Note on rings in which every proper left-ideal is cyclic

by

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We shall call an arbitrary ring \( R \) cyclic if the additive group \( R^+ \) is cyclic. The ring \( J \) of rational integers is obviously cyclic. Starting from the fundamental property of the ring \( J \) we introduce the following

Definition. An arbitrary ring \( R \) is called a ring with property \( F \), if every proper left-ideal \( L \) of \( R \) is cyclic. For example any cyclic ring and any skew-field have the property \( F \).

Theorem. An arbitrary ring \( R \) has the property \( F \) if and only if \( R \) is a skew-field, or a cyclic ring, or a zero-ring of type \( p^m \) or else an arbitrary ring of order \( p^2 \) where \( p \) is a prime.

Remark. A skew-field, as a ring without proper left-ideals, can have an arbitrary infinite cardinal, but the order of a finite ring with property \( P \) is necessarily \( p^2 \). For example \( R(p) = \{ x, y \} \) is a non-commutative ring with property \( P \) and of order \( p^2 \) where \( p \) is a prime number and \( p = p = y = x = y = x = y = x = y = 0, y = 0 \). We remark that the theorem is a generalization of Lemma 1 (see [7]). The notions of modern algebra can be found in the books [1], [3], [4] and [5], where we omit terminology remarks. Now we verify five Lemmas.

Lemma 1. A ring without proper left-ideals is a skew-field or else a zero-ring of prime-number-order.

Proof. If there exists an element \( 0 \neq a \in R \) for which \( Ra \neq R \), then \( Ra = 0 \), and thus the zero-ring \( (a) \neq 0 \), being a left-ideal, coincides with \( R \) and \( O(R) = p \). But if for any \( 0 \neq a \in R \) the element \( Ra \neq R \) holds, then \( R \) has no divisors of zero and by the single equation \( a = a \) we see that \( e \in R \) is the unity of \( R \). The solvability of all equations \( yb = e \) trivially implies by the associativity law the skew-field behaviour of \( R \).

Remarks. From this short proof we see that only the rings of order \( p \) are without proper subrings; moreover the solvability of all equations \( yb = a \) in a ring means the solvability of all equations \( b = a \) in the same ring \( (b \neq 0) \); and finally we observe that we can similarly prove that if in a ring \( R \) there exists an element \( a \neq 0 \) which is not a right an-
nihilator of \( R \) and if for this element with any \( 0 \neq b \in R \) the element \( Rab = Ra \) holds, then \( R \) is a skew-field (see [8]).

Lemma 2. A ring \( R \) with mixed group \( R^+ \) cannot have the property \( P \).

Proof. We assume that \( R \) is a ring with property \( P \) and with mixed group \( R^+ \). Let \( T \) be the cyclic torsion ring of order \( n \in J \) in \( R \). Since \( (aR) T = (aR) T = 0 \) and \( nR = T = 0 \), then exists a non-cyclic two-sided ideal \( D = nR + T \) (as a ring-theoretical direct sum) in \( R \), which by property \( P \) implies \( R = nR + T \) (without the use of the fundamental theorem of [8]). Then \( nR \) is a cyclic ideal in \( R \), consequently \( nR + T = R \). If \( nR = (aR) \), where naturally \( O(r) = \infty \), we obtain \( aR = R \), and \( T = 0 \).

Lemma 3. A ring \( R \) with property \( P \) but without divisors of zero is a skew-field or else an infinite cyclic ring.

Proof. If \( 0 \neq a \in R \), then \( Ra = 0 \). If for every \( 0 \neq a \in R \) it is \( Ra = R \), then \( R \) is cyclic. In the case \( Ra = R \) the ring \( R \) is itself cyclic by the property \( P \) and by \( (aR) = aR \), and obviously \( O(R) = \infty \).

Lemma 4. A ring with property \( P \), containing divisors of zero and being of characteristic 0, cannot have an algebraically closed additive group.

Proof. We suppose that \( R \), being a ring with divisors of zero and having an algebraically closed additive group \( R^+ \), is of characteristic 0. Then \( (R^+) \) cannot be simultaneously cyclic and algebraically closed, and therefore \( R^+ = 0 \) or else \( R^+ = R \). By Lemma 1 and by our hypothesis there exists a \( a \neq 0 \) right-annihilator of \( R_0 \), i. e., \( RZ \neq 0 \). Then the set \( Z = 0 \) of all right-annihilators of \( R \) is a two-sided ideal in \( R \), whose additive group \( Z^+ \) is a subring subgroup in \( R^+ \). The ideal \( Z^+ \) cannot be cyclic, since \( Z^+ \) is likewise algebraically closed, therefore \( R = Z \) and \( R \) is a zero-ring. But in an algebraically closed group there exists a subgroup which is not cyclic, and this contradiction proves our Lemma.

Lemma 5. Let \( F \) be a (finite or infinite) elementary \( p \)-ring for which \( F^1 = 0 \) and \( O(F) > p \), and let moreover \( J \) be a two-sided ideal of order \( p \) in a ring \( R \). If \( E = F \), then \( R \) is without property \( P \).

Proof. We shall assume that \( R \) has the property \( P \) and we shall show a contradiction. The complete endomorphism ring of \( J^+ \) has the order \( p \), an by the endomorphism \( j \rightarrow j \), \( j \in J \), \( r \in R \) of \( J^+ \) we have a ring-theoretical homomorphism \( r \rightarrow s \), of \( R \) into the complete endomorphism ring of \( J^+ \). The kernel of this mapping \( r \rightarrow s \) is an ideal \( N \), for which \( R = 0 \) and \( O(R/N) > p \) holds. Consequently by \( O(R) > p \) and by property \( P \) obviously \( R = N \); therefore \( R \) is a zero-ring. But then likewise \( E = F \) is a zero-ring, which contradicts our hypothesis.

Now we give an elementary
Proof of Theorem. Let \( R \) be a ring with property \( P \). By Lemma 3 we can suppose the existence of divisors of zero. If \( R \) contains an element of infinite order, then by Lemma 2 and 4 there exists a number \( u \in J \) for which \( 0 \neq a \in \text{R}_a \). But by \( R^+ \supseteq (aR)^+ \) and by property \( P \), \( R \) is cyclic.

If \( R^+ \) is a torsion group, then a ring-theoretical direct decomposition \( R = \bigoplus E_p \), where the ideal \( E_p \) is generated by all elements of \( p \)-power order of \( R \). If \( R^p \neq R \), then \( R \) is a finite cyclic ring. Now let \( R \) be a \( p \)-ring in which \( R' \) is generated by all elements of order \( p \) of \( R \). If \( R' \neq R \), then \( R \) is cyclic or else of type \( p^n \) because in both cases \( R' \) is cyclic [2]. Finally we assume that \( R' = R \). By the existence of divisors of zero, by \( pR = 0 \), by Lemma 1 and by property \( P \) the existence of a left-ideal \( L \) of order \( p \) of \( R \) is necessarily ensured. Now we show the impossibility of \( O(R) \supseteq p^2 \). It is clear that \( Lr \) is a left-ideal in \( R \) \((r \in E)\). If there exists an element \( 0 \neq r \in R \) for which \( Lr \neq 0 \) and \( Lr \cdot Lr = 0 \) holds, then for the left-ideal \( D = (L, Lr) \) it is \( D = D_r \), i.e., \( O(R) = p^2 \). But if \( Lr \subseteq L \) for all \( r \in R \), the subring \( L \) is a two-sided ideal in \( R \). Then \( R/L \) has the property \( P \) and consequently has no proper left-ideals. By \( O(R) \supseteq p^2 \) we can assume that \( R/L \) is a skew-field, and thus not a zero-ring, but has the property \( P \). By \( O(R/L) \supseteq p^2 \) and by Lemma 5 we have obtained a contradiction, which completes the proof.

References


Errata to the paper "On the \( \varepsilon \)-theorems"

(Fundamenta Mathematicae 43, p. 158-165)

by

H. Rasiowa (Warszawa)

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156\(\alpha\) | theories | theories since the non-enumerable case follows immediately from the enumerable one
161\(\alpha\) | a consistent | a consistent, enumerable
161\(\alpha\) | cf. [6] | cf. [8], or an extension in a Boolean algebra of all subrings of a set.
163\(\alpha\) | in algebra \( R \) \( L \) | in algebra \( R \) of sets of \( e \)-theorem 3.1 (with property \( R \)).
163\(\alpha\) | \( f \) | of \( e \)-theorem 3.1 (with property \( R \)).
164\(\alpha\) | \( e \)-theorem | \( e \)-theorem 3.1 (with property \( R \)).

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