Note on rings in which every proper left-ideal is cyclic

by

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We shall call an arbitrary ring R *cyclic* if the additive group R^+ is cyclic. The ring J of rational integers is obviously cyclic. Starting from the fundamental property of the ring J we introduce the following

Definition. An arbitrary ring R is called a *ring with property P*, if every proper left-ideal L of R is cyclic. For example any cyclic ring and any skew-field have the property P.

THEOREM. *An arbitrary ring R has the property P if and only if R is a skew-field, or a cyclic ring, or a zero-ring of type p^∞ or else an arbitrary ring of order p^2 (where p is a prime).*

Remark. A skew-field, as a ring without proper left-ideals, can have an arbitrary infinite cardinal, but the order of a finite ring with property P is necessarily p^2. For example $R(p)=\{x,y\}$ is a non-commutative ring with property P and of order p^2 where p is a prime number and $px=py=x^2=xy=yx-x=y^2-y=0$. We remark that the theorem is a generalization of Lemma 1 (see [7]). The notions of modern algebra can be found in the books [1], [3], [4] and [5], therefore we omit terminological remarks. Now we verify five Lemmas.

LEMMA 1. *A ring without proper left-ideals is a skew-field or else a zero-ring of prime-number-order.*

Proof. If there exists an element $0\neq a\in R$ for which $Ra\neq R$, then $Ra=0$, and thus the zero-ring $\{a\}\neq 0$, being a left-ideal, coincides with R and $O(R)=p$. But if for any $0\neq a\in R$ the element $Ra=R$ holds, then R has no divisors of zero and by the single equation $ea=a$ we see that $e\in R$ is the unity of R. The solvability of all equations $yb=e$ trivially implies by the associativity law the skew-field behaviour of R.

Remarks. From this short proof we see that only the rings of order p are without proper subrings; moreover the solvability of all equations $yb=a$ in a ring implies the solvability of all equations $bx=a$ in the same ring ($b\neq 0$); and finally we observe that we can similarly prove that if in a ring R there exists an element $a\neq 0$ which is not a right an-

nihilator of R and if for this element with any $0\neq b\in R$ the element $Rab=Ra$ holds, then R is a skew-field (see [6]).

LEMMA 2. *A ring R with mixed group R^+ cannot have the property P.*

Proof. We assume that R is a ring with property P and with mixed group R^+. Let T be the cyclic torsion ring of order $n\in J$ in R. Since $(nR)\cdot T=T\cdot(nR)=0$ and $nR\cap T=0$, there exists a non-cyclic two-sided ideal $D=nR+T$ (as a ring-theoretical direct sum) in R, which by property P implies $R=nR+T$ (without the use of the fundamental theorem of [8]). Then n^2R is likewise a cyclic ideal in R, consequently $n^2R+T=R$. If $nR=\{nr\}$, where naturally $O(r)=\infty$, we obtain $nr=k(n^2r)+t$ ($k\in J$, $t\in T$), *i. e.*, $n=kn^2$ and $n=1$, $T=0$.

LEMMA 3. *A ring R with property P but without divisors of zero is a skew-field or else an infinite cyclic ring.*

Proof. If $0\neq a\in R$, then $Ra\neq 0$. If for every $0\neq a\in R$ it is $Ra=R$, thus, by Lemma 1, R is a skew-field. In the case $Ra\neq R$ the ring R is itself cyclic by the property P and by $(Ra)^+\simeq R^+$, and obviously $O(R)=\infty$.

LEMMA 4. *A ring with property P, containing divisors of zero and being of characteristic* 0, *cannot have an algebraically closed additive group.*

Proof. We suppose that R, being a ring with divisors of zero and having an algebraically closed additive group R^+, is of characteristic 0. Then $(Rb)^+$ cannot be simultaneously cyclic and algebraically closed, and therefore $Rb=0$ or else $Rb=R$. By Lemma 1 and by our hypothesis there exists a $z\neq 0$ right-annihilator of R, *i. e.*, $Rz=0$. Then the set $Z\neq 0$ of all right-annihilators of R is a two-sided ideal in R, whose additive group Z^+ is a serving subgroup in R^+. The ideal Z cannot be cyclic, since Z^+ is likewise algebraically closed, therefore $R=Z$ and R is a zero-ring. But in an algebraically closed group there exists a subgroup which is not cyclic, and this contradiction proves our Lemma.

LEMMA 5. *Let F be a (finite or infinite) elementary p-ring for which $F^2\neq 0$ and $O(F)\geqslant p^2$, and let moreover $\mathcal{J}$ be a two-sided ideal of order p in a ring R. If $R/\mathcal{J}\simeq F$, then R is without property P.*

Proof. We shall assume that R has the property P and we shall show a contradiction. The complete endomorphism ring of $\mathcal{J}^+$ has the order p, an byd the endomorphism $j\to j\varepsilon_r=jr$ ($j\in\mathcal{J}$, $r\in R$) of $\mathcal{J}^+$ we have a ring-theoretical homomorphism $r\to\varepsilon_r$ of R into the complete endomorphism ring of $\mathcal{J}^+$. The kernel of this mapping $r\to\varepsilon_r$ is an ideal N, for which $RN=0$ and $O(R/N)\leqslant p$ holds. Consequently by $O(R)\geqslant p^3$ and by property P obviously $R=N$; therefore R is a zero-ring. But then likewise $R/\mathcal{J}\simeq F$ is a zero-ring, which contradicts our hypothesis.

Now we give an elementary

Proof of Theorem. Let R be a ring with property P. By Lemma 3 we can suppose the existence of divisors of zero. If R contains an element of infinite order, then by Lemma 2 and 4 there exists a number $n \in J$ for which $0 \subset nR \subset R$. But by $R^+ \simeq (nR)^+$ and by property P, R is cyclic.

If R^+ is a torsion group, then a ring-theoretical direct decomposition $R = \sum_p R_p$ holds, where the ideal R_p is generated by all elements of p-power order of R. If $R \neq R_p$, then R is a finite cyclic ring. Now let R be a p-ring in which R' is generated by all elements of order p of R. If $R' \neq R$, then R is cyclic or else of type p^∞ because in both cases R' is cyclic [2]. Finally we assume that $R' = R$. By the existence of divisors of zero, by $pR = 0$, by Lemma 1 and by property P the existence of a left-ideal L of order p of R is necessarily ensured. Now we show the impossibility of $O(R) \geqslant p^3$. It is clear that Lr is a left-ideal in R $(r \in R)$. If there exists an element $0 \neq r \in R$ for which $Lr \neq 0$ and $L \cap Lr = 0$ holds, then for the left-ideal $D = \{L, Lr\}$ it is $R = D$, *i. e.*, $O(R) = p^2$. But if $Lr \subseteq L$ for all $r \in R$, the subring L is a two-sided ideal in R. Then R/L has the property P and consequently has no proper left-ideals. By $O(R) \geqslant p^3$ we can assume that R/L is a skew-field, and thus not a zero-ring, but has the property P. By $O(R/L) \geqslant p^2$ and by Lemma 5 we have obtained a contradiction, which completes the proof.

References

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Errata to the paper "On the ε-theorems"

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Page	for	read
156_{15}	of [10]	of [10] and [13]
156_{11}	theories	theories since the non-enumerable case follows immediately from the enumerable one
161_4	a consistent	a consistent, enumerable
161_1	cf. [8]	cf. [8], or an extension in a Boolean algebra of all subsets of a set, cf. [13].
162^{10}	in algebra B	in algebra B of sets
162^{17}	f	of
164^{17}	ε-theorem	ε-theorem 5.1 (with open α).