On the topological structure of infinite Galois groups *

by

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Let K be a field and L its normal algebraical extension. The Galois group $G(L/K)$ considered as a topological space (see [3], [2], p. 187-190 and [4], p. 239-240) is bicompact and 0-dimensional. If L is a denumerable extension of K, then the space $G(L/K)$ is separable and dense in itself and hence homeomorphic with the Cantor discontinuum, *i. e.*, with the Cartesian product of denumerably many finite spaces.

The aim of this note is to generalize this fact to the of arbitrary infinite extensions: we shall show that $G(L/K)$ is homeomorphic with the Cartesian product (endowed with the ordinary Tichonoff topology) of a suitable number of finite spaces.

We shall use the following notation. The ground field will be denoted by K and the letters $L, M, N, \ldots$ with or without indices will denote normal extensions of K. The Galois group of M with respect to K will be denoted by $G(M/K)$. If $\varphi \in G(M/K)$ and N is a field intermediate between K and M, then we denote by $\varphi|N$ the restriction of φ to N. If X is a subset of $G(M/K)$, then we denote by $X|N$ the set of automorphisms $\varphi|N$ where $\varphi \in X$. If M_1 and M_2 are two extensions of K, then we denote by (M_1, M_2) the compositum of M_1 and M_2, *i. e.*, the smallest field generated by M_1 and M_2.

1. In this section we assume that L is a normal, algebraical, and separable extension of K. We further assume that $K \subset M \subset L$, $K \subset N \subset L$, and that N is finite over K.

LEMMA 1 (see [2], p. 149 or [1], p. 68). $G(N/M \cap N) = G((M, N)/M)\big|N$.

Proof. Denote by H the group of the right-hand side of the equation to be proved. It is obviously contained in $G(N/M \cap N)$ and hence, by the fundamental theorem of the Galois theory, has the form $G(N/P)$ where P is a subfield of N. Since for every a in $N - M$ there is a φ in H such that $\varphi(a) \neq a$, it follows that $P = M \cap N$. Lemma 1 is thus proved.

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LEMMA 2. *Let $G(N/K) = A_1 \cup A_2 \cup \ldots \cup A_s$ be a partition of the group $G(N/K)$ into disjoint sets each of which has exactly one element in common with each co-set of $G(N/M \cap N)$ in $G(N/K)$; further let φ be an element of $G(M/K)$ and i an integer $\leqslant s$. Under these assumptions there is a ψ in $G((M, N)/K)$ such that $\psi|M = \varphi$ and $\psi|N \in A_i$.*

Proof. It is evident that there are ψ in $G((M, N)/K)$ satisfying the equation $\psi|M = \varphi$ and that if ψ satisfies this equation, then so does $\psi\delta$ where $\delta \in G((M, N)/M)$. Hence it is sufficient to show that if δ runs over $G((M, N)/M)$, then $\psi\delta|N$ runs over a co-set of $G(N/M \cap N)$ in $G(N/K)$. This, however, follows from the observation that $\{\psi G((M, N)/M)\}\big|N$ is a co-set of $G(N/M \cap N)$ in $G(N/K)$ because, in view of lemma 1,

$$\{\psi G(M, N)/M)\}\big|N = (\psi|N)\, G((M, N)/M)\big|N = (\psi|N)\, G(N/M \cap N)\,.$$

2. Again let L be a normal, algebraical, and separable extension of K and let $\{K_\xi^*\}_{\xi<\alpha}$ be a transfinite sequence of finite extensions containing exactly once each finite extension of K. We define by transfinite induction two sequences $\{K_\eta\}$ and $\{M_\eta\}$ of subfields of L:

$M_0 = K_0 = K$,

$M_{\eta+1} = (M_\eta, K_\eta)$,

$M_\lambda = \bigcup_{\eta<\lambda} M_\eta$ for limit numbers λ,

$K_\eta = K_\eta^*$ where ξ is the smallest ordina that $K_\xi^* \not\subset M_\eta$ or $K_\eta = K$ if no such ξ exists.

The least ordinal η such that $M_\eta = L$ is denoted by η_0.

For each $\eta < \eta_0$ we select a finite family of disjoint sets $\{A_i^\eta\}_{i \in Z_\eta}$ (where Z_η is a set of indices) covering the group $G(K_\eta/K)$ and satisfying the condition that each A_i^η has exactly one element in common with each co-set of the group $G(K_\eta/K_\eta \cap M_\eta)$ in $G(K_\eta/K)$. We consider next the Cartesian product $P_{\eta<\eta_0} Z_\eta$ endowed with the ordinary Tichonoff topology. With this notation we have the following

THEOREM 1. *The space $G(L/K)$ endowed with the Krull topology is homeomorphic with $P_{\eta<\eta_0} Z_\eta$.*

Proof. For an arbitrary $\varphi \in G(L/K)$ we denote by f_φ the element of $P_{\eta<\eta_0} Z_\eta$ such that $f_\varphi(\eta)$ is the unique element i of Z_η satisfying the formula $\varphi|K_\eta \in A_i^\eta$. From lemma 2 it follows by an easy induction that each element of $P_{\eta<\eta_0} Z_\eta$ can be represented in the form f_φ for a suitable φ in $G(L/K)$.

If φ and ψ are different elements of $G(L/K)$, then $f_\varphi \neq f_\psi$. Indeed, let ζ be the smallest ordinal such that $\varphi|M_\zeta \neq \psi|M_\zeta$. Obviously ζ is not a limit number; let $\zeta - 1$ be the precedessor of ζ. Since $\varphi|M_{\zeta-1} = \psi|M_{\zeta-1}$

and $M_\zeta = (M_{\zeta-1}, K_{\zeta-1})$, we obtain $\varphi|K_{\zeta-1} \neq \psi|K_{\zeta-1}$. At the same time $\varphi|K_{\zeta-1}$ and $\psi|K_{\zeta-1}$ belong to the same co-set of the group $G(K_{\zeta-1}/K_{\zeta-1} \cap M_{\zeta-1})$ in the group $G(K_{\zeta-1}/K)$ since φ and ψ act identically on the elements of $M_{\zeta-1}$. Each set $A_i^{\zeta-1}$ having exactly one element in common with each co-set of the group $G(K_{\zeta-1}/K_{\zeta-1} \cap M_{\zeta-1})$ in the group $G(K_{\zeta-1}/K)$, we infer that if $\varphi|K_{\zeta-1} \in A_i^{\zeta-1}$ and $\psi|K_{\zeta-1} \in A_j^{\zeta-1}$, then $i \neq j$. Hence $f_\varphi(\zeta-1) \neq f_\psi(\zeta-1)$, *i. e.*, $f_\varphi \neq f_\psi$.

In order to show that the mapping $f\colon \varphi \to f_\varphi$ is continuous it is sufficient to show that for each $\xi < \eta_0$ and each $i \in Z_\xi$ the counter-image of the set $\{k \in P_{\eta<\eta_0} Z_\eta\colon k(\xi) = i\}$ is open in $G(L/K)$. This counter-image, however, is equal to the set $\{\varphi \in G(L/K)\colon f_\varphi(\xi) = i\}$, *i. e.*, to the set $\{\varphi \in G(L/K)\colon \varphi|K_\xi \in A_i^\xi\}$, which is obviously open.

The continuity of the inverse mapping f^{-1} results from the continuity of f and the bicompactness of $G(L/K)$. Theorem 1 is thus proved.

3. In this section we shall generalize theorem 1 as follows:

THEOREM 2. *If L is a normal, algebraical extension of K, then there is an ordinal η_0 such that the space $G(L/K)$ is homeomorphic to the Cartesian product $P_{\eta<\eta_0} T_\eta$ where each T_η is a space with exactly two elements.*

Proof. If L is separable over K, then theorem 2 follows from theorem 1 and the well-known fact that spaces $P_{\eta<\eta_0} Z_\eta$ and $P_{\eta<\eta_0} T_\eta$ are homeomorphic.

If L is not separable over K, then we denote by K_1 the field of invariants of $G(L/K)$, *i. e.*, the set of elements a of L such that $\varphi(a) = a$ for each φ in $G(L/K)$. Since L is a separable extension of K_1 (see [2], p. 145 or [1], p. 45-46), theorem 1 is applicable to the group $G(L/K_1)$ and thus it is sufficient to show that the spaces $G(L/K)$ and $G(L/K_1)$ are homeomorphic. This, however, is obvious, since both spaces contain exactly the same elements, are bicompact, and sets which are open in $G(L/K)$ are open in $G(L/K_1)$.

References

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On a problem of W. Sierpiński on the congruence of sets

by

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1. Introduction

W. Sierpiński [5], [6] has raised the question of the existence in Euclidean n-dimensional space E_n of point sets which are congruent to several subsets obtained by the removal of a single point. In [5] he proved that a set S in E_1 can contain at most one point p so that $S-\{p\} = S$. He also gave a wrong proof of the existence in E_2 of a set S containing two points p, q so that $S-\{p\} = S-\{q\} = S$. This error was recognized by J. Mycielski and discussed by W. Sierpiński [7]. In section 2 we prove that no such set exists, and that the above-mentioned result for E_1 is therefore valid for E_2. In section 3 we show that in E_3 there exist sets S congruent to every maximal proper subset; that is $S \cong S-\{x\}$ for all x. This has been accomplished by J. Mycielski [3] and our proof is included only because it may be somewhat simpler. We shall call such sets Sierpiński sets. This completes the solution of Sierpiński's problem.

Finally we discuss the underlying group-theoretical ideas. The author wishes to express his thanks to the referee, J. Mycielski, for his valuable corrections and improvements.

2. The two-dimensional case

THEOREM 1. *A point set S in the Euclidean plane E_2 can contain at most one point p so that $S-\{p\}$ is congruent to S.*

Proof. Assume that there are two points $p, q \in S$ so that $S \cong S-\{p\} \cong S-\{q\}$. Let φ, ψ be isometries so that $\varphi S = S-\{p\}$, $\psi S = S-\{q\}$. Then the following relations must hold:

1° $\varphi x \in S$, $\psi x \in S$ for every $x \in S$,

2° $\varphi^{-1} x \in S$ for every $x \neq p$, $x \in S$; $\varphi^{-1} p \notin S$,

3° $\psi^{-1} x \in S$ for every $x \neq q$, $x \in S$; $\psi^{-1} q \notin S$.