On the extending of models (IV) *

Infinite sums of models

par

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In this paper we consider the problem how to characterize those elementarily definable classes $\mathfrak{A}$ of models which have the following property:

for every increasing sequence of models $\{\mathfrak{M}_n\}_{n=1,2,...}$

$$\text{if} \quad \mathfrak{M}_n \in \mathfrak{A} \quad \text{for} \quad n=1,2,..., \quad \text{then} \quad \sum_{n=1}^{\infty} \mathfrak{M}_n \in \mathfrak{A}.$$

We call such classes *σ-classes* of models. The class of all groups is, for example, a σ-class of models. The sum of an arbitrary increasing sequence of groups is a group.

The theorems concerning the problem of σ-classes are given in § 3. But the central point of our paper consists in lemma 4 given in § 2.

1. Terms and notation

In what follows we are concerned with elementary theories (with identity) E_1, E_2, E_3, E_4, which are nearly analogous to those investigated previously [1]). For reasons of simplicity, in the theory E_1 we have, as extralogical constants, one sign of relation r and one sign of function f only. In the theory E_2, apart from signs of the theory E_1, there are some new signs q_i $(i=1,2,...)$. The signs q_i may be individual constants or signs of function. Therefore the models in the theories E_1 and E_2 are of the forms

$$(\Delta) \qquad \langle A, R, F\rangle \quad \text{and} \quad \langle A, R, F, Q_1, Q_2, ...\rangle$$

respectively. If one introduces into E_1 and E_2 a family of individual constants $\{g_a\}$ where the index a runs over the set A, then one obtains

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[1]) See [3], chapters 1 and 2. We assume here notions and notation used in that paper with a few exceptions mentioned explicitely in the text above.

the theories E_3 and E_4. It is clear that in the theories E_3 and E_4 we may formulate, by considering the constants g_a as names of elements $a \in A$, the descriptions (see [3], § 4) of models (Δ) for E_1 and E_2 respectively.

If $\mathfrak{M}_k$ is a model for E_k $(k=1,2)$, then $E_k(\mathfrak{M}_k)$ is the set of all sentences $\alpha \in E_k$ fulfilled in $\mathfrak{M}_k$; analogously $E_{k+2}(\mathfrak{M}_k)$ is the set of all sentences $\alpha \in E_{k+2}$ fulfilled in $\mathfrak{M}_k$ $(k=1,2)$.

The sequence of models $\{\mathfrak{M}_n\}_{n=1,2,...}$ is called *increasing* if for every n, $\mathfrak{M}_n$ is a submodel of $\mathfrak{M}_{n+1}$. If $\{\mathfrak{M}_n\}_{n=1,2,...}$ is an increasing sequence of models, then the sum $\sum_{n=1}^{\infty}\mathfrak{M}_n$ is the least model of which every model $\mathfrak{M}_n$ is a submodel. It is clear that for every increasing sequence of models $\{\mathfrak{M}_n\}_{n=1,2,...}$ the sum $\sum_{n=1}^{\infty}\mathfrak{M}_n$ exists, for the operations in $\mathfrak{M}_n$ are finitary and the sequence is infinite.

We limit our considerations to the field of sentences, *i. e.* well-formed formulas without free variables. A sentence $\alpha \in E_k$ is called $\prod$-sentence, $\prod\sum$-sentence or $\sum\prod$-sentence if there is a sentence $\beta \in E_k$ of the form

$$\prod_{x_1}\cdots\prod_{x_n}\gamma(x_1,...,x_n),$$

$$\prod_{x_1}\cdots\prod_{x_n}\sum_{y_1}\cdots\sum_{y_m}\gamma(x_1,...,x_n,y_1,...,y_m),$$

$$\sum_{x_1}\cdots\sum_{x_n}\prod_{y_1}\cdots\prod_{y_m}\gamma(x_1,...,x_n,y_1,...,y_m)$$

respectively, containing only the indicated quantifiers and no free variables and such that the equivalence $\alpha\equiv\beta$ is a tautology, *i. e.* $(\alpha\equiv\beta)\in Cn(0)$.

Let $\{\mathfrak{M}_n\}_{n=1,2,...}$ be an increasing sequence of models in the theory E_1. We call a sentence $\alpha \in E_1$ *persistent* in this sequence if the sentence α fulfils the condition:

$$\text{if for every } n \quad \alpha \in E_1(\mathfrak{M}_n), \quad \text{then} \quad \alpha \in E_1\Big(\sum_{n=1}^{\infty}\mathfrak{M}_n\Big).$$

It is quite clear that every $\prod$-sentence is persistent in every increasing sequence of models. The same holds for $\prod\sum$-sentences also. On the other hand there are $\sum\prod$-sentences which are not persistent in some increasing sequences of models [2]). Making use of the theorem on extending of models with secondary conditions [3] we can show that for

[2]) The axiomatics for ordered sets with a least element is a $\sum\prod$-sentence which is not persistent in the sequence of models $\mathfrak{M}_n=\{1,2,...,n\}$.

every sentence α which is not both $\prod\sum$-sentence and $\sum\prod$-sentence there is an increasing sequence of models in which either α or α' is not persistent [3]).

We mention here the following result obtained by Ryll-Nardzewski: if $\{\mathfrak{M}_n\}_{n=1,2,\ldots}$ is an increasing sequence of models and $\alpha\in E_1(\mathfrak{M}_n)$ for every $n=1,2,\ldots$, then there is a model $\mathfrak{M}$ such that $\sum_{n=1}^{\infty}\mathfrak{M}_n\subset\mathfrak{M}$ and $\alpha\in E_1(\mathfrak{M})$ [4]).

2. Lemmas

LEMMA 1. *Let D be a non-empty additive and multiplicative subset of some Boolean algebra B and let $\{b_k\}_{k=1,2,\ldots}$ be a sequence of elements of B such that $b_k\leqslant b_{k+1}$ for every $k=1,2,\ldots$ If the sequence $\{b_k\}_{k=1,2,\ldots}$ satisfies the condition*

$$(*)\qquad \text{if}\quad b_k\leqslant d\leqslant b_{k+m}\quad\text{then}\quad d\notin D,$$

then there exist in B two prime ideals, J_1 and J_2, such that

1) *$b_k\in J_1$ for every $k=1,2,\ldots$,*
2) *$b_1'\in J_2$ (consequently $b_k'\in J_2$),*
3) *$J_1\cap D\subset J_2$.*

Proof. Let $J(A)$ be the least ideal containing the set

$$A=\underset{x\in B}{E}(b_1\leqslant x'\in D).$$

We are going to prove that $b_k'\notin J(A)$. Let us suppose, on the contrary, that $b_k'\in J(A)$. This means that $b_k'\leqslant a_1+\ldots+a_n$ where $a_i\in A$, and consequently $b_1\leqslant a_i'\in D$. Therefore $b_1\leqslant a_1'\cdot\ldots\cdot a_n'\leqslant b_k$, which contradicts $(*)$. Let J_1 be a prime ideal such that $J(A)\subset J_1$ and $b_k\in J_1$ for every $k=1,2,\ldots$ and let $J(J_1\cap D)$ be the least ideal containing the set $J_1\cap D$. If we suppose that $b_1\in J(J_1\cap D)$, then we have $b_1\leqslant d=d_1+\ldots+d_n$ for some $d_1,\ldots,d_n\in J_1\cap D$. It follows that $d\in J_1\cap D$, and consequently $d\in J_1$, $d'\notin J_1$ and $d'\notin A$. On the other hand we have $d'\in A$, which follows from

[3]) We construct two consistent systems X_1 and X_2 such that $\alpha\in X_1$, $\alpha'\in X_2$ and $O\cap X_1=O\cap X_2$, where O is the set of all $\prod$-sentences. Then one can extend every model of X_1 to the model of X_2 and conversely. It follows that there is an increasing sequence of models $\{\mathfrak{M}_n\}_{n=1,2,\ldots}$ such that $\alpha\in X_1\subset E_1(\mathfrak{M}_{2n-1})$, $\alpha'\in X_2\subset E_1(\mathfrak{M}_{2n})$, and $\sum_{n=1}^{\infty}\mathfrak{M}_n=\sum_{n=1}^{\infty}\mathfrak{M}_{2n-1}=\sum_{n=1}^{\infty}\mathfrak{M}_{2n}$.

[4]) Obviously, every $\prod$-sentence belonging to the system $Cn(\alpha)$ is fulfilled in the model $\sum_{n=1}^{\infty}\mathfrak{M}_n$. Therefore we can make the required extension.

the previously stated relation $b_1\leqslant d\in D$. Therefore $b_1\notin J(J_1\cap D)$. If we extend the ideal $J(J_1\cap D)$ to a prime ideal J_2 such that $b_1\notin J_2$, then we see that the ideals J_1 and J_2 satisfy our lemma.

LEMMA 2. *Let $Z=Cn(Z)$ be a consistent system and let $\{\beta_k\}_{k=1,2,\ldots}$ be a sequence of sentences in E_1 such that $(\beta_{k+1}\to\beta_k)\in Cn(Z)$ for every $k=1,2,\ldots$ If this sequence satisfies the condition*

$$(**)\qquad \textit{if}\ (\beta_{k+m}\to\delta)\wedge(\delta\to\beta_k)\in Cn(Z)\ \textit{then}\ \delta\ \textit{is not a}\ \prod\sum\textit{-sentence},$$

then there exist in E_1 two consistent complete systems X, Y containing the system Z and such that

1) *$\beta_k\in X$ for every $k=1,2,\ldots$,*
2) *$\beta_1'\in Y$ (consequently $\beta_k'\in Y$),*
3) *every $\prod\sum$-sentence belonging to X belongs to Y also.*

Proof. Our lemma follows immediately from lemma 1 in view of the fact that complete consistent systems containing a system Z are prime ideals in the field of sentences modulo Z and that the set of all $\prod\sum$-sentences is additive and multiplicative.

LEMMA 3. *If $\mathfrak{M}=\langle A,R,F\rangle$ is a model in E_1 and $X=Cn(X)$, $Y=Cn(Y)$ are two consistent systems in E_1 such that $Y\subset E_1(\mathfrak{M})$ and every $\prod\sum$-sentence belonging to X belongs to Y also, then the set $X\cup Z$ is consistent, where Z is the set of all $\prod$-sentences in E_3 which are fulfilled in $\mathfrak{M}$.*

Proof. The sets X,Z are, of course, multiplicative. Therefore if the set $X\cup Z$ is inconsistent, then there is a sentence $\alpha\in X$ and a $\prod$-sentence

$$\prod_{x_1}\ldots\prod_{x_n}\eta(g_{a_1},\ldots,g_{a_m},x_1,\ldots,x_n)$$

belonging to $Z\subset E_3(\mathfrak{M})$ where $a_1,\ldots,a_m\in A$ and such that

$$\Big(\alpha\wedge\prod_{x_1}\ldots\prod_{x_n}\eta(g_{a_1},\ldots,g_{a_m},x_1,\ldots,x_n)\Big)'$$

is a tautology. Consequently the sentence

$$\alpha\to\sum_{x_1}\ldots\sum_{x_n}\eta'(g_{a_1},\ldots,g_{a_m},x_1,\ldots,x_n)$$

is a tautology. In view of the fact that $\alpha\in X$ and the constants g_{a_k} $(k=1,2,\ldots,m)$ do not occur in α, it follows that the sentence

$$\prod_{y_1}\ldots\prod_{y_m}\sum_{x_1}\ldots\sum_{x_n}\eta'(y_1,\ldots,y_m,x_1,\ldots,x_n)$$

belongs to X and consequently to Y. But this is impossible, because the contradictory sentence

$$\sum_{y_1}\cdots\sum_{y_m}\prod_{x_1}\cdots\prod_{x_n}\eta(y_1,\ldots,y_m,x_1,\ldots,x_n)$$

is fulfilled in $\mathfrak{M}$.

LEMMA 4. *Let $Z=Cn(Z)$ be a consistent system in E_1 and let $\{\beta_k\}_{k=1,2,\ldots}$ be a sequence of sentences in E_1 such that $(\beta_{k+1}\to\beta_k)\in Cn(Z)$ for every $k=1,2,\ldots$ If this sequence satisfies the condition $(**)$ of lemma 2, then there exists an increasing sequence of models $\{\mathfrak{M}_n\}_{n=1,2,\ldots}$ such that:*

1) $Z\subset E_1(\mathfrak{M}_n)$ i. e. $\mathfrak{M}_n\in\mathfrak{A}(Z)$ for all n,

2) $\beta_k\in E_1(\mathfrak{M}_n)$ for all k,n

and

3) $\beta_1\notin E_1\big(\sum_{n=1}^{\infty}\mathfrak{M}_n\big)$.

Proof. We take a sequence $\{\beta_k\}_{k=1,2,\ldots}$ of sentences in E_1. Let $\gamma_k\in E_1$ be a sentence of normal form in E_1 such that the equivalence $\beta_k'\equiv\gamma_k$ is a tautology, and let $\gamma_k^*\in E_2$ be the generalization of the open solution [5]) of γ_k. If, for example, β_k is of the form

$$\prod_x\sum_y\prod_z a(x,y,z)$$

where $a(x,y,z)$ is an open formula, then the sentences $\gamma_k\in E_1$ and $\gamma_k^*\in E_2$ are of the form

$$\sum_x\prod_y\sum_z a'(x,y,z)\quad\text{and}\quad\prod_y a'(q_1,y,q_2(y))$$

respectively.

Let $Z=Cn(Z)$ be a consistent system in E_1 and let the sequence $\{\beta_k\}_{k=1,2,\ldots}$ satisfy the conditions:

$$(\beta_{k+1}\to\beta_k)\in Cn(Z)\quad\text{for all } k,$$

$$\text{if } (\beta_{k+m}\to\delta)\wedge(\delta\to\beta_k)\in Cn(Z) \text{ then } \delta \text{ is not a } \prod\sum\text{-sentence}.$$

According to lemma 2 there are two consistent systems X_1,X_2 in E_1 (where X_1 is complete) such that $Z\subset X_1\cap X_2$, $\beta_k\in X_1$ for all $k=1,2,\ldots$, $\beta_1'\in X_2$ and

$(**)$ every $\prod\sum$-sentence belonging to X_1 belongs to X_2.

[5]) For the notion of the open solution see [2], chapter 27. The open solution = „aufgelöste Form" in [1].

The system $Y=Cn(X_2\cup\{\gamma_1^*\})$ is consistent in E_2 and contains the system Z. It follows that there is a model

$$\mathfrak{M}_1=\langle A_1,R_1,F_1,Q_{11},Q_{12},\ldots\rangle$$

in the class $\mathfrak{A}(Z)$ such that $Y\subset E_2(\mathfrak{M}_1)$ and consequently $X_2\subset E_1(\mathfrak{M}_1)$.

We shorten the model $\mathfrak{M}_1$, *i. e.* we consider the model

$$\mathfrak{M}_2=\langle A_2,R_2,F_2\rangle$$

such that $A_2=A_1$, $R_2=R_1$, $F_2=F_1$. Evidently $X_2\subset E_1(\mathfrak{M}_2)$.

We join to the theory E_1 a family of individual constants $\{g_a\}_{a\in A_1=A_2}$. We obtain in this way the theory E_3. Let us consider now the set Z_0 of all $\prod$-sentences in E_3 which are fulfilled in the model $\mathfrak{M}_2$. By lemma 3 we see that the set $X_1\cup Z_0$ is consistent. We prove that

$(^{**}_{**})$ every $\prod$-sentence in E_1 belonging to the set $Cn(X_1\cup Z_0)$ belongs to X_2.

Evidently every $\prod$-sentence is a $\prod\sum$-sentence and therefore, by $(^{**}_{*})$ every $\prod$-sentence belonging to X_1 belongs to X_2. Suppose now that there is a $\prod$-sentence a in E_1 such that $a\in Cn(X_1\cup Z_0)$ and $a\notin X_2$. Consequently $a\notin X_1$. On the other hand it follows by the multiplicativity of the set Z_0 that there is in $Z_0\subset E_3(\mathfrak{M}_2)$ a $\prod$-sentence ξ of the form

$$\prod_{x_1}\cdots\prod_{x_n}\eta(g_{a_1},\ldots,g_{a_m},x_1,\ldots,x_n)$$

for example, such that $(\xi\to a)\in Cn(X_1)$. The constants $g_{a_1},\ldots,g_{a_m}$ do not occur either in a or in the sentences belonging to X_1. Thus we have $(\xi^*\to a)\in Cn(X_1)$ where ξ^* is the sentence in E_1 of the form

$$\sum_{y_1}\cdots\sum_{y_m}\prod_{x_1}\cdots\prod_{x_n}\eta(y_1,\ldots,y_m,x_1,\ldots,x_n).$$

But $\xi^*\in Cn(Z_0)$. So we infer from the completeness of the system X_1 and from the consistency of the set $X_1\cup Z_0$ that $\xi^*\in X_1$. Consequently $a\in X_1$, which is impossible.

Let us return now to the model $\mathfrak{M}_2$. We apply to it the theorem on the existence of extensions of models with secondary conditions (Theorem 3.1 in [3]). Then, making use of $(^{**}_{**})$, we infer that there exists a model

$$\mathfrak{M}_3=\langle A_3,R_3,F_3\rangle$$

in E_3 such that $X_1\cup Z_0\subset E_3(\mathfrak{M}_3)$ and $\mathfrak{M}_2$ is a submodel of it. Evidently $X_1\subset E_1(\mathfrak{M}_3)$.

Finally we prove that there is a model

$$\mathfrak{M}_4 = \langle A_4, R_4, F_4, Q_{41}, Q_{42}, \dots \rangle$$

which is an extension of the two models, $\mathfrak{M}_1$ and $\mathfrak{M}_3$, and such that $Y \subset E_2(\mathfrak{M}_4)$, and consequently $X_2 \subset E_1(\mathfrak{M}_4)$. Let us assume, on the contrary, that such a model does not exist. We then construct from E_2 the theory E_4 by joining to it a family of individual constants $\{g_a\}_{a \in A_3}$. In E_4 we can formulate the descriptions of the models $\mathfrak{M}_1$ and $\mathfrak{M}_3$, denoted by $D(\mathfrak{M}_1)$ and $D(\mathfrak{M}_3)$. From the assumption it follows that the set

$$Y \cup D(\mathfrak{M}_1) \cup D(\mathfrak{M}_3),$$

i. e. the set

$$Y \cup \big(D(\mathfrak{M}_3) - D(\mathfrak{M}_1)\big) \cup D(\mathfrak{M}_1)$$

is inconsistent. Therefore there exist some sentences $\alpha_1, \dots, \alpha_k$ belonging to $D(\mathfrak{M}_3) - D(\mathfrak{M}_1)$ such that $\alpha' \in Cn\big(Y \cup D(\mathfrak{M}_1)\big)$ where α stands for the conjunction $\alpha_1 \wedge \dots \wedge \alpha_k$. Let us write the sentence α' in the form

$$\eta(g_{a_1}, \dots, g_{a_m}, g_{a_{m+1}}, \dots, g_{a_{m+n}})$$

where $a_1, \dots, a_m \in A_1 = A_2$ and $a_{m+1}, \dots, a_{m+n} \in A_3 - A_1$ indicate all constants g occurring in α'. From the fact that constants g_a such that $a \in A_3 - A_1$ do not occur in the sentences of the set $Y \cup D(\mathfrak{M}_1)$ we conclude that the sentence

$$\prod_{x_1} \cdots \prod_{x_n} \eta(g_{a_1}, \dots, g_{a_m}, x_1, \dots, x_n)$$

belongs to the set $Cn\big(Y \cup D(\mathfrak{M}_1)\big)$, and consequently to the set $E_3(\mathfrak{M}_1)$ also. In view of the fact that $E_3(\mathfrak{M}_1) \subset E_3(\mathfrak{M}_2)$ it follows that this sentence is satisfied in $\mathfrak{M}_2$. On the other hand this sentence is not satisfied in $\mathfrak{M}_3$ because $\alpha_1, \dots, \alpha_k \in D(\mathfrak{M}_3)$ and consequently $\alpha' \notin E_4(\mathfrak{M}_3)$. Thus we arrive at the false conclusion that there is a $\prod$-sentence in $E_3(\mathfrak{M}_2)$, (*i. e.* a sentence belonging to the set Z_0) which is not satisfied in $\mathfrak{M}_3$.

Starting from the model $\mathfrak{M}_4$ we obtain the models $\mathfrak{M}_5, \mathfrak{M}_6, \mathfrak{M}_7$ in the same manner as we have obtained the models $\mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_4$ from the model $\mathfrak{M}_1$. If we repeat this reasoning, we shall arrive at an increasing sequence of models $\{\mathfrak{M}_n\}_{n=1,2,\dots}$ such that

$$\gamma_1^* \in Y \subset E_2(\mathfrak{M}_{3n-2}) \quad \text{and} \quad \beta_1' \in X_2 \subset E_1(\mathfrak{M}_{3n-2}),$$
$$\beta_1' \in X_2 \subset E_1(\mathfrak{M}_{3n-1}) \quad \text{and} \quad \beta_k \in X_1 \subset E_1(\mathfrak{M}_{3n}) \quad \text{for every} \quad k = 1, 2, \dots$$

In view of the fact that $Z \subset X_1 \cap X_2$ it follows that all models $\mathfrak{M}_{3n}$ belong to the class $\mathfrak{A}(Z)$. It remains to prove that $\beta_1 \notin E_1\big(\sum_{n=1}^{\infty} \mathfrak{M}_{3n}\big)$. If $\sum_{n=1}^{\infty} \mathfrak{M}_{3n} = \langle A, R, F \rangle$ then $\sum_{n=1}^{\infty} \mathfrak{M}_{3n-2} = \langle A, R, F, Q_1, Q_2, \dots \rangle$ and therefore $E_1\big(\sum_{n=1}^{\infty} \mathfrak{M}_{3n-2}\big) = E_1\big(\sum_{n=1}^{\infty} \mathfrak{M}_{3n}\big)$. The sentence γ_1^* is a $\prod$-sentence and consequently $\gamma_1^* \in E_2\big(\sum_{n=1}^{\infty} \mathfrak{M}_{3n-2}\big)$. It follows that $\beta_1' \in E_1\big(\sum_{n=1}^{\infty} \mathfrak{M}_{3n-2}\big)$, $\beta_1 \notin E_1\big(\sum_{n=1}^{\infty} \mathfrak{M}_{3n-2}\big)$ and finally $\beta_1 \notin E_1\big(\sum_{n=1}^{\infty} \mathfrak{M}_{3n}\big)$, q. e. d.

3. Theorems

THEOREM 1. *All $\prod\sum$-sentences, and only those sentences are persistent in every increasing sequence of models.*

Proof. It is obvious that all $\prod\sum$-sentences are persistent in all increasing sequences of models. It remains to prove that all persistent sentences are $\prod\sum$-sentences. To prove this we take a sentence β_0 which is not a $\prod\sum$-sentence. We apply the lemma 4 putting $Z = Cn(0) =$ set of tautologies and $\beta_k = \beta_0$ for all k. It follows that there exists an increasing sequence of models $\{\mathfrak{M}_n\}_{n=1,2,\dots}$ such that $\beta_0 \in E_1(\mathfrak{M}_n)$ for all n and $\beta_0 \notin E_1\big(\sum_{n=1}^{\infty} \mathfrak{M}_n\big)$. Therefore β_0 is not persistent.

THEOREM 2. *Let $\mathfrak{A}(X)$ be an elementarily definable class of models. The class $\mathfrak{A}(X)$ is a σ-class if and only if there exists a set Y of $\prod\sum$-sentences such that $\mathfrak{A}(X) = \mathfrak{A}(Y)$ or, which is equivalent, $Cn(X) = Cn(Y)$.*

Proof. Suppose that there exists a set Y of $\prod\sum$-sentences such that $\mathfrak{A}(X) = \mathfrak{A}(Y)$ and take an increasing sequence of models $\{\mathfrak{M}_n\}_{n=1,2,\dots}$ such that $\mathfrak{M}_n \in \mathfrak{A}(X)$ for all n. It follows that $Y \subset E_1(\mathfrak{M}_n)$ for all n and consequently $Y \subset E_1\big(\sum_{n=1}^{\infty} \mathfrak{M}_n\big)$. In other words $\sum_{n=1}^{\infty} \mathfrak{M}_n \in \mathfrak{A}(Y) = \mathfrak{A}(X)$. Therefore $\mathfrak{A}(X)$ is a σ-class.

To prove the necessity of our condition we consider the set $Y_0 = \underset{\alpha \in Cn(X)}{E}$ (α is a $\prod\sum$-sentence). Clearly $Cn(Y_0) \subset Cn(X)$. We prove that $Cn(X) \subset Cn(Y_0)$. Suppose on the contrary that $Cn(X) - Cn(Y_0) = X_0 \neq 0$. Let $\{\alpha_n\}_{n=1,2,\dots}$ be the sequence of all elements of X_0 and β_k the conjunction $\alpha_1 \wedge \dots \wedge \alpha_k$. The system $Cn(Y_0)$ and the sequence $\{\beta_k\}_{k=1,2,\dots}$ satisfy the assumptions of lemma 4. Namely, it is obvious that $(\beta_{k+1} \to \beta_k) \in Cn(Y_0)$. On the other hand, if δ is a $\prod\sum$-sentence and $(\beta_{k+m} \to \delta) \wedge (\delta \to \beta_k) \in Cn(Y_0)$, then $\delta \in Cn(Y_0 \cup \{\beta_{k+m}\}) \subset Cn(X)$. Consequently $\delta \in Y_0$ and $\beta_k \in Cn(Y_0)$. But this is a contradiction because $\beta_k \in Cn(X) - Cn(Y_0)$. Therefore by lemma 4, there is an increasing sequence of models $\{\mathfrak{M}_n\}_{n=1,2,\dots}$

such that $Cn(Y_0) \subset E_1(\mathfrak{M}_n)$ for all n and $\beta_k \in E_1(\mathfrak{M}_n)$ for all k, n but $\beta_1 \notin E_1(\sum_{n=1}^{\infty} \mathfrak{M}_n)$. It follows that $X \subset E_1(\mathfrak{M}_n)$ for every n and $X \not\subset E_1(\sum_{n=1}^{\infty} \mathfrak{M}_n)$. In other words, all models $\mathfrak{M}_n$ belong to the class $\mathfrak{A}(X)$ but $\sum_{n=1}^{\infty} \mathfrak{M}_n \notin \mathfrak{A}(X)$. The class $\mathfrak{A}(X)$ is not a σ-class, q. e. d.

THEOREM 3. *Let $\mathfrak{A}_0 = \mathfrak{A}(Z)$ be a σ-class of models definable by the set of axioms $Z \subset E_1$ and let β be a sentence in E_1. The class $\mathfrak{A}_1 = \mathfrak{A}(Z + \{\beta\})$ is a σ-class if and only if the sentence β is a $\prod\sum$-sentence over the axiomatics Z, i. e. if there is a $\prod\sum$-sentence γ such that $\beta \equiv \gamma \in Cn(Z)$.*

Proof. According to theorem 2 we can assume that every sentence belonging to the set Z is a $\prod\sum$-sentence. If $\mathfrak{A}(Z + \{\beta\})$ is a σ-class, then it follows, from theorem 2 again, that there is a set Y of $\prod\sum$-sentences such that $Cn(Z + \{\beta\} = Cn(Y)$. Therefore $\beta \in Cn(Y)$. Consequently $\beta \in Cn(\gamma) \subset Cn(Z + \{\gamma\})$ where $\gamma = \gamma_1 \wedge \ldots \wedge \gamma_n$ for some $\gamma_i \in Y$. It is clear that γ is a $\prod\sum$-sentence. On the other hand, $\gamma \in Cn(Y) \subset Cn(Z + \{\beta\})$. Therefore $\beta \equiv \gamma \in Cn(Z)$.

Suppose now that γ is a $\prod\sum$-sentence and $\beta \equiv \gamma \in Cn(Z)$. It follows that $\mathfrak{A}_1 = \mathfrak{A}(Z + \{\beta\}) = \mathfrak{A}(Z + \{\gamma\})$. If $\mathfrak{M}_n \in \mathfrak{A}_1$ for $n = 1, 2, \ldots$, then $\mathfrak{M}_n \in \mathfrak{A}_0 = \mathfrak{A}(Z)$ and $\gamma \in E_1(\mathfrak{M}_n)$ for $n = 1, 2, \ldots$ In view of the fact that $\mathfrak{A}_0$ is a σ-class and γ is a $\prod\sum$-sentence we infer that $\sum_{n=1}^{\infty} \mathfrak{M}_n \in \mathfrak{A}_0$ and $\gamma \in E_1(\sum_{n=1}^{\infty} \mathfrak{M}_n)$. It follows that $\sum_{n=1}^{\infty} \mathfrak{M}_n \in \mathfrak{A}(Z + \{\gamma\})$, *i. e.* $\mathfrak{A}_1$ is a σ-class.

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On the definitions of computable real continuous functions

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In this paper I shall prove the equivalence of some definitions of computable real continuous functions. Let us assume the following abbreviations: $\mathcal{N}$ = the set of natural numbers, $\mathcal{J}$ = the set of all integers, $\mathcal{R}$ = the set of real numbers, $\mathfrak{F} = \mathcal{J}^{\mathcal{J}}$ (the class of functions defined over the set $\mathcal{J}$ and assuming the values from $\mathcal{J}$), $\mathfrak{Com}$ = the class of computable (general recursive) integral functions, $\mathfrak{Com} \subset \mathfrak{F}$, $\mathcal{K}$ = the class of computable functionals in the sense of [1] (defined over the n-tuples of the elements of $\mathfrak{F}$, and the k-tuples of the elements of $\mathcal{J}$ and assuming the integral values. We shall often use the expression $A(a, f)$ as an abbreviation of: $a \in \mathcal{R}$, $f \in \mathfrak{F}$ and for any $n \in \mathcal{N}$

$$\left| a - \frac{f(n)}{n+1} \right| < \frac{1}{n+1}.$$

Latin letters will be used in such a manner that always $i, k, l, m, n \in \mathcal{N}$, $p, q, r, s, t, u, x, y, z \in \mathcal{J}$, $a, b, c, d, e \in \mathcal{R}$.

Let r_n be a recursive enumeration of all rationals without repetitions. Let $\mathrm{No}(p, q)$ be the recursive converse function of the function r_n. This means that

(1) $$r_{\mathrm{No}(p,q)} = \frac{p}{q}.$$

We assume that $p/0 = 0$. Instead of $\mathrm{No}(p, q)$ we shall often write $\mathrm{No}(p/q)$. Let us set

(2) $$W_n(k) = W(n, k) = (\mu x)\left[\left| r_n - \frac{x}{k+1} \right| < \frac{1}{k+1}\right],$$

No, $W_n \in \mathfrak{Com}$. We obviously have

(3) $$\left| r_n - \frac{W(n,k)}{k+1} \right| < \frac{1}{k+1} \quad \text{for all} \quad n, k \in \mathcal{N}.$$