Abelian groups that are direct summands of every abelian group which contains them as pure subgroups

by

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It is a well known theorem of Baer [1], that an abelian group G is a direct summand of every abelian group containing it if and only if G is a divisible group, *i. e.* if $G=nG$ for every positive integer n. It is easy to see that divisibility of G is a necessary condition. A direct summand is a pure subgroup and only divisible groups are pure subgroups of every abelian group containing them. It is also known that there exist groups which are not divisible but are direct summands of every abelian group which contains them as pure subgroups. *E. g.*, according to a theorem of Prüffer, all groups of bounded order have this property.

It is the object of this note to characterize the class **D** of those groups which are direct summands of every abelian group which contains them as pure subgroups. The main result is contained in the following:

THEOREM 1. *A group G belongs to* **D** *if and only if G is a direct summand of a group H which admits a compact topology.*

The above-mentioned theorems of Prüffer and Baer follow from this result, but not directly.

1. Lemmas. A subgroup G of an abelian group H is *pure* if for any integer n and any element x in H, $nx=g\in G$ implies $ny=g$ with y in G.

The following three lemmas are known:

(1.1) *If for every prime number p and every positive integer n, $p^n x=g\in G$ implies $p^n y=g$ with $y\in G$, then G is a pure subgroup.*

(1.2) *Let F be a subgroup of G and G a subgroup of H. If F is pure in H, then F is pure in G.*

(1.3) *If G is a pure subgroup of the abelian group H and H/G is a direct (discrete) sum of cyclic groups, then G is a direct summand of H.*

A subset A of the group H is called a *selection* in H/G, if for every class $h+G$ there is exactly one element a in A with $a+G=h+G$. If A

is a subgroup of H and a selection in H/G, then A is called a *representation* of H/G.

(1.4) *The subgroup G is a direct summand of the abelian group H if and only if there exists a representation of H/G.*

Let T be an arbitrary set and for every t in T let G_t be a group. The set of all functions $x(t)=x_t$ defined on T with values x_t in G is called the *complete direct sum* of the groups G_t and is denoted by $\sum_t^* G_t$. If $G=G_t$ for all t in T, then the complete direct sum is denoted by G^T.

The following lemma results directly from the definition:

(1.5) *If $G_t=A_t+B_t$ for all t in T, then*

$$\sum_t^* G_t=\sum_t^* A_t+\sum_t^* B_t .$$

A topological group G is *compact* if

(1.6) For every sequence of non-empty, closed sets $G\supset F_1\supset F_2\supset\ldots\supset F_\xi\supset\ldots$, $\xi<\alpha$, the intersection of all F_ξ is non-empty: $\bigcap_\xi F_\xi\neq 0$.

From the well known theorem of Tychonoff on the product (in our case the complete direct sum) of compact spaces it follows that:

(1.7) *If G_t are compact topological groups, then $\sum_t^* G_t$ is also a compact topological group in the product topology.*

It is also known that the product topology has the following property:

(1.8) *If G is a topological group, then the set*

$$Z(t,s,r)=\{x\in G^T:\ x_t+x_s-x_r=g\}$$

(where $g\in G$) is closed in G^T.

An example of a compact abelian group is the group K of all complex numbers z with $|z|=1$. All finite groups are also compact groups in the discrete topology. Therefore:

(1.9) *A complete direct sum of finite groups and of groups of the type K admits a compact topology.*

The additive group of rational numbers R^+, and the Prüffer groups C_{p^∞} admit no compact topology, but it is known that

(1.10) *Both R^+ and C_{p^∞} (for every prime number p) are direct summands in K.*

Finally we note a lemma on class **D**:

(1.11) *If H is in* **D** *and G is a direct summand of H, then G is in* **D**.

Proof. Let $H=G+G'$ and let G be a pure subgroup of F. Evidently H is a pure subgroup of $F+G'$ and as H is in **D**, then we have $F+G'=F'+H=F'+G+G'$. This gives us $F=F'+G$ as desired.

2. Now we shall prove the following theorem:

(2.1) *If G is a pure subgroup of H and G admit a compact topology, then G is a direct summand in H.*

Proof. We denote H/G by $\mathfrak{H}$ and let A be a selection in $\mathfrak{H}$. The group G^A is concerned as a compact topological group in the product topology of G.

For x in G^A we denote by xA the set of all elements $a-x_a$ with a in A. It is clear that each selection of $\mathfrak{H}$ may be written in the form xA with a suitable x in G^A. For a subgroup $H_0/G=\mathfrak{H}_0$ of $\mathfrak{H}$ we denote by $R(\mathfrak{H}_0)$ the set of all elements x in G^T such that $xA\frown H_0$ is a representation of $\mathfrak{H}_0$. It is easy to see that

(2.2) *$R(\mathfrak{H}_0)\neq 0$ if and only if G is a direct summand of H_0.*

(2.3) *If $\mathfrak{H}_0\subset\mathfrak{H}_1\subset\ldots\subset\mathfrak{H}_\xi\subset\ldots,\xi<a$, are subgroups of $\mathfrak{H}$, then $R(\mathfrak{H}_0)\supset R(\mathfrak{H}_1)\supset\ldots\supset R(\mathfrak{H}_\xi)\supset\ldots$, and $\bigcap_\xi R(\mathfrak{H}_\xi)=R(\bigcup_\xi\mathfrak{H}_\xi)$.*

Lemma (2.2) follows from (1.4) and (2.3) direct from the definitions.

Let $H_0/G=\mathfrak{H}_0$ be, as before, a subgroup of $\mathfrak{H}$, and let a,b,c be elements in $A\frown H_0$ with $(a+G)+(b+G)=c+G$. Then $a+b-c=g$ and g is in G. It follows from (1.8) that the set $Z(a,b,c)=\{x\in G^A\colon x_a+x_b-x_c=g\}$ is closed in G^A. But $R(\mathfrak{H}_0)$ is the intersection of all $Z(a,b,c)$ with $g=x_a+x_b-x_c$ in G and therefore $R(\mathfrak{H}_0)$ is a closed set of G^A. This yields the following lemma:

(2.4) *For every subgroup $\mathfrak{H}_0$ of $\mathfrak{H}$, $R(\mathfrak{H}_0)$ is a closed set in G^A.*

In order to prove (2.1) we proceed by induction with respect to the power of the generators of $\mathfrak{H}$. If $\mathfrak{H}$ is finitely generated, then (2.1) follows from (1.3). If the power of the generators of $\mathfrak{H}$ is $\aleph_\lambda$, then $\mathfrak{H}$ is a union of a sequence $\mathfrak{H}_0\subset\mathfrak{H}_1\subset\ldots\subset\mathfrak{H}_\xi\subset\ldots,\xi<a$, of subgroup: $\mathfrak{H}=\bigcup_\xi\mathfrak{H}_\xi$, where $\mathfrak{H}_\xi$ are groups with the power of generators $<\aleph_\lambda$. From the assumptions and (2.2) it follows that $R(\mathfrak{H}_\xi)\neq 0$ and from (2.3), (2.4), (1.6) and (2.2) it follows that G is a direct summand of H.

3. In this section the following theorem will be proved:

THEOREM 2. *Each abelian group G may be embedded in a complete direct sum H of groups of type C_{p^a} (p is a prime and $a=1,2,\ldots$ or ∞), in such a way that G is pure in H.*

This theorem is a generalization of a result of Birkhoff [3], the method being a modification of Birkhoff's method.

Proof of theorem 2. It is known that for every element g in G there exists a maximal subgroup G_g in G which does not contain g, and that in such a case group G/G_g is of type C_{p^a} and $g+G_g$ is of order p. By p-height of an element g in G (p is a prime number) we denote such a positive integer $H_p(g)=n$, that g is divisible in G by p^n but not by p^{n+1}; if such a number does not exist, then $H_p(g)=\infty$. It is easy to see that in the type C_{p^a} of group G/G_g a is not smaller than $H_p(g)+1$. We shall prove that if $H_p(g)<\infty$, then such a G_g may be found that G/G_g is precisely of type C_{p^a} with $a=H_p(g)+1$.

In fact, if $H_p(g)=n<\infty$, then in the group $\mathfrak{H}=G/p^{n+1}G$ all elements are of order $\leqslant p^{n+1}$ and $\bar{g}=g+(p^{n+1}G)$ precisely of order p. Let $\mathfrak{H}_{\bar{g}}=G_g/p^{n+1}G$ be a maximal subgroup of $\mathfrak{H}$ which does not contain $\bar{g}$; then $\mathfrak{H}/\mathfrak{H}_{\bar{g}}=G/G_g=C_{p^a}$, $g+G_g$ is of order p and as $\mathfrak{H}$ is a group with orders of elements bounded by p^{n+1}, then $a\leqslant n+1$ and finally $a=n+1$.

Let $P(g)$ be the set of all prime numbers p with $H_p(g)<\infty$. We have demonstrated that

(3.1) *For every g in G and p in $P(g)$ there exists a homomorphism $h(G)=C_{p^a}$, such that $a=H_p(g)+1$ and $h(g)\neq 0$.*

Now let $\{h_t\}$, $t\in T$, be a class of homomorphisms of G, such that

(1) For each t in T, $h_t(G)=C_{p^a}$.

(2) For each g in G there exists a h_t with $h_t(g)\neq 0$.

(3) For each g in G and p in $P(g)$ there exists a homomorphism $h_t(G)=C_{p^a}$ with $h_t(g)\neq 0$ and $a=H_p(g)+1$.

The existence of such a class follows from (3.1) and from the properties of maximal groups G_g.

Now we form the complete direct sum $H=\sum_t^* h_t(G)$ and we assign to each element g in G the element $x=\varphi(g)$ in H with $x_t=h_t(g)$ for each element t in T. It follows from (2) that φ is an isomorphism. Let be given $x=\varphi(g)$ in $\varphi(G)$ and y in H such that $p^ky=x$. If p is not in $P(g)$ then $H_p(g)=\infty$, if p is in $P(g)$ then for some suitable t we have $h_t(g)\neq 0$, $h(G)=C_{p^a}$ and $a=H_p(g)+1$ (this follows from (3)), as $p^ky=x$, then $p^ky_t=x_t=h_t(g)$. This gives $k\leqslant H_p(g)$. Therefore in both cases there exists an element g' in G with $p^kg'=g$ and in consequence we have $p^k\varphi(g')=\varphi(g)$. From (1.1) it follows that $\varphi(G)$ is a pure subgroup of H. In view of (1) H is a complete direct sum of groups C_{p^a}. The proof of theorem 2 is finished.

4. Proof of theorem 1. Let H admit a compact topology and let $H=G+G'$. From (2.1) it follows that H belongs to **D** and from (1.11) it follows that G also belongs to **D**.

Now let G belong to **D**. In view of lemma 2 G is a pure subgroup of the complete direct sum $H=\sum_t^* C_{p_t^{a_t}}$. G is in **D** therefore $H=G+G'$. For t in T with $a_t<\infty$ let $G_t=\{0\}$ and for t with $a_t=\infty$ let G_t be such a group that $C_{p_t^{a_t}}+G_t=K$ (see (1.10)). We have

$$\sum_t^*(C_{p_t^{a_t}}+G_t)=\sum_t^* C_{p_t^{a_t}}+\sum_t^* G_t=H+\sum_t^* G_t=G+(G'+\sum_t^* G_t),$$

therefore G is a direct summand of $\sum_t^*(C_{p_t^{a_t}}+G_t)$. But from (1.9) it follows that $\sum_t^*(C_{p_t^{a_t}}+G_t)$ admits a compact topology.

5. Corollaries. Let $\{h_t\}$ be a family of homomorphisms which fulfils conditions (1)-(3) and let φ be the isomorphism of G in $H=\sum_t^* h_t(G)$ defined above. We have the following corollary:

(5.1) *G belongs to* **D** *if and only if $\varphi(G)$ is a direct summand of H.*

From (1.5) and (1.7) it follows that if each G_t is a direct summand in a group which admits a compact topology, then $\sum_t^* G_t$ is also such a group. Therefore from theorem 1 we have:

(5.2) *If each G_t belongs to* **D**, *then $\sum_t^* G_t$ also belongs to* **D**.

As we know, for every compact topological group H and every natural number n, the subgroup nH is closed in H. Therefore if H admits a compact topology, then nH also admits a compact topology. If $H=G+G'$, then $nH=nG+nG'$. This yields the following corollaries:

(5.3) *If G belongs to* **D** *then for every n the group nG also belongs to* **D**.

(5.4) *If G belongs to* **D** *and $n_1, n_2, \ldots$ is a sequence of non-negative integers, then the group $\bigcap_k (n_1\ldots n_k)G$ also belongs to* **D**.

The group $\bigcap_k p^k G$ consists of those elements of G which are of infinite p-height. Therefore:

(5.5) *If G is in* **D**, *then the subgroup of elements of infinite p-height is also in* **D**.

6. Another proof of (2.1). The proof of lemma (2.1), which is essential to our reasoning, can be simplified by a theorem due to S. Gacsályi.

Let U be a system of equations of the form

$$\sum_{s\in T} n_{ts}x_s=g_t$$

where $t\in T$, $g_t\in G$, n_{ts} are integers and for fixed t, $n_{ts}\neq 0$ for a finite number of s in T only. The system U is of power $\mathfrak{m}$, if T is of power $\mathfrak{m}$.

Let V be a subsystem of system U. By a solution of V in G we understand an element x in G^T such that each equation of V is satisfied by x_s. Gacsályi [4] has shown that:

(6.1) *A subgroup G of H is pure if and only if every finite system U (with coefficients g_t in G) solvable in H is also solvable in G.*

(6.2) *A subgroup G of H is a direct summand if and only if every system U solvable in H is also solvable in G.*

It is easy to see that if G is a topological group, then the set of all solutions of a system V is closed in G^T. The set of all solutions of U is obviously the intersection of all sets of solutions of finite subsystems V of U. Therefore if G admits a compact topology, then G has the following property, which we call the finite intersection property for linear equations:

If every finite subsystem of U is solvable in G, then U is solvable in G.

Now we turn to the proof of (2.1). Let G be a pure subgroup of H and let G admit a compact topology. It follows from (6.1) that every finite subsystem of a system U solvable in H is also solvable in G, and, since G has the finite intersection property for linear equations, also the whole system U has a solution in G; therefore it follows from (6.2) that G is a direct summand in H.

In fact we have proved a more general theorem, namely, that every group with the finite intersection property is a direct summand of each group in which it is a pure subgroup. Therefore every group with the finite intersection property is in **D**. It is easy to see that the converse also holds, and thus we have another characterization of the group in **D**:

THEOREM 3. *A group G is in* **D** *if and only if G has the finite intersection property for linear equations.*

7. Remark. I. Kaplansky in his book [5] has considered the algebraical structure of compact groups (p. 54-56). He has introduced the notion of algebraically compact groups. In the paper [2] S. Balcerzyk has shown that the algebraically compact groups of Kaplansky are precisely those which are in **D**.

References

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On algebraically compact groups of I. Kaplansky

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In his paper [2] J. Łoś considers a class of abelian groups, that are direct summands of every abelian group which contains them as pure subgroups. This class is denoted by **D**. Łoś has proved the following propositions, giving an interesting characterization of this class:

(1) *A group G belongs to* **D** *if and only if G is a direct summand of a group which admits a compact* ($=$ *bicompact*) *topology.*

(2) *A group G belongs to* **D** *if and only if G is a direct summand of group H of a form*

$$H=\sum_{p}^{*}\sum_{t_p\in T_p}^{*}C_{p^{\alpha_{t_p}}}.$$

In his book [1] Kaplansky introduces the notion of algebraically compact (abelian) group.

The purpose of this paper is to prove that class **D** is identical with the class of algebraically compact groups. It gives another proof of Kaplansky's theorem, stating that every group which admits a compact topology is algebraically compact.

For a prime integer p let R_p be the ring of p-adic integers, and M a R_p-module with no element of infinite height, (*i. e.*, the module satisfies the condition $\bigcap_{n=1}^{\infty}p^nM=\{0\}$). Taking submodules p^nM as neighbourhoods of 0 we get a p-adic topology in M.

Let us repeat Kaplansky's definition:

An abelian group G is *algebraically compact* if it has the form

$$G=C+\sum_{p}^{*}D_p,$$

C being a divisible group and D_p a module over R_p, with no element of infinite p-height and complete in its p-adic topology (complete direct sum over all prime integers).

The algebraical structure of algebraically compact groups is fully known.