On a problem of W. Sierpiński on the congruence of sets

by

E. G. Straus (Princeton, N. J.)

1. Introduction

W. Sierpiński [5], [6] has raised the question of the existence in Euclidean n-dimensional space E_n of point sets which are congruent to several subsets obtained by the removal of a single point. In [5] he proved that a set S in E_1 can contain at most one point p so that $S-\{p\}=S$. He also gave a wrong proof of the existence in E_2 of a set S containing two points p,q so that $S-\{p\}=S-\{q\}=S$. This error was recognized by J. Mycielski and discussed by W. Sierpiński [7]. In section 2 we prove that no such set exists, and that the above-mentioned result for E_1 is therefore valid for E_2. In section 3 we show that in E_3 there exist sets S congruent to every maximal proper subset; that is $S\cong S-\{x\}$ for all x. This has been accomplished by J. Mycielski [3] and our proof is included only because it may be somewhat simpler. We shall call such sets Sierpiński sets. This completes the solution of Sierpiński's problem.

Finally we discuss the underlying group-theoretical ideas. The author wishes to express his thanks to the referee, J. Mycielski, for his valuable corrections and improvements.

2. The two-dimensional case

THEOREM 1. *A point set S in the Euclidean plane E_2 can contain at most one point p so that $S-\{p\}$ is congruent to S.*

Proof. Assume that there are two points $p,q\in S$ so that $S\cong S-\{p\}\cong S-\{q\}$. Let φ,ψ be isometries so that $\varphi S=S-\{p\}$, $\psi S=S-\{q\}$. Then the following relations must hold:

1° $\varphi x\in S$, $\psi x\in S$ for every $x\in S$,

2° $\varphi^{-1}x\in S$ for every $x\neq p$, $x\in S$; $\varphi^{-1}p\notin S$,

3° $\psi^{-1}x\in S$ for every $x\neq q$, $x\in S$; $\psi^{-1}q\notin S$.

LEMMA 1. *$\varphi^n p \neq p$ and $\psi^n q \neq q$ for all $n = 1, 2, \ldots$* (in particular φ, ψ are of infinite order).

Proof. If $\varphi^n p = p$, $n > 0$ then $\varphi^{n-1} p = \varphi^{-1} p \in S$ contradicting condition 2°. The proof for $\psi^n q \neq q$ is analogous.

LEMMA 2. *At least one of φ, ψ is a rotation.*

Proof. If neither φ nor ψ is a rotation then they are either translations or orientation reversing transformations. We consider the various possible cases:

(i) φ, ψ both *translations*. Then $\varphi\psi = \psi\varphi$ and since $\psi^{-1} p \in S$ we must have $\psi^{-1} p = p$ since otherwise $\varphi^{-1}\psi^{-1} p = \psi^{-1}\varphi^{-1} p \in S$ implies $\varphi^{-1} p \in S$, a contradiction. But ψ is a translation and thus has no fixed points.

(ii) *φ a translation, ψ orientation reversing.* In this case, using the complex number z to denote a point in E_2 we can write $\psi(z) = a\bar{z} + b$ with $|a| = 1$. Hence

$$\psi(\psi(z)) = a\overline{(a\bar{z} + b)} + b = |a|^2 z + a\bar{b} + b = z + a\bar{b} + b\,.$$

In other words ψ^2 is a translation, and $\varphi\psi^2 = \psi^2\varphi$. Now $\psi^{-2}\varphi^{-1} q \in S$ would lead to $\varphi^{-1}\psi^{-2} q \in S$ and hence $\psi^{-1} q \in S$, a contradiction, so that we must have $\psi^{-1}\varphi^{-1} q = q$ that is $\varphi\psi q = q$. Since ψ^2 commutes with $\varphi\psi$ we see that $\varphi\psi$ has the fixed points $\psi^{2n} q$. Hence $\varphi\psi$ must be a reflection on a line and $\varphi\psi\varphi\psi = 1$. But this implies $\psi\varphi\psi p = \varphi^{-1} p \in S$, a contradiction.

(iii) *φ, ψ both orientation reversing.* As in (ii) we see that φ^2, ψ^2 are translations and thus $\varphi^2\psi^2 = \psi^2\varphi^2$. Thus again $\varphi^{-2}\psi^{-2} p \in S$ would lead to the contradiction $\psi^{-2}\varphi^{-2} p \in S$ and hence $\varphi^{-1} p \in S$. Considering the possible reasons for $\varphi^{-2}\psi^{-2} p \notin S$ we have:

(α) $\psi^{-2} p = p$ impossible since ψ^{-2} is a translation.

(β) $\psi^{-2} p = \varphi p$ or $p = \psi^2\varphi p$. Since φ^2 commutes with ψ^2 this shows that $\psi^2\varphi$ has the fixed points $\varphi^{2n} p$ and hence $\psi^2\varphi\psi^2\varphi = 1$ which leads to the contradiction $\varphi^{-1} p = \psi^2\varphi\psi^2 p \in S$.

(γ) $\psi^{-1} p = q$ and similarly $\varphi^{-1} q = p$ so that $\psi\varphi p = p$, $\varphi\psi q = q$.

Now we must have $\varphi^{-2}\psi^2 p \in S$ since $\psi^2 p \neq p$ as ψ^2 is a translation, and $\psi^2 p = \varphi p$ or $\varphi^{-1}\psi^2 p = p$ would give to $\varphi^{-1}\psi^2$ the fixed points $\varphi^{2n} p$ so that $\varphi^{-1}\psi^2\varphi^{-1}\psi^2 = 1$, leading to the contradiction $\psi^{-2} q = \varphi^{-1}\psi^2\varphi^{-1} q = \varphi^{-1}\psi^2 p \in S$. Thus to avoid the contradiction $\psi^{-2}\varphi^{-2}\psi^2 p = \varphi^{-2} p \in S$, we must have either

(δ) $\varphi^{-2}\psi^2 p = q = \varphi p$, which leads to $\psi^2 p = \varphi^3 p$ and hence $\psi^2\varphi^{-1} p = \varphi^{-1}\psi^2 p$ so that we must have either $\varphi^{-1}\psi^2 p = q$, that is, $\psi^2 p = p$, a contradiction; or $\varphi^{-1}\psi^2 p = \psi q$, that is $\psi^2 p = q$ or $\psi^3 q = q$ in contradiction to Lemma 1, or

(ε) $\varphi^{-2}\psi^2 p = \psi q = p$ which implies that the translation $\varphi^{-2}\psi^2$ is the identity.

We now combine the relations $\psi\varphi p = p$; $\varphi^2 = \psi^2$ setting

$$\varphi(z) = a\bar{z} + b\,, \quad \psi(z) = c\bar{z} + d\,, \quad p = 0\,, \quad |a| = |c| = 1\,.$$

Then

$$\psi(\varphi(0)) = c\bar{b} + d = 0\,, \quad \varphi(\varphi(0)) = a\bar{b} + b = \psi(\psi(0)) = c\bar{d} + d\,.$$

Substituting $d = -c\bar{b}$ in the second equation we obtain

$$a\bar{b} + b = -|c|^2 b - c\bar{b} = -b - c\bar{b}$$

or $2b = -(a + c)\bar{b}$. Now $|a + c| = 2$ and $|a| = |c| = 1$ so that we must have $a = c$ and $b + a\bar{b} = 0$. But this means $\varphi^2 p = p$, a contradiction.

Without loss of generality we may now assume that φ is a rotation.

LEMMA 3. *If ψ is a rotation then its center differs from that of φ.*

Proof. If φ, ψ had the same center then they would commute and $\varphi^{-1}\psi^{-1} p = \psi^{-1}\varphi^{-1} p \in S$ with $\psi^{-1} p \neq p$ would imply $\varphi^{-1} p \in S$, a contradiction. But $\psi^{1-} p = p$ would imply $\varphi p = p$, a contradiction.

Remark. Lemmas 1-3 imply that $\varphi^m\psi^n \neq \psi^n\varphi^m$ for all $m, n = \pm 1, \pm 2, \ldots$

We now know that the orbit of a point $x \in E_2$ under φ lies on a circle $C(\varphi, x)$ (possibly of radius 0) and that the orbit of x under ψ lies on a curve $C(\psi, x)$ which is either a circle not concentric with $C(\varphi, x)$ or a straight line or a pair of parallel straight lines.

The commutator of two orientation preserving isometries is always a translation, so that the commutator of two commutators is the identity. Applying this fact to the commutators $\alpha\beta\alpha^{-1}\beta^{-1}$ and $\beta\alpha^{-1}\beta^{-1}\alpha$ we obtain

$$\alpha\beta\alpha^{-1}\beta^{-1}\beta\alpha^{-1}\beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-1}\alpha^{-1}\beta\alpha\beta^{-1} = \alpha\beta\alpha^{-2}\beta^{-1}\alpha\beta\alpha\beta^{-1}\alpha^{-2}\beta\alpha\beta^{-1} = 1\,,$$

(see also [1], [3]). Substituting $\beta = \psi^{-n}$, $\alpha = \varphi^m$ we obtain

$$\varphi^{-m} = \psi^{-n}\varphi^{-2m}\psi^n\varphi^m\psi^{-n}\varphi^m\psi^n\varphi^{-2m}\psi^{-n}\varphi^m\psi^n\,. \tag{1}$$

We now set $p_0 = p$, $p_1 = \psi^n p$, $p_2 = \varphi^m p_1, \ldots, p_{11} = \psi^{-n} p_{10} = \varphi^{-m} p$. If we can show that $p_i \in S$ $(i = 0, \ldots, 11)$ for suitable positive choices of m, n then we obtain a contradiction.

We again divide the problem into two cases.

Case 1. *ψ a rotation.* Since both φ and ψ are of infinite order we may assume m, n so that φ^m, ψ^n are nearly identity rotations and we write $\varphi^m(z) = (1 + \varepsilon_1)(z - z_1) + z_1$, $\psi^n(z) = (1 + \varepsilon_2)(z - z_2) + z_2$ where $z_1 \neq z_2$ and $\varepsilon_1, \varepsilon_2$ are small complex numbers so that $|1 + \varepsilon_1| = |1 + \varepsilon_2| = 1$. Without loss of generality we further assume $p = 0$, $q = 1$ and obtain

$$p_0 = 0\,, \quad p_1 = -\varepsilon_2 z_2; \quad p_2 = -\varepsilon_1 z_1 - \varepsilon_2 z_2 - \varepsilon_1\varepsilon_2 z_2\,,$$
$$p_3 = -\varepsilon_1 z_1 - \varepsilon_1\bar{\varepsilon}_2(z_1 - z_2)\,, \quad p_4 = -\bar{\varepsilon}_1 z_1 + \bar{\varepsilon}_1\bar{\varepsilon}_2(z_1 - z_2) + \bar{\varepsilon}_1^2\bar{\varepsilon}_2(z_1 - z_2)\,,$$
$$p_5 = -\bar{\varepsilon}_1 z_1 - \varepsilon_2 z_2 - \bar{\varepsilon}_1\varepsilon_2(2z_1 - z_2) - \bar{\varepsilon}_1^2\varepsilon_2(z_1 - z_2)\,,$$

$$p_6 = -\varepsilon_2 z_2 + (\varepsilon_1 - \bar{\varepsilon}_1)\varepsilon_2(z_1 - z_2), \qquad p_7 = -\bar{\varepsilon}_2(\varepsilon_1 - \bar{\varepsilon}_2)(z_1 - z_2),$$
$$p_8 = -\varepsilon_1 z_1 - 2\varepsilon_1\bar{\varepsilon}_2(z_1 - z_2) - \bar{\varepsilon}_2\varepsilon_1^2(z_1 - z_2),$$
$$p_9 = -\varepsilon_1 z_1 - \varepsilon_2 z_2 + \varepsilon_1\varepsilon_2(z_1 - 2z_2) + \varepsilon_1^2\varepsilon_2(z_1 - z_2),$$
$$p_{10} = -\bar{\varepsilon}_1 z_1 - \varepsilon_2 z_2 - \bar{\varepsilon}_1\varepsilon_2 z_1, \qquad p_{11} = -\bar{\varepsilon}_1 z_1.$$

We can now choose $\varepsilon_1, \varepsilon_2$ so that p_2, p_6, p_{10} are not on $C(\psi, q) = \{z \,\big|\, |z - z_2| = |1 - z_2|\}$ and p_3, p_9 are not on $C(\varphi, p) = \{z \,\big|\, |z - z_1| = |z_1|\}$. All we have to show is that none of the five equations can be an identity in $\varepsilon_1, \varepsilon_2$ for any permissible choice of z_1, z_2.

(i)
$$|p_2 - z_2|^2 = |z_2|^2 + \varepsilon_1 z_1\bar{z}_2 + \bar{\varepsilon}_1\bar{z}_1 z_2 + |\varepsilon_1|^2|z_1|^2 + \ldots$$
$$= |z_2|^2 + \varepsilon_1 z_1(\bar{z}_2 - \bar{z}_1) + \bar{\varepsilon}_1\bar{z}_1(z_2 - z_1) + \ldots = |1 + z_2|^2$$

where the three dots indicate terms involving ε_2. This cannot be an identity in ε_1 since $z_1 \neq 0$ and $z_1 \neq z_2$.

(ii)
$$|p_6 - z_2|^2 = |z_2|^2 - (\varepsilon_1 - \bar{\varepsilon}_1)\varepsilon_2(1 + \bar{\varepsilon}_2)(z_1 - z_2)\bar{z}_2 -$$
$$- (\bar{\varepsilon}_1 - \varepsilon_1)\bar{\varepsilon}_2(1 + \varepsilon_2)(\bar{z}_1 - \bar{z}_2)z_2 + (\varepsilon_1 - \bar{\varepsilon}_1)(\bar{\varepsilon}_1 - \varepsilon_1)\varepsilon_2\bar{\varepsilon}_2|z_1 - z_2|^2$$
$$= |z_2|^2 + (\varepsilon_1 - \bar{\varepsilon}_1)[\bar{\varepsilon}_2(z_1 - z_2)\bar{z}_2 - \varepsilon_2(\bar{z}_1 - \bar{z}_2)z_2 +$$
$$+ (\varepsilon_1 - \bar{\varepsilon}_1)(\varepsilon_2 + \bar{\varepsilon}_2)|z_1 - z_2|^2].$$

If this were identically $|1 - z_2|^2$ then we would have

$$|z_2|^2 = |1 - z_2|^2 \quad \text{and} \quad \bar{\varepsilon}_2(z_1 - z_2)\bar{z}_2 - \varepsilon_2(\bar{z}_1 - \bar{z}_2)z_2 +$$
$$+ (\varepsilon_1 - \bar{\varepsilon}_1)(\varepsilon_2 + \bar{\varepsilon}_2)|z_1 - z_2|^2 \equiv 0 \text{ in } \varepsilon_1, \varepsilon_2.$$

This is impossible since $z_1 \neq z_2$.

(iii)
$$|p_{10} - z_2|^2 = |1 + \varepsilon_2|^2|\bar{\varepsilon}_1 z_1 + z_2|^2$$
$$= \varepsilon_1\bar{\varepsilon}_1|z_1|^2 + \varepsilon_1\bar{z}_1 z_2 + \bar{\varepsilon}_1 z_1\bar{z}_2 + |z_2|^2$$
$$= \varepsilon_1\bar{z}_1(z_2 - z_1) + \bar{\varepsilon}_1 z_1(\bar{z}_2 - \bar{z}_1) + |z_2|^2$$

which cannot be identically $|1 - z_2|^2$ since $z_1 \neq 0$ and $z_1 \neq z_2$.

(iv)
$$|p_3 - z_1|^2 = |z_1|^2 + \varepsilon_1\bar{\varepsilon}_2(z_1 - z_2)\bar{z}_1 + \bar{\varepsilon}_1\varepsilon_2(\bar{z}_1 - \bar{z}_2)z_1 + \varepsilon_1\bar{\varepsilon}_1\varepsilon_2\bar{\varepsilon}_2|z_1 - z_2|^2$$
$$= |z_1|^2 + \varepsilon_1\varepsilon_2|z_1 - z_2|^2 + \ldots$$

which cannot be identically $|z_1|^2$ since $z_1 \neq z_2$.

(v)
$$|p_9 - z_1|^2 = |z_1|^2 + \varepsilon_2 z_2(\bar{z}_1 - \bar{z}_2) + \bar{\varepsilon}_2\bar{z}_2(z_1 - z_2) + \varepsilon_1\bar{\varepsilon}_2\bar{z}_1(z_1 - z_2) + \bar{\varepsilon}_1\varepsilon_2 z_1(\bar{z}_1 - \bar{z}_2)$$

which cannot be identically $|z_1|^2$ since the terms in $\varepsilon_1\bar{\varepsilon}_2$ and $\bar{\varepsilon}_1\varepsilon_2$ would vanish only for $z_1 = 0$ or $z_1 = z_2$ both of which are impossible.

Having chosen $\varepsilon_1, \varepsilon_2$ — that is, m, n — so that $p_2, p_6, p_{10} \notin C(\psi, q)$ and $p_3, p_4 \notin C(\varphi, p)$ we know that $p_i \in S$ $(i = 1, \ldots, 11)$. In other words $\varphi^{-m}p \in S$ for some (in fact arbitrarily large) m, a contradiction.

Case 2. ψ^2 *a translation* (this includes both the case ψ a translation and ψ orientation reversing). Here the identity (1) can be replaced by the simpler identity

$$\varphi^m\psi^{2n}\varphi^{-m}\psi^{-2n}\varphi^m\psi^{-2n}\varphi^{-m}\psi^{2n} = 1, \tag{2}$$

which expresses the fact that the translations ψ^{2n} and $\varphi^m\psi^{-2n}\varphi^{-m}$ commute. We again introduce the notations $p_0 = p, p_1 = \psi^{2n}p_0, \ldots, p_7 = \psi^{2n}p_6 = \varphi^{-m}p$ so that $p_i \in S$ $(i = 1, \ldots, 7)$ would lead to a contradiction. We set $\psi^{2n}z = z + a$, $\varphi^m z = (1 + \varepsilon)(z - z_0) + z_0$ $(a \neq 0, |1 + \varepsilon| = 1)$; $p = 0$, $q = 1$. Then we obtain

$$p_0 = 0, \quad p_1 = a, \quad p_2 = a + \bar{\varepsilon}(a - z_0), \quad p_3 = \bar{\varepsilon}(a - z_0),$$
$$p_4 = -\varepsilon a, \quad p_5 = -(1 + \varepsilon)a, \quad p_6 = -a - \bar{\varepsilon}z_0, \quad p_7 = -\bar{\varepsilon}z_0.$$

Now we can select positive m, n so that $p_1, p_5 \notin C(\varphi, p)$ and $p_2 \notin C(\psi, q)$. The first part, $p_1, p_5 \notin C(\varphi, p)$, can be attained simply by choosing n and hence $|a|$ so large that the circle $|z| = |a|$ does not intersect the circle $C(\varphi, p)$. The second part, $p_2, p_4 \notin C(\psi, q)$, is obtained by choosing ε so that $\arg\big(\bar{\varepsilon}(a - z_0) - 1\big) \neq \arg a$, $\arg\big(\bar{\varepsilon}(a - z_0) - \psi(1)\big) \neq \arg a$, $\arg(-\varepsilon a - 1) \neq \arg a$, $\arg\big(-\varepsilon a - \psi(1)\big) \neq \arg a$. Since we may choose $a \neq 0$, $a \neq z_0$ these inequalities can be satisfied. We thus have $p_i \in S$ for $i = 1, \ldots, 7$, a contradiction.

As pointed out by Sierpiński [7] the reason for his error was his neglect of the identities (1), (2) and his consequently futile attempt to construct two independent elements of the group of rigid motions of E_2. We shall now show that Sierpiński's method is valid whenever the group of motions contains two independent elements (and hence infinitely many independent elements). For another method of accomplishing these results see [3].

3. The three dimensional case

The following extension of my original theorem was suggested by J. Mycielski.

THEOREM 2. *Let G be a free group of rank $\geqslant 2$ and $\mathfrak{m}$ a cardinal so that $\overline{\overline{G}}^{\mathfrak{m}} = \overline{\overline{G}}$. Then there exists a subset U of G with $\overline{\overline{U}} = \overline{\overline{G}}$ so that for each $Q \subset U$ with $\overline{\overline{Q}} \leqslant \mathfrak{m}$ there exist a $\varphi_Q \in G$ so that*

$$\varphi_Q U = U - Q.$$

In other words the removal from U of any set of cardinality less than $\mathfrak{m}$ yields a set which is a translation of U by an element of G.

Proof. Let $\Phi = \{\varphi_a\}$ be a set of generators of G. We may assume $\overline{\overline{\Phi}} = \overline{\overline{G}}$. By G_a, ${}_aG$ we denote those elements of G whose reduced expression in terms of the φ does not have a φ_a as rightmost term, respectively as leftmost term. Let $\mathfrak{G}=\{Q_a\}$ be a well ordering of the subsets $Q\subset G$ with $\bar{\bar{Q}}<\mathfrak{m}$ so that the order type of $\mathfrak{G}$ is the initial order type of $\bar{\bar{G}}$. Since $\mathfrak{m}<\bar{\bar{G}}$. We can make a one-to-one association between $\mathfrak{G}$ and elements of Φ so that if Q_a corresponds to φ_a then $Q_a \subset_a G$. We now define

(3) $$U=G-\bigcup_a\{g_a\varphi_a^{-n}q_a\ \ q_a\in Q_a;\ g_a\in G_a;\ n=1,2,\dots\}=G-\bigcup_a V_a\,.$$

Multiplication by φ_a maps all the sets on the right of (3) into themselves except the set V_a for which $\varphi_a V_a=V_a\cup Q_a$. Thus

$$\varphi_a U=U-Q_a\,.$$

Remark. The set $G-U$ is thus a nontrivial set for which the *addition* of any $Q\subset G$ with $\bar{\bar{Q}}\leqslant\mathfrak{m}$ can be attained by a translation by an element of G.

COROLLARY. (See also [3], Theorem 8 for fuller exploitation of Theorem 2). *There is a point set M on a sphere S in E_3 so that for every $x\in M$ there is a rotation φ of S so that $\varphi M=M-\{x\}$.*

Proof. The group of rotations of S in E_3 contains a denumerable free subgroup G' of infinitely many generators (see [4]). Thus, according to Theorem 2 there exists a denumerable set of rotations H so that for every $\psi\in H$ there is a $\varphi\in G'$ with $\varphi H=H-\{\psi\}$.

Since G' is denumerable the set of points of S which are fixed points of an element of G' other than the identity is denumerable. Thus there exists a point $x_0\in S$ which is not a fixed point of any element of G' other than the identity. Let $M=Hx_0$, that is the set of all images of x under rotations of H. Then every $x\in M$ has a unique representation as $\psi x_0, \psi\in H$. Let φ be chosen so that $\varphi H=H-\{\psi\}$ then $\varphi M=M-\{x\}$.

4. The group-theoretical background

From Theorem 2 we see that the existence of Sierpiński sets in a group G follows from the existence of two independent elements in G. Kuranishi [1] has shown that such pairs of elements exist in every nonsolvable Lie group. If G is a group of motions in a space S, and the fixed point of an element of G other than the identity form a nowhere dense set in S, then we can apply the method of the Corollary to Theorem 2 to obtain a Sierpiński set in S with respect to the motions of G.

The nonexistence of independent elements in a Lie group G implies the existence of a nontrivial monomial $M(x,y)=\prod x^{m_i}y^{n_i}$ which is the identity for all elements of G (see [1]) such an "identical relation" surely exists in all nilpotent groups and hence for all solvable Lie groups. However even the commutativity of the group does not preclude the existence of sets which are congruent under that group to more than one maximal subset. In particular we have the following

THEOREM 3. *There exists a set S in E_2 so that $(0,1)\in S$, $(1,0)\in S$ and $S-\{(0,1)\}$, $S-\{(1,0)\}$ are images of S under diagonal matrices.*

Proof. Let $S=\{(0,2^n),(2^n,0)\mid n=0,1,\dots\}$ and let $\varphi=\begin{pmatrix}2&0\\0&1\end{pmatrix}$, $\psi=\begin{pmatrix}1&0\\0&2\end{pmatrix}$ then obviously

$$\varphi S=S-\{(1,0)\},\qquad \psi S=S-\{(0,1)\}\,.$$

We may still conjecture that some converse to Theorem 2 holds:

Conjecture. A group contains a Sierpiński set only if it contains a free group of rank 2.

References

[1] M. Kuranishi, *On everywhere dense imbedding of free groups in Lie groups*, Nagoya Math. J. 2 (1951), p. 63-71.

[2] Jan Mycielski, *Generalizations of the theorems on paradoxical decompositions of the sphere*, Bull. Acad. Pol. Sci. 3 (1955), p. 199-200.

[3] — *About sets with strange isometrical properties*, Fund. Math. 42 (1955), p. 1-10.

[4] J. von Neumann, *Zur allgemeinen Theorie des Maßes*, Fund. Math. 13 (1929), p. 73-116.

[5] W. Sierpiński, *Sur un ensemble plan singulier*, Fund. Math. 37 (1950), p. 1-4.

[6] — *Sur quelques problèmes concernant la congruence des ensembles de points*, Elem. Math. 5 (1950), p. 1-4.

[7] — *Sur une relation entre deux substitutions linéaires*, Fund. Math. 41 (1954), p. 1-5.

[8] — *Quelques résultats et problèmes concernant la congruence des ensembles de points*, Le Matematiche 10 (1955), p. 71-79.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES AND
THE INSTITUTE FOR ADVANCED STUDY

Reçu par la Rédaction le 3.12.1955