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On computable sequences

by

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A real number α $(0<\alpha<1)$ is said to be *computable* (cf. Robinson [9], Rice [8]) if there is a general recursive function φ such that

(i) $|\alpha-\varphi(n)/n|<1/n$ for $n=1,2,\ldots$

This definition is equivalent to each of the following ones [1]):

(ii) There is a general recursive function ψ such that

$$\alpha=\sum_{n=1}^{\infty}\psi(n)/10^n \quad \text{and} \quad \psi(n)<10 \quad \text{for} \quad n=1,2,\ldots$$

(iii) The relation R which p bears to q if and only if $p/q<\alpha$ is general recursive. (In other words the function ϑ such that $\vartheta(p,q)\leqslant 1$ and $\{\vartheta(p,q)=1\}\equiv\{p/q<\alpha\}$ is general recursive [2]).)

Several other equivalent formulations of (i) are known.

Let us now pass from numbers to sequences. If we replace in the definitions given above α by α_k and φ,ψ,ϑ by $\varphi_k,\psi_k,\vartheta_k$ where the index k runs over integers and if we further require that these functions be general recursive in all variables (including "k"), then we obtain three definitions of what may be called *computable sequences*. It will be proved below that no two of these definitions and of a couple of others, which we shall formulate later, are equivalent.

There is no doubt that of these various definitions the one which best expresses the existence of an algorithm permitting one to calculate uniformly the terms of a sequence with any desired degree of accuracy is that which corresponds to (i). The other definitions represent merely a mathematical curiosity. It seems to us, however, that the following circumstance deserves emphasis: if we replace in the definitions (i)-(iii)

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[1]) The equivalence of these definitions has been first observed by Robinson [9]. Cf. further Rice [8] and Myhill [6].

[2]) These definitions have been formulated by Mazur [3]. The definition given by Rice [8] is equivalent to the first of these definitions.

of computable numbers the general recursive functions φ,ψ,ϑ by the primitive recursive ones, we obtain definitions which are not equivalent to each other and the logical relations which hold between those definitions are exactly the same as the logical relations which hold between the definitions of computable sequences in which general recursive functions are used (cf. Specker [10] and Péter [7], p. 185 seq.).

It remains an open problem whether this is a coincidence or a special case of a general phenomenon whose causes ought to be discovered.

1. In what follows we shall use lower case Roman type to denote integers $\geqslant 0$ and lower case Greek type to denote general recursive functions. Sequences of real numbers are denoted by symbols $\{\mathfrak{a}_k\}$ where $\mathfrak{a}_k$ is the kth term of the sequence. We assume once for all that $0<\mathfrak{a}_k<1$.

We introduce several classes of sequences [3]):

$$\{\mathfrak{a}_k\}\in C_1\equiv\sum_{\varphi}\prod_{k}\prod_{n\geqslant 1}\{|\mathfrak{a}_k-\varphi(n,k)/n|<1/n\},$$

$$\{\mathfrak{a}_k\}\in C_{2p}\equiv\sum_{\psi}\Big\{\prod_{k}\Big[\mathfrak{a}_k=\sum_{n=1}^{\infty}\psi(n,k)/p^n\Big]\prod_{n,k}[\psi(n,k)<p]\Big\},$$

$$\{\mathfrak{a}_k\}\in C_3\equiv\sum_{\xi}\prod_{p>1}\Big\{\prod_{k}\Big[\mathfrak{a}_k=\sum_{n=1}^{\infty}\xi(n,p,k)/p^n\Big]\prod_{n,k}[\xi(n,p,k)<p]\Big\},$$

$$\{\mathfrak{a}_k\}\in C_4\equiv\sum_{\vartheta}\prod_{p,q,k}\big(\{[\vartheta(p,q,k)=1]\equiv[p/q<\mathfrak{a}_k]\}[\vartheta(p,q,k)\leqslant 1]\big),$$

$$\{\mathfrak{a}_k\}\in C_5\equiv\sum_{\zeta}\prod_{p,q,k}\big(\{[\zeta(p,q,k)=1]\equiv[p/q>\mathfrak{a}_k]\}[\zeta(p,q,k)\leqslant 1]\big).$$

LEMMA 1 [4]). *If* $|\mathfrak{a}_k-\lambda_1(n,k)/\lambda_2(n,k)|<\mu_1(n)/\mu_2(n)$ *for* $n,k=1,2,\ldots$ *and* $\lim_{n\to\infty}\mu_1(n)/\mu_2(n)=0$, *then* $\{\mathfrak{a}_k\}\in C_1$.

Proof. For an arbitrary n there is a t such that $\mu_1(t)/\mu_2(t)<1/2n$. The function $\nu(n)=\min_t\{\mu_1(t)/\mu_2(t)<1/2n\}$ is thus general recursive. Putting $\sigma_i(n,k)=\lambda_i(\nu(n),k)$ for $i=1,2$ we obviously have $|\mathfrak{a}_k-\sigma_1(n,k)/\sigma_2(n,k)|<1/2n$. If $\varphi(n,k)$ is the integer nearest to $n\sigma_1(n,k)/\sigma_2(n,k)$, then φ is a general recursive function and $|n\sigma_1(n,k)/\sigma_2(n,k)-\varphi(n,k)|<1/2$. Combining the inequalities thus obtained we have $|\mathfrak{a}_k-\varphi(n,k)/n|<1/n$ and hence $\{\mathfrak{a}_k\}\in$

[3]) We use symbols $\sum$, $\prod$, $\vee$, $\cdot$, $\supset$ and $\equiv$ as synonymous with the words "there is", "for every", "or", "and", "implies" and "is equivalent to". The dot denoting conjunction is often omitted.

[4]) This lemma is due to Mazur [3].

THEOREM 1. $C_3\subset C_{2p}\subset C_1$ *for each* $p>1$.

Proof. The formula $C_3\subset C_{2p}$ is evident. If $\{\mathfrak{a}_k\}\in C_{2p}$, then

$$x\mathfrak{a}_k=p^{-x}\sum_{n=1}^{x}\psi(n,k)xp^{x-n}+x\sum_{n=x+1}^{\infty}\psi(n,k)p^{-n}.$$

The function $\alpha(x,k)=[p^{-x}\sum_{n=1}^{x}\psi(n,k)xp^{x-n}]$ is obviously general recursive and satisfies the inequalities

$$|x\mathfrak{a}_k-\alpha(x,k)|<1+x\sum_{n=x+1}^{\infty}\psi(n,k)p^{-n}<1+x/p^x.$$

On account of lemma 1 this proves that $\{\mathfrak{a}_k\}\in C_1$.

THEOREM 2. $C_4\subset C_3$ *and* $C_5\subset C_3$.

Proof. If $\{\mathfrak{a}_k\}\in C_5$, then

$$\{x=[q\mathfrak{a}_k]\}\equiv\{x\leqslant q\mathfrak{a}_k<x+1\}\equiv\{x=\min_{y\leqslant q}(\zeta(y,q,k)=1)-1\}.$$

These equivalences prove that the function $\alpha(q,k)=[q\mathfrak{a}_k]$ is general recursive. Now we put

$$\xi(1,p,k)=\alpha(p,k),$$
$$\xi(n+1,p,k)=\alpha(p^{n+1},k)-p\alpha(p^n,k),$$

and easily obtain $\mathfrak{a}_k=\sum_{n=1}^{\infty}\xi(n,p,k)/p^n$ and $\xi(n,p,k)<p$.

If $\{\mathfrak{a}_k\}\in C_4$, then we denote by $(\mathfrak{a})$ the least integer x for which $x+1\geqslant\mathfrak{a}$ and obtain

$$\{x=(q\mathfrak{a}_k)\}\equiv\{x=\min_{y<q}(y+1\geqslant q\mathfrak{a}_k)\}\equiv\{x=\min_{y<q}(\vartheta(y+1,q,k)=0)\}.$$

Thus the function $\alpha(q,k)=(q\mathfrak{a}_k)$ is general recursive. Defining ξ in the same way as above we obtain the desired function.

THEOREM 3. *If* $p,q\geqslant 1$ *and a power of* q *is divisible by* p, *then* $C_{2q}\subset C_{2p}$.

Proof. Assume that $sp=q^{n_0}$ and $\{\mathfrak{a}_k\}\in C_{2q}$ and let

$$\mathfrak{a}_k=\sum_{n=1}^{\infty}\psi(n,k)/q^n\quad\text{where}\quad\psi(n,k)<q\quad\text{for}\quad n,k=1,2,\ldots$$

We put

$$\psi'(j,k)=\sum_{t=1}^{n_0}\psi(t+jn_0,k)q^{n_0-t},\qquad j=0,1,2,\ldots,$$

and define by induction the functions ψ_0 and ϱ as follows:

$$[\psi'(0,k)/s]=\psi_0(1,k), \qquad \psi'(0,k)-s\psi_0(1,k)=\varrho(1,k),$$

$$\left[\left(\psi'(j,k)+q^{n_0}\varrho(j,k)\right)/s^{j+1}\right]=\psi_0(j+1,k),$$

$$\psi'(j,k)+q^{n_0}\varrho(j,k)-s^{j+1}\psi_0(j+1,k)=\varrho(j+1,k).$$

It follows from these definitions that $\varrho(j,k)$ is the rest of a division of an integer by s^j and hence $\varrho(j,k)<s^j$.

We shall show that $\psi_0(j,k)<p$.

Indeed, for $j=1$ we have the inequality

$$\psi'(0,k)\geqslant s\psi_0(1,k),$$

which on account of the formula

$$q^{n_0}-1=(q-1)(q^{n_0-1}+q^{n_0-2}+\ldots+1)\geqslant\psi'(j,k)$$

proves that $q^{n_0}=sp>s\psi_0(1,k)$.

For $j>1$ the required inequality results from the identity

$$\psi'(j,k)+q^{n_0}\varrho(j,k)=s^{j+1}\psi_0(j+1,k)+\varrho(j+1,k)$$

by means of the following calculations:

$$s^{j+1}\psi_0(j+1,k)\leqslant\psi'(j,k)+q^{n_0}\varrho(j,k)\leqslant q^{n_0}-1+q^{n_0}(s^j-1)$$
$$=q^{n_0}s^j-1=s^{j+1}p-1<s^{j+1}p.$$

It remains to prove that

$$\mathfrak{a}_k=\sum_{n=1}^{\infty}\psi_0(n,p)/p^n.$$

In order to obtain this formula we shall first prove that

$$\mathfrak{a}_k=\sum_{n=1}^{j}\psi_0(n,k)/p^n+\varrho(j,k)/q^{jn_0}+\sum_{n=j}^{\infty}\psi'(n,k)/q^{(n+1)n_0}.$$

The verification of this formula for $j=1$ is immediate. Assuming that it holds for an integer j we obtain

$$\mathfrak{a}_k=\sum_{n=1}^{j}\psi_0(n,k)/p^n+\varrho(j,k)/q^{jn_0}+\psi'(j,k)/q^{(j+1)n_0}+\sum_{n=j+1}^{\infty}\psi'(n,k)/q^{(n+1)n_0}$$

$$=\sum_{n=1}^{j}\psi_0(n,k)/p^n+\left(\psi'(j,k)+q^{n_0}\varrho(j,k)\right)/q^{(j+1)n_0}+\sum_{n=j+1}^{\infty}\psi'(n,k)/q^{(n+1)n_0}$$

$$=\sum_{n=1}^{j}\psi_0(n,k)/p^n+s^{j+1}\psi_0(j+1,k)/q^{(j+1)n_0}+\varrho(j+1,k)/q^{(j+1)n_0}+$$
$$+\sum_{n=j+1}^{\infty}\psi'(n,k)/q^{(n+1)n_0}$$

$$=\sum_{n=1}^{j+1}\psi_0(n,k)/p^n+\varrho(j+1,k)/q^{(j+1)n_0}+\sum_{n=j+1}^{\infty}\psi'(n,k)/q^{(n+1)n_0}.$$

The formula is thus proved for an arbitrary j. It follows from this formula that

$$\left|\mathfrak{a}_k-\sum_{n=1}^{j}\psi_0(n,k)/p^n\right|<s^j/q^{jn_0}+(q^{n_0}-1)\sum_{n=j}^{\infty}q^{-(n+1)n_0}=p^{-j}+q^{-jn_0},$$

which proves that $\{\mathfrak{a}_k\}\in C_{2p}$.

2. We shall now show that inclusions in theorems 1-3 cannot be replaced by equations.

THEOREM 4. *$C_1\neq C_{2p}$ for each $p>1$.*

Proof[5]. In order to facilitate the reading of the subsequent formulas we first give an outline of the proof.

Let X_1,X_2 be two disjoint recursively enumerable sets which cannot be separated by means of recursive sets (cf. Kleene [1]). The problem whether $k\in X_i$ is equivalent to the problem whether there exists an integer y such that $a_i(k,y)=0$ $(i=1,2)$. We now define a rational number $\mathfrak{a}_k=\lim_{n\to\infty}\mathfrak{a}_{kn}$ in the following manner:

For a given n we test the integers $y\leqslant n$ and try to find among them the smallest integer for which $a_1(k,y)=0$ or $a_2(k,y)=0$. If there is no such y, then we put $\mathfrak{a}_{kn}=1/p-1/p^n$. If t_k is the least $y\leqslant n$ such that $a_i(k,y)=0$ then we put $\mathfrak{a}_{kn}=1/p\pm1/p^{t_k+2}$ where we take the $+$ sign if $k\in X_1$ and the $-$ sign if $k\in X_2$. It is evident that $\mathfrak{a}_k<1/p$ or $\mathfrak{a}_k>1/p$ according as $k\in X_2$ or $k\in X_1$. Hence the first digit in the development of $\mathfrak{a}_k$ on the scale p is 0 for $k\in X_1$ and 1 for $k\in X_2$. The development of $\mathfrak{a}_k$ on the scale p cannot, therefore, be recursive.

We now give an exact proof. Let a_1,a_2 be general recursive functions such that $k\in X_i\equiv\sum_y[a_i(k,y)=0]$ and let

$$\beta_i(k,n)=\min\left(1,\min_{z\leqslant n}a_i(k,z)\right), \qquad i=1,2,$$

$$\gamma_i(k,n)=\min_{z\leqslant n}\left(\beta_i(k,z)=0\right), \qquad i=1,2,$$

[5]) The proof was already sketched in Mostowski [4]. We give here a detailed proof.

$$\delta(k,n)=(p^n-1)+[1-\beta_1(k,n)][1+p^{n-1-\gamma_1(k,n)}]+[1-\beta_2(k,n)][1-p^{n-1-\gamma_2(k,n)}],$$
$$\mathfrak{a}_k=\lim_{n\to\infty}\{\delta(k,n)/p^{n+1}\}.$$

If $k\in X_1$, then there is a least s_{1k} such that $\alpha_1(k,s_{1k})=0$ and there is no s such that $\alpha_2(k,s)=0$. Hence $\beta_1(k,y)=1$ and $\gamma_1(k,y)=0$ for $y<s_{1k}$ and $\beta_1(k,y)=0$ and $\gamma_1(k,y)=s_{1k}$ for $y\geqslant s_{1k}$; finally $\beta_2(k,y)=1$ and $\gamma_2(k,y)=0$ for all y. It follows that if $n\geqslant s_{1k}$, then $\delta(k,n)=p^n-1+$ $+1+p^{n-1-s_{1k}}=p^n+p^{n-1-s_{1k}}$ and hence $\mathfrak{a}_k=1/p+1/p^{s_{1k}+2}$.

If $k\in X_2$, we similarly find $\mathfrak{a}_k=1/p-1/p^{s_{2k}+2}$ where s_{2k} is the least s such that $\alpha_2(k,s_{2k})=0$.

If $k\notin X_1\cup X_2$, then $\beta_1(k,y)=\beta_2(k,y)=1$ for all y and hence $\mathfrak{a}_k=p^{-1}$.

Thus $\mathfrak{a}_k=p^{-1}+\varepsilon_k p^{-t_k-2}$ where $\varepsilon_k=1,0,-1$ according as $k\in X_1$, $k\notin X_1\cup X_2$, $k\in X_2$ and where t_k is an integer $\geqslant 0$.

We shall now show that $\{\mathfrak{a}_k\}\in C_1$.

From the definitions it follows that

$$\mathfrak{a}_k-\delta(k,n)/p^{n+1}=\varepsilon_k p^{-t_k-2}-p^{-n-1}-[1-\beta_1(k,n)][p^{-n-1}+p^{-2-\gamma_1(k,n)}]-$$
$$-[1-\beta_2(k,n)][p^{-n-1}-p^{-2-\gamma_2(k,n)}].$$

If $\varepsilon_k=0$, then $\beta_1(k,n)=\beta_2(k,n)=1$ and hence

$$\mathfrak{a}_k-\delta(k,n)/p^{n+1}=-p^{-n-1}.$$

If $\varepsilon_k=1$ and $n<s_{1k}$, then $t_k=s_{1k}$ and

$$\mathfrak{a}_k-\delta(k,n)/p^{n+1}=p^{-s_{1k}-2}+p^{-n-1}<2p^{-n-1}.$$

If $\varepsilon_k=1$ and $n\geqslant s_{1k}$, then

$$\mathfrak{a}_k-\delta(k,n)/p^{n+1}=p^{-s_{1k}-2}-p^{-n-1}-p^{-n-1}-p^{-s_{1k}-2}=-2p^{-n-1}.$$

If $\varepsilon_k=-1$ and $n<s_{2k}$, then $t_k=s_{2k}$ and

$$\mathfrak{a}_k-\delta(k,n)/p^{n+1}=-p^{-s_{2k}-2}+p^{-n-1},$$

whence

$$0<\mathfrak{a}_k-\delta(k,n)/p^{n+1}<p^{-n-1}.$$

If $\varepsilon_k=-1$ and $n\geqslant s_{2k}$, then

$$\mathfrak{a}_k-\delta(k,n)/p^{n+1}=-p^{-s_{2k}-2}-p^{-n-1}-p^{-n-1}+p^{-s_{2k}-2}=-2p^{-n-1}.$$

Thus for all k and n

$$|\mathfrak{a}_k-\delta(k,n)/p^{n+1}|<2p^{-n-1},$$

which proves that $\{\mathfrak{a}_k\}\in C_1$.

The assumption that $\{\mathfrak{a}_k\}\in C_{2p}$ leads to a contradiction. Indeed, if $\mathfrak{a}_k=\sum_{n=1}^{\infty}\psi(k,n)/p^n$, then $\psi(k,1)=0$ for $k\in X_1$ and $\psi(k,1)=1$ for $k\in X_2$. The sets X_1,X_2 are thus separated by the set $\underset{k}{E}[\psi(k,1)=0]$ and hence ψ cannot be a recursive function.

THEOREM 5. *If no power of q is divisible by p, then $C_{2q}\not\subset C_{2p}$.*

The idea of this proof is as follows. We develop $1/p$ on the scale q

$$1/p=\sum_{n=1}^{\infty}\varkappa(n)/q^n$$

and show first that $0<\varkappa(n)<q$ for infinitely many n. Let $n_1,n_2,\ldots$ be a sequence of (not necessarily all) values of n which satisfy this inequality. We now consider the same sets X_1,X_2 and functions α_1,α_2 as in the previous proof and put $\mathfrak{a}_k=\sum_{n=1}^{\infty}\psi(k,n)/q^n$ where $\psi(k,n)=\varkappa(n)$ except when n is the first term n_h of the sequence $n_1,n_2,\ldots$ such that $\alpha_1(k,h)=0$ or $\alpha_2(k,h)=0$. In this exceptional case we put $\psi(k,n)=\varkappa(n)\mp 1$ according as $\alpha_1(k,h)=0$ or $\alpha_2(k,h)=0$. It follows that $\mathfrak{a}_k<1/p$ for $k\in X_1$ and $\mathfrak{a}_k>1/p$ for $k\in X_2$, and hence the development of $\mathfrak{a}_k$ on the scale p is not recursive.

The exact proof runs as follows. We may assume that for infinitely many n $\varkappa(n)>0$. Since the development of $1/p$ is periodical, we can represent $1/p$ in the form

$$1/p=\varkappa(1)/q+\ldots+\varkappa(n_0)/q^{n_0}+\sum_{j=1}^{\infty}q^{-js}[\varkappa(n_0+1)/q^{n_0+1}+\ldots+\varkappa(n_0+s)/q^{n_0+s}]$$

where $\varkappa(n_0+1),\ldots,\varkappa(n_0+s)$ form the period of the development. In view of the assumption made above not all integers $\varkappa(n_0+1),\ldots,\varkappa(n_0+s)$ vanish. If all of them were $=q-1$ we should have

$$1/p=\varkappa(1)/q+\ldots+\varkappa(n_0)/q^{n_0}+1/q^{n_0}$$

where $\varkappa(n_0)<q-1$ (since the period begins after the n_0th term). Thus we should obtain

$$1/p=\{\varkappa(1)q^{n_0-1}+\ldots+\varkappa(n_0-1)q+[\varkappa(n_0)+1]\}/q^{n_0}$$

and q^{n_0} would be divisible by p.

It follows that there is an integer r, $1\leqslant r\leqslant s$, such that

$$0<\varkappa(n_0+jr)<q\quad\text{for}\quad j=1,2,\ldots$$

Now let $\alpha_i, \beta_i, \gamma_i$ $(i=1,2)$ be the same functions that we considered in the proof of theorem 4 and let

$$\psi(k,n)=\begin{cases}\varkappa(n)+(-1)^i & \text{if}\quad r\,|\,n-n_0,\ \beta_i(k,(n-n_0)/r)=0\\ & \quad\text{and}\ (n-n_0)/s=\gamma_i(k,(n-n_0)/r),\\ \varkappa(n) & \text{in all remaining cases,}\end{cases}$$

$$\mathfrak{a}_k=\sum_{n=1}^{\infty}\psi(k,n)/q^n.$$

It is obvious that $\{\mathfrak{a}_k\}\in C_{2q}$. We shall show that $\{\mathfrak{a}_k\}\notin C_{2p}$.

If $k\in X_1$ and $s_{1k}=\min_y[\alpha_1(k,y)=0]$, then $\beta_2(k,n)=1$ for all n and $\gamma_1(k,n)=0$ or 1 according as $n<s_{1k}$ or $n\geqslant s_{1k}$. From the definition of the function ψ it follows that

$$\psi(k,n)=\begin{cases}\varkappa(n) & \text{for}\quad n\neq n_0+s\cdot s_{1k},\\ \varkappa(n)-1 & \text{for}\quad n=n_0+s\cdot s_{1k},\end{cases}$$

and hence $\mathfrak{a}_k=1/p-1/q^{s_{1k}}$. If $k\in X_2$ and $s_{2k}=\min_y[\alpha_2(k,y)=0]$, then we find similarly that $\mathfrak{a}_k=1/p+1/q^{s_{2k}}$. If $k\notin X_1\cup X_2$, then $\mathfrak{a}_k=1/p$.

Suppose now that $\mathfrak{a}_k=\sum_{n=1}^{\infty}\lambda(k,n)/p^n$ where $\lambda(k,n)<p$. Hence

$$p\mathfrak{a}_k=\lambda(k,1)+\sum_{n=1}^{\infty}\lambda(k,n+1)/p^n$$

and we obtain $\lambda(k,1)=1$ for $k\in X_2$, $\lambda(k,1)=0$ for $k\in X_1$. It follows that the function λ cannot be recursive.

In order to express conveniently the content of theorems 3 and 5 we denote by $p_1,p_2,\ldots$ the sequence of primes and by Z_p the set of j's such that $p_j|p$. The class of all sets Z_p is identical with the class of all finite sets of integers. From theorems 3 and 5 we obtain the following

COROLLARY I. *The family of classes C_{2^p} ordered by the relation of inclusion is similar to the class of all sets Z_p ordered by the same relation.*

Another corollary from theorems 1 and 5 is

COROLLARY II. *$C_{2p}\neq C_3$ for each $p>1$.*

Indeed, $C_3\subset C_{2p}$ for each $p>1$ and no class C_{2p_0} is contained in the common part of classes C_{2p}, $p=2,3,\ldots$

THEOREM 6. *$C_4\neq C_3\neq C_5$.*

Proof. Let σ be a recursive function such that the set

$$Z=\underset{k}{E}\sum_n[\sigma(k,n)=0]$$

is non-recursive [6]) and, for $i=1,2$, let

$$\mathfrak{a}_k^{(i)}=\begin{cases}1/2 & \text{for}\quad k\notin Z,\\ 1/2+(-1)^i/2^s & \text{for}\quad k\in Z,\ s=\min[\sigma(k,n)=0].\end{cases}$$

The assumption that $\{\mathfrak{a}_k^{(1)}\}\in C_5$ leads to a contradiction since

$$k\in Z\equiv\{\mathfrak{a}_k^{(1)}<1/2\}\equiv\{\zeta(1,2,k)=1\}.$$

Similarly $\{\mathfrak{a}_k^{(2)}\}\in C_4$ is impossible since we should then have

$$k\in Z\equiv\{\mathfrak{a}_k^{(2)}>1/2\}\equiv\{\vartheta(1,2,k)=1\}.$$

It remains to prove that $\{\mathfrak{a}_k^{(i)}\}\in C_3$ for $i=1,2$. We put

$$\gamma_1(1,2m)=m-1,\quad \gamma_1(n,2m)=2m-1\quad\text{for}\quad n>1,$$
$$\gamma_2(1,2m)=m,\quad \gamma_2(n,2m)=0\quad\text{for}\quad n>1,$$
$$\gamma_i(n,2m+1)=m\quad\text{for}\quad n=1,2,\ldots\ \text{and}\ i=1,2.$$

It is obvious that

$$1/2=\sum_{n=1}^{\infty}\gamma_i(n,p)/p^n\quad\text{for}\quad i=1,2.$$

We shall show that there exist primitive recursive functions $\lambda_i(n,p,s)$ such that for $i=1,2$

(1) $$1/2+(-1)^i/2^s=\sum_{n=1}^{\infty}\lambda_i(n,p,s)/p^n,\quad 0\leqslant\lambda_i(n,p,s)<p,$$

(2) $$\text{if}\quad n<(s-1)/\lg_2 p,\quad\text{then}\quad \lambda_i(n,p,s)=\gamma_i(n,p).$$

In order to prove this we distinguish two cases:

Case I. $p=2m$. We put $1/2^s=\sum_{n=1}^{\infty}\beta(n,p,s)/p^n$ where β is primitive recursive and $\beta(n,p,s)\neq 0$ for infinitely many n. If n_0 is the least n for which $\beta(n,p,s)\neq 0$, then $n_0>s/\lg_2 p$. On putting

$$\lambda_i(n,p,s)=\gamma_i(n,p)+(-1)^i\beta(n,p,s)$$

we obtain functions satisfying (1) and (2). The verification is immediate; we remark only that the inequality $0\leqslant\lambda_i(1,p,s)<p$ follows from the fact that $\beta(1,2m,s)\leqslant m-1$, since otherwise we should have $1/2^s>\beta(1,2m,s)/2m\geqslant m/2m=1/2$.

[6]) The existence of such a function has been established by Specker [10].

Case II. $p=2m+1$. Evidently there are primitive recursive functions λ_i satisfying (1). We show that (2) is automatically satisfied. Indeed, if n_0 is the least integer for which the conclusion of (2) fails, then

$$1/2^s=\Big|\sum_{n=1}^{\infty}[\lambda_i(n,p,s)-\gamma_i(n,p)]/p^n\Big|$$
$$\geqslant|m-\lambda_i(n_0,p,s)|/p^{n_0}-\sum_{n=n_0+1}^{\infty}|m-\lambda_i(n,p,s)|/p^n$$
$$\geqslant 1/p^{n_0}-(m/p^{n_0+1})\sum_{n=0}^{\infty}1/p^n=1/2p^{n_0}$$

whence $n_0\geqslant(s-1)/\lg_2 p$.

Now we put

$$(3)\qquad \xi_i(n,p,k)=\begin{cases}\gamma_i(n,p) & \text{if } \prod\limits_{u\leqslant n\lg_2 p+1}[\sigma(k,u)\neq 0],\\ \lambda_i(n,p,s) & \text{if } \sum\limits_{u\leqslant n\lg_2 p+1}[\sigma(k,u)=0] \text{ and } s=\min[\sigma(k,u)=0].\end{cases}$$

If $k\notin Z$, then $\xi_i(n,p,k)=\gamma_i(n,p)$ for all n and hence

$$(4)\qquad \sum_{n=1}^{\infty}\xi_i(n,p,k)/p^n=\alpha_k^{(i)}.$$

If $k\in Z$ and $s_k=\min\limits_u[\sigma(k,u)=0]$, then according to (3) $\xi_i(n,p,k)=\gamma_i(n,p)$ for $s_k>n\lg_2 p+1$. Formula (2) proves that for these values of n we have also $\gamma_i(n,p)=\lambda_i(n,p,s_k)$. Hence

$$\xi_i(n,p,k)=\lambda_i(n,p,s_k).$$

The same equation holds for the remaining values of n as we immediately see from (3). Formula (1) proves therefore that (4) holds also in the present case. From (4) we immediately obtain $\{\alpha_k^{(i)}\}\in C_3$ and theorem 6 is thus proved.

THEOREM 7. $C_4-C_5\neq 0\neq C_5-C_4$.

Proof. Let ϱ be a primitive recursive function such that the set

$$Z=\mathop{E}_{k}\sum_{x_0}\prod_{x>x_0}[\varrho(k,x)=0]$$

belongs to the class $P_2^{(1)}$ but not to the class $Q_2^{(1)}$ and that $\varrho(k,x)\leqslant 1$ for all k and x [7]. Put

$$\alpha_k=\sum_{n=1}^{\infty}\varrho(k,n)/n!.$$

[7] The existence of a function ϱ with these properties has been established by Markwald [2] and Mostowski [5].

We shall show that $\{\alpha_k\}\in C_5$. Indeed,

$$\{p/q\leqslant\alpha_k\}\equiv\Big\{p/q\leqslant\sum_{n=1}^{\infty}\varrho(k,n)/n!\Big\}$$
$$\equiv p(q-1)!\leqslant\sum_{n=1}^{q}(q!/n!)\varrho(k,n)+\sum_{n=q+1}^{\infty}(q!/n!)\varrho(k,n).$$

Since

$$\sum_{n=q+1}^{\infty}(q!/n!)\varrho(k,n)\leqslant q![1/(q+1)!+1/(q+2)!+\ldots]$$
$$=\big(1/(q+1)\big)[1+1/(q+2)+\ldots]<\big(1/(q+1)\big)\sum_{n=0}^{\infty}1/(q+1)^n=1/q\leqslant 1,$$

we obtain

$$\{p/q\leqslant\alpha_k\supset p(q-1)!\}\leqslant\Big\{\sum_{n=1}^{q}(q!/n!)\varrho(k,n)\Big\}.$$

The converse implication being obvious, we obtain the equivalence

$$\{p/q\leqslant\alpha_k\}\equiv\Big\{p(q-1)!\dot{-}\sum_{n=1}^{q}(q!/n!)\varrho(k,n)=0\Big\}$$

and hence $\{p/q>\alpha_k\}\equiv\{\zeta(p,q,k)=1\}$ where

$$\zeta(p,q,k)=\operatorname{sgn}\Big[p(q-1)\dot{-}\sum_{n=1}^{q}(q!/n!)\varrho(k,n)\Big].$$

If $\{\alpha_k\}$ were in C_4, we should have the equivalence

$$\{\alpha_k \text{ is rational}\}\equiv\sum_{p,q}[(p/q\leqslant\alpha_k)(p/q\geqslant\alpha_k)]\equiv\sum_{p,q}[\zeta(p,q,k)=\vartheta(p,q,k)=0].$$

Since $k\in Z$ if and only if α_k is rational, the above equivalence would prove that Z is a recursively enumerable set. Hence $\{\alpha_k\}\in C_5-C_4$.

Now we put $\alpha_k'=1-\alpha_k$. Since

$$\{p/q<\alpha_k'\}\equiv\{(q-p)/q>\alpha_k\}\equiv\{\zeta(q-p,q,k)=1\},$$

we obtain $\{\alpha_k'\}\in C_4$. On the other hand $\{\alpha_k'\}\notin C_5$ since $\{k\in Z\}\equiv\{\alpha_k' \text{ is rational}\}$.

Theorem 7 is thus proved.

3. Let $\alpha_k=\sum\limits_{n=1}^{\infty}f(k,n)/p^n$ and let g and h be functions such that $\{p/q<\alpha_k\}\equiv\{g(p,q,k)=1\}$ and $\{p/q>\alpha_k\}\equiv\{h(p,q,k)=1\}$. We know from theorems proved in sections 1 and 2 that if $\{\alpha_k\}\in C_1$, then the functions f,g,h do not need to be general recursive. We shall prove here two theorems which characterize to a certain extent the nature of these functions.

THEOREM 8 [8]. *$\{\alpha_k\} \in C_1$ if and only if the ternary relations R_+ and R_- defined by means of the equivalences*

$$R_\pm(p,q,k) \equiv p/q \gtrless \alpha_k$$

are recursively enumerable.

Proof. If $|\alpha_k - \varphi(n,k)/n| < 1/n$, then

$$R_\pm(p,q,k) \equiv \sum_n [p/q \gtrless \varphi(n,k)/n \pm 1/n],$$

and hence relations $R_\pm$ are recursively enumerable.

Assume now that

$$R_\pm(p,q,k) \equiv \sum_n A_\pm(p,q,k,x)$$

where $A_\pm$ are recursive relations.

We shall construct a general recursive sequence of intervals $I_{nk} = \langle \varphi_k(n)/3^n, \psi_k(n)/3^n \rangle$ whose lengths tend recursively to 0 and such that $\alpha_k \in I_{nk}$ for each n and k.

The plan of this construction is as follows: We start with the interval $I_{ok} = \langle 0,1 \rangle$. Let us now assume that I_{nk} is defined for a value of n. We subdivide it into three equal parts $I_{nk}^{(1)} = \langle p_1/q_1, p_2/q_2 \rangle$, $I_{nk}^{(2)} = \langle p_2/q_2, p_3/q_3 \rangle$, $I_{nk}^{(3)} = \langle p_3/q_3, p_4/q_4 \rangle$ where $p_1/q_1 < p_2/q_2 < p_3/q_3 < p_4/q_4$ and determine the least integer $x = \Phi_k(n+1)$ for which

$$A_-(p_2,q_2,k,x) \vee A_+(p_3,q_3,k,x).$$

Such an integer x exists since if $\alpha_k < p_3/q_3$, then $R_+(p_3,q_3,k)$ is true and if $\alpha_k \geqslant p_3/q_3$, then $R_-(p_2,q_2,k)$ is true. Hence $\Phi_k(n+1)$ can be defined by means of an effective min-operation and $\Phi_k(n+1)$ is a (general) recursive function of k and n. It is now sufficient to take $I_{k,n+1} = I_{kn}^{(1)} \cup I_{kn}^{(2)}$ or $I_{k,n+1} = I_{kn}^{(2)} \cup I_{kn}^{(3)}$ according as $A_+(p_3,q_3,k,\Phi_k(n+1))$ or $A_-(p_2,q_2,k,\Phi_k(n+1))$ is true.

We give now an exact proof. We define by induction three functions φ, ψ, Φ as follows:

$$\varphi_k(0) = 0, \qquad \psi_k(0) = 1, \qquad \Phi_k(0) = 0,$$

(5) $$\Phi_k(n+1) = (\min_x)\left[A_-\big(2\varphi_k(n)+\psi_k(n), 3^{n+1}, k, x\big) \vee A_+\big(\varphi_k(n)+2\psi_k(n), 3^{n+1}, k, x\big)\right],$$

$$\varphi_k(n+1) = \begin{cases} 2\varphi_k(n)+\psi_k(n), \\ 3\varphi_k(n), \\ 2\varphi_k(n)+\psi_k(n), \end{cases} \qquad \psi_k(n+1) = \begin{cases} 3\psi_k(n), \\ \varphi_k(n)+2\psi_k(n), \\ \varphi_k(n)+2\psi_k(n), \end{cases}$$

[8] This theorem is related to but not identical with a theorem of Myhill [6].

according as

(6) $$A_-\big(2\varphi_k(n)+\psi_k(n), 3^{n+1}, k, \Phi_k(n+1)\big) \text{ and non-}A_+\big(\varphi_k(n)+2\psi_k(n), 3^{n+1}, k, \Phi_k(n+1)\big),$$

(7) $$\text{non-}A_-\big(2\varphi_k(n)+\psi_k(n), 3^{n+1}, k, \Phi_k(n+1)\big) \text{ and } A_+\big(\varphi_k(n)+2\psi_k(n), 3^{n+1}, k, \Phi_k(n+1)\big),$$

(8) $$A_-\big(2\varphi_k(n)+\psi_k(n), 3^{n+1}, k, \Phi_k(n+1)\big) \text{ and } A_+\big(\varphi_k(n)+2\psi_k(n), 3^{n+1}, k, \Phi_k(n+1)\big).$$

This kind of definitions does not lead outside the class of general recursive functions if the min-operation in (5) is effective, *i. e.*, if for each n there is an x satisfying the condition given in (5). In order to show this we first prove that

(9) $$\varphi_k(n)/3^n < \alpha_k < \psi_k(n)/3^n.$$

For $n=0$ these inequalities are evident. Let us assume their validity for an n and consider the intervals

$$I_{nk}^{(1)} = \underset{u}{E}\{3\varphi_k(n)/3^{n+1} < u \leqslant [2\varphi_k(n)+\psi_k(n)]/3^{n+1}\},$$

$$I_{nk}^{(2)} = \underset{u}{E}\{[2\varphi_k(n)+\psi_k(n)]/3^{n+1} < u \leqslant [\varphi_k(n)+2\psi_k(n)]/3^{n+1}\},$$

$$I_{nk}^{(3)} = \underset{u}{E}\{[\varphi_k(n)+2\psi_k(n)]/3^{n+1} < u < 3\psi_k(n)/3^{n+1}\}.$$

α_k is in one of these intervals. If it is in $I_{nk}^{(1)}$, then we have case (7) and hence $\varphi_k(n+1) = 3\varphi_k(n)$, $\psi_k(n+1) = \varphi_k(n)+2\psi_k(n)$, which proves that (9) is true for the number $n+1$. If α_k is in $I_{nk}^{(2)}$, then we have case (8) and we prove similarly that (9) is true for the number $n+1$. Finally if α_k is in $I_{nk}^{(3)}$, then (6) is satisfied and we again obtain (9) for the number $n+1$.

Since

$$\varphi_k(n+1) = 2\varphi_k(n)+\psi_k(n) \quad \text{or} \quad \psi_k(n+1) = \varphi_k(n)+2\psi_k(n),$$

the formula $\varphi_k(n+1)/3^{n+1} < \alpha_k < \psi_k(n+1)/3^{n+1}$ proves that

$$R_-\big(2\varphi_k(n)+\psi_k(n), 3^{n+1}, k\big) \quad \text{or} \quad R_+\big(\varphi_k(n)+2\psi_k(n), 3^{n+1}, k\big).$$

Hence the min-operation in (5) is effective and Φ, φ, ψ are general recursive.

From the inductive definitions of these functions we obtain

$$0<\psi_k(n+1)-\varphi_k(n+1)<2[\psi_k(n)-\varphi_k(n)],$$

whence $0<\psi_k(n)-\varphi_k(n)<2^n$ and finally

$$|\alpha_k-\varphi_k(n)/3^n|<(2/3)^n.$$

According to lemma 1 this formula proves that $\{\alpha_k\}\in C_1$. Theorem 8 is thus proved.

THEOREM 9. *If* $\{\alpha_k\}\in C_1$, *then the relation* R *defined by means of the equivalence*

$$R(m,n,p,k)\equiv\{m=[p^n\alpha_k]\}$$

belongs to the smallest field of sets generated by $P_1^{(4)}$.

Proof. If $|\alpha_k-\varphi(n,k)/n|<1/n$, then

$$R(m,n,p,k)\equiv\{m\leqslant p^n\alpha_k<m+1\}\equiv\Big\{\sum_y\big[p^n(\varphi(y,k)+1)<y(m+1)\big]\Big\}\cdot\cdot\prod_y\big[p^n(\varphi(y,k)+1)\geqslant ym\big]\Big\}.$$

From theorem 9 it follows that if $\{\alpha_k\}\in C_1$, then $\alpha_k=\sum_{n=1}^{\infty}f(k,n)/p^n$ where f is a function obtainable by multiplications and subtractions from functions whose graphs belong to the smallest field of sets generated by recursively enumerable sets.

4. Let C_1^0 be the class of primitive recursive real numbers α $(0<\alpha<1)$ (cf. Specker [10] and Péter [7], p. 185 seq.), C_{2p}^0 the class of real numbers which possess a primitive recursive development $\alpha=\sum_{n=1}^{\infty}\psi(n)/p^n$ $(0\leqslant\psi(n)<p)$, C_3^0 the class of real numbers α such that $\alpha=\sum_{n=1}^{\infty}\xi(n,p)/p^n$ $(0\leqslant\xi(n,p)<p)$ for each $p>1$. Finally let C_4^0 and C_5^0 be classes of real numbers α $(0<\alpha<1)$ such that relations $p/q\lesseqgtr\alpha$ are primitive recursive.

THEOREM 10. $C_1^0\supsetneqq C_{2p}^0\supsetneqq C_4^0=C_5^0=C_3^0$.

The first two inclusions were proved by Specker for $p=10$. Changing slightly his construction we obtain a proof valid for an arbitrary $p>1$. Since the next equation is evident, it remains only to prove that $C_3^0=C_5^0$. If $\alpha=p/q$, then $\alpha\in C_3^0$ and $\alpha\in C_4^0$, we can therefore assume that α is irrational. The development of α on the scale p is given by the formula $\alpha=\sum_{n=1}^{\infty}\xi(n,p)/p^n$ where $\xi(1,p)=[p\alpha]$ and $\xi(n,p)=[p^n\alpha]-p[p^{n-1}\alpha]$ for $n>1$. If $\alpha\in C_5$, then $[p^n\alpha]$ is a primitive recursive function of p and n and hence $\alpha\in C_3$. If $\alpha\in C_3$, then $\xi(1,p)$ is a primitive recursive function and hence $p/q>\alpha$ is a primitive recursive relation since $\{p/q>\alpha\}\equiv\{p>[q\alpha]\}\equiv\{p>\xi(1,q)\}$.

THEOREM 11. *If* $p,q>1$ *and a power of* q *is divisible by* p, *then* $C_{2q}^0\subset C_{2p}^0$.

In order to prove this theorem we repeat the proof of theorem 3 supressing the argument k in all the functions considered and assuming ψ to be primitive recursive.

It remains an open question whether a theorem converse to theorem 11 is also true. Another open question is whether C_3 is the common part of C_{2p} $(p=2,3,\ldots)$ and whether C_3^0 is the common part of $C_{2p}^0(p=2,3,\ldots)$.

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