Remarks on Henkin's paper: Boolean representation through propositional calculus *

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Remark 1. In the paper mentioned Henkin deals with the so called Gödel-Malcev propositional theorem and shows, that it is equivalent without recourse to the axiom of choice (in the sequel we abbreviate: equivalent) to the Boolean representation theorem [1]. It will be observed that this is an easy consequence of the well known facts concerning the structure of the Boolean algebra of the propositional calculus.

As we know, the Boolean representation theorem follows from (and is equivalent to) the following lemma:

In every Boolean algebra there is a prime ideal,

whereas the Gödel-Malcev propositional theorem expressed in terms of ideal theory runs as follows:

Every ideal of a Boolean algebra of the propositional calculus is contained in a prime ideal.

Obviously it must only be shown, that the Gödel-Malcev theorem implies the representation theorem. But, as is well known (see e. g. Rieger [6], p. 51)], Boolean algebras of the propositional calculus are

free Boolean algebras. The propositional variables are free generators. Therefore every Boolean algebra B may be represented as the quotient algebra A/I of a suitable Lindenbaum algebra A. The Gödel-Malcev theorem implies that the ideal I may be extended to a prime ideal I'. Evidently I'/I is a prime ideal of A/I [2].

Remark 2. In the final part of the paper mentioned Henkin deals with a problem which can be formulated as follows:

Let U be a class of ideals of a Boolean algebra B. With the help of the axiom of choice it may be shown that there exists a class U' of power equal to that of U, such that for every ideal I in U there exists a prime ideal I' in U', such that $I \subset I'$. Can this theorem be proved from the Gödel-Malcev theorem or from the lemma on the existence of prime ideals, without recourse to the axiom of choice?

It will be shown that the positive answer follows in a very easy way from the equivalence of the lemma on the existence of ideals and the axiom of choice for bicompact spaces. This equivalence is demonstrated in [4].

From the lemma on the existence of prime ideals it follows that the Stone space $S(B)$ of prime ideals of an algebra B is a non-empty bicompact space. It is easy to see that for every ideal I of B, the set $P(I)$ of prime ideals I' with $I \subset I'$ is a closed set in $S(B)$. Therefore $P(I)$ is a compact space. For every I in U we have chosen an ideal I' from $P(I)$ and thus obtained the set U'.

References

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[4] — and C. Ryll-Nardzewski, *Effectiveness of the representation theory for Boolean algebras*, Fund. Math. 41 (1954), p. 49-56.

[5] H. Rasiowa and R. Sikorski, *A proof of the completeness theorem of Gödel*, Fund. Math. 37 (1950), p. 193-200.

[6] L. Rieger, *On free* $\aleph_\xi$*-complete Boolean algebras*, Fund. Math. 38 (1951), p. 35-52.

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* Fundamenta Mathematicae 41 (1954), p. 89-96.

[1] The Gödel-Malcev functional theorem is equivalent to the Boolean representation theorem. Henkin in the paper [1] has shown that the Gödel-Malcev theorem implies the representation theorem (it is my error that this fact is not mentioned in paper [2]). In what regards the converse implication, Henkin supposes that it follows from the results of Rasiowa nad Sikorski [5]. He writes: "By a slight change in their argument, it can be turned into a proof that the Boolean representation theorem implies the Gödel-Malcev theorem" (see p. 89). In my opinion that is not so. The proof of Rasiowa and Sikorski makes use of the method of category (see lemma IV on p. 197) and therefore it holds only in the case when a denumerable theory is considered (*i. e.* a theory with a denumerable set of primitive signs only). In case of non-denumerable theories the method of Rasiowa and Sikorski is applicable to open theories only (*i. e.* to theories without quantifiers). In order to prove the Gödel-Malcev theorem for non-denumerable theories it is therefore necessary to show that for theories with quantifiers this theorem follows if it is supposed to be true for open theories. This is demonstrated in my paper [3] and my second paper [2] involves this remark.

[2] In the case of the Gödel-Malcev functional theorem, this proof is not applicable for two reasons. First: the Lindenbaum functional algebras are not free Boolean algebras. Secondly: not every prime ideal in a Lindenbaum functional algebra permits the construction of a model (or of a valuation function).