proof. Remark 4.6 shows that (i) implies (ii). If $a \geqslant 1$, then by (iii), a belongs to no maximal ideal; it follows that (a) cannot be proper, whence a is a unit; thus, (iii) implies (iv). The final statement of the theorem is an immediate consequence of Theorem 4.3.

On referring to Theorem 4.4, we obtain:

THEOREM 4.8. *Every semi-simple $L_{\mathfrak{M}}$-ring with identity is a lattice, $|a|^2 = a^2$ for every $a \in A$ (whence $1 \geqslant 0$), and for every $M \in \mathfrak{M}$, we have $a_M \geqslant 0$ or $\leqslant 0$ according as* $\sup(a, 0) \equiv |a|$ *or* $\equiv 0\,(M)$.

In any $L_{\mathfrak{S}}$-ring, we define

$$\mathfrak{S}^+(a) = \{S \in \mathfrak{S}\colon\ a_S > 0\}.$$

We assume now that $\mathfrak{S}$ is a structure space.

THEOREM 4.9. *If A is an $L_{\mathfrak{S}}$-ring, then $\mathfrak{S}^+(a)$ is an open set in $\mathfrak{S}$ $(a \in A)$.*

Proof. From Theorem 4.4 (iii), we find that $\mathfrak{S}^+(a) = \mathfrak{C}\mathfrak{S}(\sup(2a, 0))$.

THEOREM 4.10. *If A is a semi-simple $L_{\mathfrak{M}}$-ring with identity, then $\mathfrak{M}$ is a Hausdorff space.*

Proof. Given distinct elements M_1, M_2 of $\mathfrak{M}$, there exists $a \in A$ satisfying $a \equiv 0\,(M_1)$, $a \equiv 2\,(M_2)$. Then $\mathfrak{M}^+(a-1)$, $\mathfrak{M}^+(1-a)$ are separating open sets.

THEOREM 4.11. *If A is a semi-simple $L_{\mathfrak{M}}$-ring with identity, then $\mathfrak{M}$ is normal in A* (see Lemma 2.8 ff.).

Proof. Let $\mathfrak{A}$, $\mathfrak{B}$ be disjoint closed subsets of $\mathfrak{M}$. For every $M_\lambda \in \mathfrak{B}$, there exists $a_\lambda \in A$ such that $a_\lambda \equiv 2\,(M_\lambda)$ and $a_\lambda \equiv 0\,(\Delta\mathfrak{A})$ (Corollary 1.7). Since $\mathfrak{M}^+(a_\lambda - 1)$ is open, there is a neighborhood $\mathfrak{U}_\lambda$ of M_λ such that $(a_\lambda)_M > 1$ for every $M \in \mathfrak{U}_\lambda$. Since $\mathfrak{B}$ is compact (Corollary 1.2), there exist indices $\lambda_1, \ldots, \lambda_n$ such that $\bigcup_{k=1}^{n} \mathfrak{U}_{\lambda_k} \supset \mathfrak{B}$. Define $a = |a_{\lambda_1}| + \ldots + |a_{\lambda_n}|$. Then $a_M > 1$ for all $M \in \Delta\mathfrak{B}$, and $a \equiv 0\,(\Delta\mathfrak{A})$. Since A is a lattice (Theorem 4.8), we may define $b = \inf(a, 1)$. Then $b \equiv 1\,(\Delta\mathfrak{B})$ and $b \equiv 0\,(\Delta\mathfrak{A})$.

THEOREM 4.12. *If A is an $\mathfrak{S}$-semi-simple $L_{\mathfrak{S}}$-ring with identity, where $\mathfrak{M} \subset \mathfrak{S} \subset \mathfrak{Q}$, then for every $a \in A$, the following are equivalent.*

(a) *The sets $\mathfrak{S}^+(a)$, $\mathfrak{S}^+(-a)$ are completely separated by A* (see Lemma 2.8 ff.).

(b) *There exists an element $k \in A$ such that $a = k|a|$ (whence $|a| = ka$).*

(c) *The ideal $(a, |a|)$ is principal.*

Furthermore, a necessary and sufficient condition that every $a \in A$ have these properties is that $N(S)$ be a prime ideal for every $S \in \mathfrak{S}$.

The proof is analogous to those of [5], Theorems 2.3 (c, d, e) and 2.5 (reference being made, at the appropriate place, to Theorem 4.8 above).

THEOREM 4.13. *Let A be a partially ordered ring satisfying* **I** *and* **II** [1], *and having an identity element. If M is a maximal ideal in A that contains no element $\geqslant 1$, then A satisfies axiom* $\mathbf{III}_M$, *whence A/M is partially ordered.*

Proof. Let a, b satisfy $a \geqslant b \geqslant 0$, $a \in M$, and suppose that $b \notin M$. Since M is maximal, there exists $x \in A$ such that $bx \equiv 1\,(M)$. Then $1 + a|x| - bx \in M$. But

$$1 + a|x| - bx = 1 + (a-b)|x| + b(|x| - x) \geqslant 1,$$

a contradiction.

This result is essentially due to Stone ([17], p. 458). Our proof is somewhat simpler than his, in that we do not make use of axiom $\mathbf{IV}_M$. Using Theorem 4.4 (i), we get:

COROLLARY 4.14. *Let A be a partially ordered ring satisfying* **I**, **II** *and* $\mathbf{IV}_{\mathfrak{M}}$. *If A has an identity element, and every element $\geqslant 1$ is a unit, then every residue class field A/M $(M \in \mathfrak{M})$ is totally ordered.*

In particular, this corollary applies to any ring $C(X)$ (continuous real-valued functions) or $C^*(X)$ (bounded continuous real-valued functions): see Remark 4.6. Of course, in these cases, more specific information is also known. For example, criteria are available to determine when $C(X)/M$ is or is not archimedean ([8], Theorems 41, 50, et al; [6], §§ 2 and 3); and $C^*(X)/M$ is always archimedean ([17], Theorem 76; see also [6], § 1).

The preceding results yield the following theorem regarding the ideals $N(S)$.

THEOREM 4.15. *Let A be a partially ordered ring. If every A/S is partially ordered, then every $A/N(S)$ is partially ordered $(S \in \mathfrak{S})$. If A satisfies* **I** *and* **II**, *A has an identity element, and every element $\geqslant 1$ is a unit, then every $A/N(M)$ is partially ordered $(M \in \mathfrak{M})$.*

If, in addition, $|a|^2 = a^2$ for every $a \in A$, then $A/N(M)$ is totally ordered whenever $N(M)$ is prime (Remark 4.6). The result in the case $A = C(X)$ was first observed by Kohls [12].

[1] M. Jerison and I have since observed that it is sufficient that A be *directed*, *i. e.*, for every $x \in A$, there exists $x' \in A$ such that $x' \geqslant x$ and $x' \geqslant 0$.

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Added in proof. The following is due to M. Jerison.

THEOREM. *Let A be an $\mathfrak{S}$-semi-simple ring with identity, where $\mathfrak{S} \supset \mathfrak{P}$; if an ideal I is contained in a unique $\mathfrak{S}$-ideal S, then $I \supset N(S)$.*

Proof. If $b \in N(S)$, then $b \in \Delta\mathfrak{C}\mathfrak{S}(a)$ for some $a \notin S$. Every $\mathfrak{S}$-ideal contains a or b, and hence any product in which both are factors; so every such product is 0. Every proper ideal is imbeddable in an $\mathfrak{S}$-ideal; hence $(I, a) = A$. Thus $1 = i + r_1a + ar_2 + r_3ar_4$ for suitable $i \in I$, $r_k \in A$; so $b = ib \in I$.

The space of prime ideals of a ring *

by

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1. Introduction. Jacobson showed [4] that the set of primitive ideals of an arbitrary ring may be made into a topological space by means of a closure operator defined in terms of intersection and inclusion relations among ideals of the ring. It was observed by McCoy in [11] that the set of generalized prime ideals defined therein may be treated in exactly the same way.

In the present paper, we shall primarily consider subspaces of this latter space, among which the space of primitive ideals is the most important. In section 2, we review the basic results of the subject and establish notation and terminology. In section 3, we present some simple extensions of the discussion of Jacobson [4] on the connection between compactness of a general space of ideals and restrictions on the ring. The following section treats the relation of other topological properties on appropriate spaces of prime ideals to algebraic conditions on a commutative ring. Section 5 is devoted to an examination of the connection between the prime and primitive ideals of an *ideal* of an arbitrary ring and those of the whole ring. These results are applied in the last section to the situation in which the ideal is viewed as given, and the containing ring is a ring with identity into which it has been imbedded by a standard process.

Several of the results of sections 3 and 4 (in particular, 3.1, 4.1 and 4.7) were obtained independently, in a slightly different form, by McKnight [12]. Since none of his results have been published, we have included a complete discussion.

2. Preliminary concepts. Throughout this paper, the word "ideal" always means "proper two-sided ideal". For the definition of a primitive ideal, see [4], and for the notion of prime ideal in an arbitrary ring,

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