On the definitions of computable real continuous functions

by

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In this paper I shall prove the equivalence of some definitions of computable real continuous functions. Let us assume the following abbreviations: $\mathcal{N}$ = the set of natural numbers, $\mathcal{I}$ = the set of all integers, $\mathcal{R}$ = the set of real numbers, $\mathfrak{F} = \mathcal{I}^{\mathcal{I}}$ (the class of functions defined over the set $\mathcal{I}$ and assuming the values from $\mathcal{I}$), $\mathfrak{Com}$ = the class of computable (general recursive) integral functions, $\mathfrak{Com} \subset \mathfrak{F}$, $\mathcal{K}$ = the class of computable functionals in the sense of [1] (defined over the n-tuples of the elements of $\mathfrak{F}$, and the k-tuples of the elements of $\mathcal{I}$ and assuming the integral values. We shall often use the expression $A(a,f)$ as an abbreviation of: $a \in \mathcal{R}$, $f \in \mathfrak{F}$ and for any $n \in \mathcal{N}$

$$\left| a - \frac{f(n)}{n+1} \right| < \frac{1}{n+1}.$$

Latin letters will be used in such a manner that always $i, k, l, m, n \in \mathcal{N}$, $p, q, r, s, t, u, x, y, z \in \mathcal{I}$, $a, b, c, d, e \in \mathcal{R}$.

Let r_n be a recursive enumeration of all rationals without repetitions. Let $\mathrm{No}(p,q)$ be the recursive converse function of the function r_n. This means that

$$r_{\mathrm{No}(p,q)} = \frac{p}{q}. \tag{1}$$

We assume that $p/0 = 0$. Instead of $\mathrm{No}(p,q)$ we shall often write $\mathrm{No}(p/q)$. Let us set

$$W_n(k) = W(n,k) = (\mu x)\left[\left| r_n - \frac{x}{k+1} \right| < \frac{1}{k+1} \right], \tag{2}$$

No, $W_n \in \mathfrak{Com}$. We obviously have

$$\left| r_n - \frac{W(n,k)}{k+1} \right| < \frac{1}{k+1} \quad \text{for all} \quad n, k \in \mathcal{N}. \tag{3}$$

We start from the definition proposed in [1]: A real function φ is said to be *computable continuous* ($\varphi \in \mathcal{K}_{\mathrm{I}}$) if and only if there exists a functional $\Phi \in \mathcal{K}$ such that for all $a \in \mathcal{R}$ and $f \in \mathfrak{F}$

(4) if $\left|\frac{f(k)}{k+1} - a\right| < \frac{1}{k+1}$ for all $k \in \mathcal{N}$,

$$\text{then} \quad \left|\frac{\Phi\langle f\rangle(k)}{k+1} - \varphi(a)\right| < \frac{1}{k+1} \quad \text{for all} \quad k \in \mathcal{N}.$$

(To simplify the proofs we shall consider only the real functions defined over the whole set $\mathcal{R}$.)

The second definition shall be an extension of the definition of Mazur: $\varphi \in \mathcal{K}_{\mathrm{II}}$ if and only if the following two conditions are satisfied:

(5) (i) For each computable sequence $\{a_n\}$, the sequence of values $\{\varphi(a_n)\}$ is also computable [1]).

(ii) φ is computably uniformly continuous with respect to the rational segments. This means that there exists a function $g \in \mathfrak{Com}$, such that

$$\text{if } r_n < a,\ b < r_m \text{ and } |a-b| < \frac{1}{g(n,m,k)} \text{ then } |\varphi(a) - \varphi(b)| < \frac{1}{k+1}$$

for all $n, m, k \in \mathcal{N}$ and $a, b \in \mathcal{R}$.

THEOREM 1. $\mathcal{K}_{\mathrm{I}} \subset \mathcal{K}_{\mathrm{II}}$.

Proof. If $\varphi \in \mathcal{K}_{\mathrm{I}}$ then there exists a functional $\Phi \in \mathcal{K}$, such that condition (4) is satisfied. A sequence $\{a_n\}$ is said to be *computable* if there exists a function $f \in \mathfrak{Com}$ such that $f_n(k) = f(n,k)$ and

(6) $$\left|\frac{f_n(k)}{k+1} - a_n\right| < \frac{1}{k+1} \quad \text{for all} \quad n, k \in \mathcal{N}.$$

Hence if the sequence $\{a_n\}$ is computable, then from (4) and (6) it follows that

(7) $$\left|\frac{\Phi\langle f_n\rangle(k)}{k+1} - \varphi(a_n)\right| < \frac{1}{k+1} \quad \text{for all} \quad n, k \in \mathcal{N}.$$

This means, according to (6), that the sequence $\{\varphi(a_n)\}$ is computable because the substitution of a computable function in a computable functional produces a computable function ([1], property 6, p. 174).

[1]) S. Mazur in an unpublished paper: *Introduction to the computable analysis*, considers the real functions satisfying only condition (5i). The main result of Mazur is that condition (5i) implies the continuity of the function φ in the set of computable numbers. In this paper we make no use of this result.

According to Theorem 2 of [1] there exists a functional $\omega_\varphi \in \mathcal{K}$ such that if $A(c, f')$, $A(d, f'')$ and $c < a$, $b < d$ and

$$|a-b| < \frac{1}{\omega_\varphi\langle f', f''\rangle(k)+1}, \quad \text{then} \quad |\varphi(a) - \varphi(b)| < \frac{2}{k+1}.$$

Setting $c = r_n$, $d = r_m$, $g(n,m,k) = \omega_\varphi\langle W_n, W_m\rangle(2k+1)+1$ we obtain (ii) from (3).

Let us modify the definition of $\mathcal{K}_{\mathrm{II}}$: $\varphi \in \mathcal{K}'_{\mathrm{II}}$ if and only if

(8) (i) the sequence $\{\varphi(r_n)\}$ is computable,

(ii) φ is computably uniformly continuous in the sense of (5ii).

THEOREM 2. $\mathcal{K}_{\mathrm{II}} \subset \mathcal{K}'_{\mathrm{II}}$.

Proof. According to (3) and (6) the sequence $\{r_n\}$ is computable. Hence by (5i) the sequence $\{\varphi(r_n)\}$ is computable [2]).

A continuous function is defined by the values at the rational points. Hence if we assume the continuity of φ, then it suffices that the condition of computably uniform continuity be formulated with respect to the sequence $\{\varphi(r_n)\}$. Thus we obtain the following definitions similar to that of E. Specker [4].

Let $\mathcal{K}_{\mathrm{III}}$ be the class of real functions satisfying the conditions:

(9) (a) φ is continuous,

(b) the sequence $\{\varphi(r_n)\}$ is computable,

(c) the sequence $\{\varphi(r_n)\}$ is computably uniformly continuous with respect to the rational segments:

$$\text{if} \quad r_n < r_l,\ r_t < r_m \text{ and } |r_l - r_t| < \frac{1}{g(n,m,k)},$$

$$\text{then} \quad |\varphi(r_l) - \varphi(r_m)| < \frac{1}{k+1}$$

for all $n, l, t, m \in \mathcal{N}$.

Evidently:

THEOREM 3. $\mathcal{K}'_{\mathrm{II}} \subset \mathcal{K}_{\mathrm{III}}$.

[2]) The computability of the sequence $\{\varphi(r_n)\}$ can be expressed in many different ways. For example by the condition that there exist two functions $f_1, f_2 \in \mathfrak{Com}$ such that for all $k, n \in \mathcal{N}$

(a) $r_{f_1(n,k)} < \varphi(r_n) < r_{f_2(n,k)}$,

(b) $|r_{f_1(n,k)} - r_{f_2(n,k)}| < 1/(k+1)$.

The discussion of different definitions of computable sequences is contained in Mostowski [3] and in an unpublished paper of Mazur (mentioned in the footnote [1])).

From (9) it follows that the condition of computably uniform continuity can be formulated without using the function φ. Namely let $\varphi\in\mathcal{K}'_{\mathrm{III}}$ if and only if there exist two functions $f, g'\in\mathfrak{Com}$ such that

(10) (a) φ is continuous,

(b) $\left|\dfrac{f(n,k)}{k+1}-\varphi(r_n)\right|<\dfrac{1}{k+1}$,

(c) if $r_n<r_l,\ r_t<r_m$ and $|r_l-r_t|<\dfrac{1}{g'(n,m,k)}$, then $|f(l,k)-f(t,k)|<3$ [3]).

Obviously (9) and (6) involve (10). Hence

THEOREM 4. $\mathcal{K}_{\mathrm{III}}\subset\mathcal{K}'_{\mathrm{III}}$.

Conversely we shall prove

THEOREM 5. $\mathcal{K}'_{\mathrm{III}}\subset\mathcal{K}'_{\mathrm{II}}$.

Proof. We shall prove the uniform computable continuity of φ. Suppose that $r_n<a<b<r_m$ and

(11) $$|a-b|<\frac{1}{g'(n,m,k)}.$$

From the continuity of φ it follows that there exist two rationals r_l and r_t such that

(12) $$a<r_l,\quad r_t<b$$

and

(13) $$|\varphi(a)-\varphi(r_l)|<\frac{1}{k+1},$$

(14) $$|\varphi(b)-\varphi(r_t)|<\frac{1}{k+1}.$$

Hence from (12), (13) and (10c) we find that

(15) $$|f(l,k)-f(t,k)|<3.$$

(13), (14), (10b) and (15) involve $|\varphi(a)-\varphi(b)|<7/(k+1)$. Thus setting $g(n,m,k)=g'(n,m,7k+6)$ we obtain (5ii).

[3]) Condition (10 b) of the computable convergence of the double sequence $f(n,k)/(k+1)$ to the sequence $\varphi(r_n)$ can be replaced by others. For example by the following two conditions:

(b′) $\varphi(r_n)=\lim_k f(n,k)/(k+1)$,

(b″) the sequence $\{a_k\}=\{f(n,k)/(k+1)\}$ is computably convergent.

The definition of $\mathcal{K}'_{\mathrm{III}}$ with (b) replaced by the conjunction of (b′) and (b″) is very similar to the definition of Specker. The main difference is that Specker considers in [4] only the primitive recursive functions.

THEOREM 6. $\mathcal{K}'_{\mathrm{II}}\subset\mathcal{K}_{\mathrm{I}}$.

Proof. Suppose that for any $k\in\mathcal{N}$

(16) $$\left|\frac{j(k)}{k+1}-a\right|<\frac{1}{k+1}.$$

Hence for all $k\in\mathcal{N}$

(17) $$j(0)-2<j(k)/(k+1)<j(0)+2.$$

Let g be the computable function satisfying (5ii). Hence the following functional ψ_1 is computable:

(18) $$\psi_1\langle j\rangle(k)=g\Big(\mathrm{No}\big((j(0)-2)/2\big),\mathrm{No}\big((j(0)+2)/1\big),k\Big).$$

From (5ii), (1), (17) and (18) it follows that for $b\in\mathcal{R}$ and $j(0)-2<b<j(0)+2$

(19) $$\text{if}\quad |a-b|<\frac{1}{\psi_1\langle j\rangle(k)},\quad\text{then}\quad|\varphi(a)-\varphi(b)|<\frac{1}{k+1}.$$

Setting in (16) $\psi_1\langle j\rangle(k)$ for k, and in (19)

$$\frac{j\big(\psi_1\langle j\rangle(k)\big)}{\psi_1\langle j\rangle(k)+1}$$

for b, we find from (16) and (19) that

(20) $$\left|\varphi(a)-\varphi\left(\frac{j\big(\psi_1\langle j\rangle(k)\big)}{\psi_1\langle j\rangle(k)+1}\right)\right|<\frac{1}{k+1}.$$

Now we shall use the fact that the sequence $\varphi(r_n)$ is computable in the sense of (6). Let us set

(21) $$\psi_2\langle j\rangle(k)=\mathrm{No}\Big(j\big(\psi_1\langle j\rangle(k)\big)/\big(\psi_1\langle j\rangle(k)+1\big)\Big),$$

(6) implies that

(22) $$\left|\varphi(r_{\psi_2\langle j\rangle(k)})-\frac{f(\psi_2\langle j\rangle(k),k)}{k+1}\right|<\frac{1}{k+1}.$$

From (1), (20), (21) and (22) we obtain

(23) $$\left|\varphi(a)-\frac{f\big(\psi_2\langle j\rangle(k),k\big)}{k+1}\right|<\frac{2}{k+1},$$

(23) involves that

(24) $$\frac{f\big(\psi_2\langle j\rangle(2k+1),2k+1\big)-1}{2(k+1)}<\varphi(a)<\frac{f\big(\psi_2\langle j\rangle(2k+1),2k+1\big)+1}{2(k+1)}.$$

If we set

(25) $\Phi\langle j\rangle(k)$

$$=(\mu x)\left[\frac{f(\psi_2\langle j\rangle(2k+1),2k+1)-1}{2(k+1)}\leqslant\frac{x}{k+1}\leqslant\frac{f(\psi_2\langle j\rangle(2k+1),2k+1)+1}{2(k+1)}\right],$$

the minimum operation in (25) is obviously effective, thus $\Phi\in\mathcal{K}$ and according to (24) and (25)

$$\left|\frac{\Phi\langle j\rangle(k)}{k+1}-\varphi(a)\right|<\frac{1}{k+1}$$

which completes the proof of (4).

Now we introduce a definition similar to that of D. Lacombe proposed in [2].

Let $\{s_n\}$ be a computable sequence of all open rational segments:

(26) $$s_n=\underset{c}{E}[r_{Kn}<c<r_{Ln}]$$

where $Kn=n-|\sqrt{n}|^2$, $Ln=|\sqrt{n}|-Kn$, $J(x,y)=(x+y)^2+x$ (the pairing functions).

Let us set $\varphi\in\mathcal{K}_{IV}$ if and only if there exists a function $f\in\mathfrak{Com}$ such that

(27) (α) if $a\in s_n$, then $\varphi(a)\in s_{f(n)}$,

(β) if $\varphi(a)\in s_m$, then there exists $n\in\mathcal{N}$ such that $a\in s_n$ and $s_{f(n)}\subset s_m$,

(γ) if $s_n\subset s_k$ and $n\geqslant k$, then $s_{f(n)}\subset s_{f(k)}$ [4]).

In order to prove the equivalence of $\mathcal{K}_I$ and $\mathcal{K}_{IV}$ we need the following

LEMMA 7. *There exist two functionals $\eta,\zeta\in\mathcal{K}$ and a function $\mathrm{pt}\in\mathfrak{Com}$, such that if $A(a,g)$ then*

(28) $$\frac{\eta\langle g\rangle(n)}{\mathrm{pt}(n)}<\frac{\eta\langle g\rangle(n+1)}{\mathrm{pt}(n+1)}<a<\frac{\zeta\langle g\rangle(n+1)}{\mathrm{pt}(n+1)}<\frac{\zeta\langle g\rangle(n)}{\mathrm{pt}(n)},$$

$$\left|\frac{\eta\langle g\rangle(n)}{\mathrm{pt}(n)}-\frac{\zeta\langle g\rangle(n)}{\mathrm{pt}(n)}\right|<\frac{1}{n+1}.$$

[4]) The definition of D. Lacombe in the original version contains the condition of monotonity:

(γ*) if $s_n\subset s_k$, then $s_{f(n)}\subset s_{f(k)}$

without the supposition that $n\geqslant k$. The proof of (27) with (γ*) is more difficult. We omit it since condition (γ) can be disregarded, as will be pointed out in Theorem 11.

Proof. First we shall define some auxiliary computable functionals, especially the functional γ such that for any function g the sequence $\gamma\langle g\rangle(n)/(n+1)$ is convergent and if $A(a,g)$ then $\gamma\langle g\rangle(n)=g(n)$. Let us set

(29) $$\Xi(g,n)\equiv\prod_{k\leqslant n}\prod_{i\leqslant k}\frac{g(i)-2}{i+1}<\frac{g(k)}{k+1}<\frac{g(i)+2}{i+1},$$

(30) $$\lambda\langle g\rangle(0)=g(0),\qquad \lambda\langle g\rangle(n+1)=\begin{cases}g(n+1) & \text{if}\quad \Xi(g,n+1),\\ \lambda\langle g\rangle(n) & \text{if}\quad \sim\Xi(g,n+1),\end{cases}$$

(31) $$\nu\langle g\rangle(0)=0,\qquad \nu\langle g\rangle(n+1)=\begin{cases}n+1 & \text{if}\quad \Xi(g,n+1),\\ \nu\langle g\rangle(n) & \text{if}\quad \sim\Xi(g,n+1)\ ^{5)},\end{cases}$$

(32) $$\gamma\langle g\rangle(n)=\begin{cases}g(n) & \text{if}\quad \Xi(g,n),\\ (\mu x)\left[\left|\frac{x}{n+1}-\frac{\lambda\langle g\rangle(n)}{\nu\langle g\rangle(n)+1}\right|<\frac{1}{n+1}\right] & \text{if}\quad \sim\Xi(g,n).\end{cases}$$

We shall prove that for all $i\in\mathcal{N}$

(33) $$\frac{\gamma\langle g\rangle(i)-2}{i+1}\leqslant\lim_n\frac{\gamma\langle g\rangle(n)}{n+1}\leqslant\frac{\gamma\langle g\rangle(i)+2}{i+1}.$$

Let us distinguish two cases: 1° For all n, $\Xi(g,n)$. Hence, according to (32), $\gamma\langle g\rangle(n)=g(n)$, and thus (33) follows from (29). 2° For some n, $\sim\Xi(g,n)$. Let $n_0=(\mu n)[\sim\Xi(g,n+1)]$. From (30), (31), (32), (1) and (2) it follows that $\lim_n\gamma\langle g\rangle(n)/(n+1)=g(n_0)/(n_0+1)$, and $\gamma\langle g\rangle(n)=g(n)$ for $n\leqslant n_0$ and $\gamma\langle g\rangle(n)=W(\mathrm{No}(g(n_0)/(n_0+1)),n)$ for $n>n_0$. Hence (33) is satisfied by $i\leqslant n_0$ according to (29) and by $i>n_0$ according to (1), (2) and (3).

From (33) it follows that

(34) $$\frac{\gamma\langle g\rangle(i)-3}{i+1}<\frac{\gamma\langle g\rangle(3(i+1))-3}{3(i+1)+1}<\lim_n\frac{\gamma\langle g\rangle(n)}{n+1}<\frac{\gamma\langle g\rangle(3(i+1))+3}{3(i+1)+1}<\frac{\gamma\langle g\rangle(i)+3}{i+1}.$$

Hence, putting $\mathrm{rt}(0)=0$, $\mathrm{rt}(n+1)=3(\mathrm{rt}(n)+1)$, we find from (34) that

(35) $$\frac{\gamma\langle g\rangle(\mathrm{rt}(n))-3}{\mathrm{rt}(n)+1}<\frac{\gamma\langle g\rangle(\mathrm{rt}(n+1))-3}{\mathrm{rt}(n+1)+1}<\lim_n\frac{\gamma\langle g\rangle(n)}{n+1}<\frac{\gamma\langle g\rangle(\mathrm{rt}(n+1))+3}{\mathrm{rt}(n+1)+1}<\frac{\gamma\langle g\rangle(\mathrm{rt}(n))+3}{\mathrm{rt}(n)+1}.$$

[5]) These three definitions are due to A. Ehrenfeucht.

Finally setting $\mathrm{st}(0)=(\mu k)[\mathrm{rt}(k)+1>6]$,

$$\mathrm{st}(n+1)=(\mu k)[k>\mathrm{st}(n) \text{ and } \big(\mathrm{rt}(k)+1\big)(n+1)>6]$$

and

$$\eta\langle g\rangle(n)=\gamma\langle g\rangle\big(\mathrm{rt}(\mathrm{st}(n))\big)-3\,,\qquad \zeta\langle g\rangle(n)=\gamma\langle g\rangle\big(\mathrm{rt}(\mathrm{st}(n))\big)+3\,,$$

$$\mathrm{pt}(n)=\mathrm{rt}\big(\mathrm{st}(n)\big)+1$$

we obtain (28) from (35).

THEOREM 8. $\mathcal{K}_{\mathrm{I}}\subset\mathcal{K}_{\mathrm{IV}}$.

Proof. If $\varphi\in\mathcal{K}_{\mathrm{I}}$, then according to Theorem 3 of [1] the function $\psi'(a,b)=\max\limits_{a\leqslant c\leqslant b}\varphi(c)$ belongs likewise to $\mathcal{K}_{\mathrm{I}}$. It is evident that similarly $\psi''\in\mathcal{K}_{\mathrm{I}}$ where $\psi''(a,b)=\min\limits_{a\leqslant c\leqslant b}\varphi(c)$. Hence there exist two functionals $\Psi',\Psi''\in\mathcal{K}$ such that according to (4) if $A(a,f')$ and $A(b,f'')$ then

$$(36)\qquad A\big(\psi'(a,b),\Psi'\langle f',f''\rangle\big)\quad\text{and}\quad A\big(\psi''(a,b),\Psi''\langle f',f''\rangle\big)\,.$$

We shall define the function f in such a manner that

$$(37)\qquad \mathop{E}_{c}[\psi''(r_{\mathrm{K}n},r_{\mathrm{L}n})<c<\psi'(r_{\mathrm{K}n},r_{\mathrm{L}n})]\subset s_{f(n)}$$

and that $s_{f(n)}$ decreases with s_n. Namely we set

$$f(n)=J\big(\mathrm{No}(\eta\langle\Psi''\langle W_{\mathrm{K}n},W_{\mathrm{L}n}\rangle\rangle(n)/\mathrm{pt}(n)),\ \mathrm{No}(\zeta\langle\Psi'\langle W_{\mathrm{K}n},W_{\mathrm{L}n}\rangle\rangle(n)/\mathrm{pt}(n))\big)\,.$$

Hence according to (26) and (1)

$$(38)\qquad s_{f(n)}=\mathop{E}_{c}\left[\frac{\eta\langle\Psi''\langle W_{\mathrm{K}n},W_{\mathrm{L}n}\rangle\rangle(n)}{\mathrm{pt}(n)}<c<\frac{\zeta\langle\Psi'\langle W_{\mathrm{K}n},W_{\mathrm{L}n}\rangle\rangle(n)}{\mathrm{pt}(n)}\right].$$

Putting in (36) $f'(k)=W_{\mathrm{K}n}(k)$ and $f''(k)=W_{\mathrm{L}n}(k)$ we obtain (37) from (3), (36), (38) and Lemma 7. Condition (27 α) follows from (26), (37) and the definitions of ψ' and ψ''. Condition (27 γ) follows from the definitions of the functions ψ' and ψ'' with the use of (26), (38), and Lemma 7. In order to prove (27 β) let us suppose that $\varphi(a)\in s_m$, $s_m=\mathop{E}_{c}[b'<c<b'']$ and

$$(39)\qquad |b'-\varphi(a)|>\frac{3}{k+1}\quad\text{and}\quad |b''-\varphi(a)|>\frac{3}{k+1}\,.$$

From the continuity of the function φ it follows that there exists a segment s_n such that

$$(40)\qquad a\in s_n\,,$$

$$(41)\qquad |\varphi(c)-\varphi(a)|<\frac{1}{k+1}\quad\text{for any}\quad c\in s_n\,.$$

We can suppose that $n>k$ because there are infinitely many segments s_n satisfying (40) and (41). From (41) and from the definitions of ψ' and ψ'' it follows that

$$(42)\qquad |\psi'(r_{\mathrm{K}n},r_{\mathrm{L}n})-\varphi(a)|<\frac{1}{k+1}\quad\text{and}\quad |\psi''(r_{\mathrm{K}n},r_{\mathrm{L}n})-\varphi(a)|<\frac{1}{k+1}\,.$$

From (38), (3), (42) and Lemma 8 we find that for any $c\in s_{f(n)}$:

$$|c-\varphi(a)|<\frac{2}{n+1}+\frac{1}{k+1}\,.$$

Hence $|c-\varphi(a)|<3/(k+1)$ because $n>k$; thus according to (39) $s_{f(n)}\subset s_m$.

Let $\mathcal{K}'_{\mathrm{IV}}$ be the class of functions satisfying the following conditions:

(43) (α) if $a\in s_n$, then $\varphi(a)\in s_{f(n)}$,

(β) if $b\neq\varphi(a)$, then there exists a number $n\in N$ such that $a\in s_n$ and $b\notin s_{f(n)}$,

where $f\in\mathfrak{Com}$. Evidently (27 β) involves (43 β). Thus we have

THEOREM 9. $\mathcal{K}_{\mathrm{IV}}\subset\mathcal{K}'_{\mathrm{IV}}$.

Now we shall prove

THEOREM 10. $\mathcal{K}'_{\mathrm{IV}}\subset\mathcal{K}_{\mathrm{I}}$.

Proof. Let us consider the computable functional Θ defined as follows:

$$(44)\ (a)\qquad \Theta\langle g\rangle(0)=(\mu k)[g(0)-3\in s_k \text{ and } g(0)+3\in s_k]\,,$$

$$(b)\qquad \Theta\langle g\rangle(m+1)=(\mu k)\left[\frac{g(m)-3}{m+1}\in s_k \text{ and } \frac{g(m)+3}{m+1}\in s_k \text{ and } s_k\neq s_{\Theta\langle g\rangle(i)} \text{ for } i\leqslant m\right].$$

From this definition it follows that if

$$(45)\qquad \frac{g(i)-2}{i+1}\leqslant a\leqslant\frac{g(i)+2}{i+1}\quad\text{for all}\quad i\in N,$$

then

(46) for each $n\in N$ if $a\in s_n$ there exists such $m\in N$ that $s_n=s_{\Theta\langle g\rangle(m)}$.

Indeed let us consider all segments s_k for $k\leqslant n$, such that $a\in s_k$. Let s be the intersection of all those segments. Hence s is an open segment and $a\in s$. Thus according to (45) there exists a number m_0 such that

$$(47)\qquad \frac{g(m)-3}{m+1}\in s\quad\text{and}\quad\frac{g(m)+3}{m+1}\in s\quad\text{for all}\quad m\geqslant m_0.$$

Now if we suppose that

$$s_n \neq s_{\Theta\langle g\rangle(m)} \quad \text{for all} \quad m \in N, \tag{48}$$

we shall obtain a contradiction. Namely (48) and (47) imply that the segment s_n satisfies condition (44b) under the operation of minimum for all $m \geqslant m_0$. Hence according to (44b) it follows that $n = \Theta\langle g\rangle(m_0+1)$ or there exists a number $k < n$ such that s_k satisfies condition (44b) under the operation of minimum and $k = \Theta\langle g\rangle(m_0+1)$. The first possibility is exclused by our assumption (48); thus, for some $k<n$, $k = \Theta\langle g\rangle(m_0+1)$. Similarly we conclude that there exists $k' < n$ such that $k' = \Theta\langle g\rangle(m_0+2)$. But from (44) it follows that for $m' \neq m''$ $\Theta\langle g\rangle(m') \neq \Theta\langle g\rangle(m'')$. Thus $k \neq k'$. Repeating this reasoning $n+1$ times we find that there exist $n+1$ numbers less than the number n. Hence condition (46) is proved.

From condition (43β) it follows that for each number $a \in \mathcal{R}$ and $k \in \mathcal{N}$ there exist $n', n'' \in \mathcal{N}$ such that

$$a \in s_{n'} \cap s_{n''} \quad \text{and} \quad \left(\varphi(a) - \frac{1}{2(k+1)}\right) \notin s_{f(n')} \quad \text{and} \quad \left(\varphi(a) + \frac{1}{2(k+1)}\right) \notin s_{f(n'')}. \tag{49}$$

Putting $a = \lim_n \gamma\langle g\rangle(n)/(n+1)$ we find from (33), (45), (46) and (49) that for each $g \in \mathfrak{F}$, and $k \in \mathcal{N}$ there exist m' and m'' such that

$$|s_{f(\Theta\langle\gamma\langle g\rangle\rangle(m'))} \cap s_{f(\Theta\langle\gamma\langle g\rangle\rangle(m''))}| < \frac{1}{k+1}$$

where $|s_k|$ = the length of the segment s_k. Hence the following functional is computable:

$$\chi_1\langle g\rangle(k) = (\mu m)\left[|s_{f(\Theta\langle\gamma\langle g\rangle\rangle(\mathrm{K}m))} \cap s_{f(\Theta\langle\gamma\langle g\rangle\rangle(\mathrm{L}m))}| < \frac{1}{k+1}\right].$$

Setting

$$\chi_2\langle g\rangle(k) = (\mu n)[s_n = s_{f(\Theta\langle\gamma\langle g\rangle\rangle(\mathrm{K}\chi_1\langle g\rangle(k)))} \cap s_{f(\Theta\langle\gamma\langle g\rangle\rangle(\mathrm{L}\chi_1\langle g\rangle(k)))}]$$

we find that $\chi_2 \in \mathcal{K}$. According to (33) and (44) $\lim \gamma\langle g\rangle(n)/(n+1) \in s_{\Theta\langle\gamma\langle g\rangle\rangle(m)}$ for any m. Thus for each $g \in \mathfrak{F}$, and $k \in \mathcal{N}$, according to (43α) and the definitions of χ_1 and χ_2 we have

$$\varphi(\lim \gamma\langle g\rangle(n)/(n+1)) \in s_{\chi_2\langle g\rangle(k)} \quad \text{and} \quad |s_{\chi_2\langle g\rangle(k)}| < \frac{1}{k+1}. \tag{50}$$

If $A(a,g)$, then $a = \lim \gamma\langle g\rangle(n)/(n+1)$. Hence (50) implies that

$$|\varphi(a) - r_{\mathrm{K}\chi_2\langle g\rangle(k)}| < \frac{1}{k+1}$$

and we obtain (4) in the same manner as in the end of the proof of Theorem 6.

It is possible to introduce many other notions of computable real functions. *E. g.*, starting from the decimal developments we can assume that a function φ is computable if and only if there exist a computable functional which produces the decimal development of value $\varphi(a)$ from the decimal development of a. From the results of Mostowski [3] concerning computable sequences it follows that this definition and a similar definition using the cuts of Dedekind lead to non-equivalent notions. But the notion of the computable continuous function considered in this paper seems to be the most natural one. The definition of computable convergence of the form

$$\left|a - \frac{f(n)}{n+1}\right| < \frac{1}{n+1} \quad \text{for any} \quad n \in \mathcal{N},$$

where $f \in \mathfrak{Com}$, is equivalent to the most general definition of computable convergence of a computable sequence of rationals:

$$|a - r_{h(n)}| < 1/(k+1) \quad \text{for} \quad n > g(k), \quad \text{for any} \quad k \in N,$$

where $f, g, h \in \mathfrak{Com}$ [6]).

References

[1] A. Grzegorczyk, *Computable functionals*, Fund. Math. 42 (1955), p. 168-202.

[2] D. Lacombe, *Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles*, C. R. Acad. Sci. Paris, *I*, t. 240 (1955), p. 2478-2480, *II*, t. 241 (1955), p. 13-14, *III*, t. 241 (1955), p. 151-153.

[3] A. Mostowski, *On computable sequences*, this volume, p. 37-51.

[4] E. Specker, *Nicht konstruktiv beweisbare Sätze der Analysis*, J. Symbolic Log. 14 (1949), p. 145-158.

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[6]) In the previous paper [1] I used the shorter name: computable real function to denote the computable real continuous functions. The present name is more convenient, because it is possible to introduce some kinds of computable real functions which are not continuous, *e. g.* Riemann's computably integrable functions and Lebesgue's computably measurable functions.