On the existence of free subgroups in topological groups

by

S. Balcerzyk (Toruń) and Jan Mycielski (Wrocław)

1. In this paper we prove the existence of free algebraic subgroups in some topological groups. Our chief results (theorems 2 and 3) state the existence of a free subgroup of the rank $2^{\aleph_0}$ in every compact, connected, non-Abelian group and in every locally compact, connected non-solvable group [1]).

These problems arise in connection with some special constructions on the sphere (see *e. g.* [12], [13]). W. Sierpiński [17] has proved that the group of rotations of the sphere contains a free subgroup of the rank $2^{\aleph_0}$. His proof is effective (*i. e.*, does not use the axiom of choice). This effective method is further developed by J. de Groot [5]. M. Kuranishi [10] proves the existence of a free subgroup of rank 2 in every semi-simple connected Lie group. Clearly our results generalizes all these theorems, but we are using the axiom of choice and the result of Kuranishi. A part of our reasoning (the proofs of Theorem 1 and Lemma 1) is analogous to a proof of the above mentioned theorem of Sierpiński given by J. de Groot and T. Dekker [6].

The authors are indebted to prof. J. Łoś for some general ideas of this paper.

2. Notions and notations. The *rank* of a free group F [of an Abelian free group F] is the cardinal number of a set of free generators of F and is, as we know, uniquely determined by F.

A set of free generators of a subgroup of a group G is called a *free set* in G.

The group generated by all the commutators in a group G is called the *commutant* of G.

The closure of the commutant of a topological group is called its *closed commutant* (it is normal because the closure of a normal subgroup is normal).

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[1]) These results have been announced in [2].

A group G is said to be *algebraically solvable* if the sequence of different consecutive commutants in G is finite and ends on $\{1\}$ [2]).

A topological group G is said to be *solvable* if the sequence of different consecutive closed commutants in G is finite and ends on $\{1\}$.

Note that a topological group is solvable in this sense if and only if it is solvable algebraically (see [4], p. 98), and that the closure of a solvable subgroup of a topological group is a solvable group.

A topological group is said to be *semi-simple* if it has no closed solvable normal subgroups except $\{1\}$.

A *word* is a function on a group of the form

$$\sigma(x_1,\dots,x_n)=x_{s_1}^{k_1}x_{s_2}^{k_2}\dots x_{s_m}^{k_m}$$

where $s_i\in\{1,2,\dots,n\}$, $k_i=\pm1$, $x_{s_i}^{k_i}\neq x_{s_{i+1}}^{-k_{i+1}}$, and $n,m\geqslant 1$.

A group G is said to be *functionally free* if for every word $\sigma(x,y)$ there exist such elements $\xi,\eta\in G$, that $\sigma(\xi,\eta)\neq 1$ (this implies for every word $\sigma(x_1,\dots,x_n)$ the existence of such $\xi,\eta\in G$, that $\sigma(\xi\eta,\xi^2\eta^2,\dots,\xi^n\eta^n)\neq 1$).

(2.1) A functionally free group is not solvable.

In fact let $x_{r_1,\dots,r_n}$ be some variables with indices $r_i=0$ or 1, and let $\varphi_n(x_{\underbrace{0,\dots,0}_{n}},\dots,x_{\underbrace{1,\dots,1}_{n}})$ be words defined by

$$\varphi_1(x_0,x_1)=x_0x_1x_0^{-1}x_1^{-1},$$

$$\varphi_{n+1}=\varphi_1\big(\varphi_n(x_{0,\dots,0,0},\dots,x_{1,\dots,1,0}),\ \varphi_n(x_{0,\dots,0,1},\dots,x_{1,\dots,1,1})\big).$$

If G is solvable and n is the number of different consecutive commutants in G, then $\varphi_n=1$ for every substitution from G.

Let us consider functions of the form

(2.2) $$\tau(x)=a_0x^{k_1}a_1x^{k_2}a_2\dots x^{k_m}a_m$$

where $a_i\in G$, k_i are integers $\neq 0$, and the following property of a group G:

(A) *If, for a function τ and a (non empty) set V open in G, $\tau(x)=1$ for every $x\in V$, then $\tau(x)=1$ for every $x\in G$.*

It was proved ([14], Theorem 1) that

(2.3) Connected locally compact groups have property (A).

3. A general theorem. Our theorem concerning the most general class of groups is the following

THEOREM 1. *Every topological group G with property* (A), *functionally free and such that G and $G\times G$ are not of the first category on themselves* [3]), *contains a free subgroup of the rank $\aleph_1$.*

[2]) 1 denotes the unity of the group.

[3]) See section 6 of this paper. The sign $\times$ denotes the Cartesian product.

Proof. Since G has property (A), we have

(3.1) For every word $\sigma(x_1,x_2)$, if $\sigma(\xi_1,\xi_2)=1$ for every $\xi_1\in V_1$, $\xi_2\in V_2$ where V_1 and V_2 are some open (non empty) sets in G, then $\sigma(\xi_1,\xi_2)=1$ for every $\xi_1,\xi_2\in G$ [4]).

Now we shall prove that

(3.2) There exists a free subgroup of G of the rank $\aleph_0$.

For every word $\sigma(x_1,x_2)\colon G\times G\to G$ the set

$$\{\langle\xi_1,\xi_2\rangle\colon\ \sigma(\xi_1,\xi_2)=1,\ \xi_1,\xi_2\in G\}=\sigma^{-1}(1)\subset G\times G$$

is closed. Owing to G being functionally free and by (3.1) $\sigma^{-1}(1)$ is also a border set in $G\times G$. Consequently the set $\bigcup_\sigma\sigma^{-1}(1)$, where σ runs over the denumerable set of all words with two arguments at most, is of the first category in $G\times G$. Consequently there exists a pair $\langle\xi_1,\xi_2\rangle\in G\times G\setminus\bigcup_\sigma\sigma^{-1}(1)$. Hence $\{\xi_1\xi_2,\xi_1^2\xi_2^2,\xi_1^3\xi_2^3,\dots\}$ is a free set in G, which proves (3.2).

(3.3) If M is a free set in G and $\overline{\overline{M}}=\aleph_0$, then there exists such a $\xi\in G\setminus M$ that $M\cup\{\xi\}$ is a free set.

Let Φ be the set of all functions $\varphi(x)$ of the form (2.2) ($\varphi\colon G\to G$), with $a_i\in[M]$ [5]). Then $\overline{\overline{\Phi}}=\aleph_0$. The sets $\{\xi\colon\ \varphi(\xi)=1,\ \xi\in G\}=\varphi^{-1}(1)\subset G$ are closed. For every φ, M being infinite, there exists such a $\zeta\in M$ that $a_i\in[M\setminus\{\zeta\}]$ $(i=0,1,\dots,m)$. Then, M being free, $\varphi(\zeta)\neq 1$. Hence by property (A), for every $\varphi\in\Phi$, $\varphi^{-1}(1)$ is a border set in G. Consequently $\bigcup_{\varphi\in\Phi}\varphi^{-1}(1)$ is of the first category in G and by hypothesis there exists a $\xi\in G\setminus\bigcup_{\varphi\in\Phi}\varphi^{-1}(1)$, which proves (3.3).

Clearly the statements (3.2) and (3.3) imply Theorem 1 (by means of the axiom of choice).

4. Somme lemmas on Lie groups. The lemmas proved here are special cases of the theorems given in the next section.

LEMMA 1. *A connected functionally free Lie group G contains a free subgroup of the rank $2^{\aleph_0}$* [6]).

Proof. G is a locally compact space, and by (2.3) G has the property (A). Then by Theorem 1

(4.1) G contains an infinite free set.

Now we shall prove that

(4.2) If M is a free set in G and $\aleph_0\leqslant\overline{\overline{M}}<2^{\aleph_0}$, then there exists such a $\xi\in G\setminus M$ that $M\cup\{\xi\}$ is a free set.

[4]) The proof is immediate. In this connection see [14], Theorem 1.

[5]) $[M]$ denotes the subgroup algebraically generated by M.

[6]) Prof. J. de Groot informs us that he has also obtained this result.

We consider the set Φ of all functions φ defined as in the proof of (3.3). Then $\overline{\overline{\Phi}} \leqslant 2^{\aleph_0}$. The mapping φ: $G \to G$ is analytic (*i. e.*, the local coordinates of a point $\varphi(\xi)$ are analytic functions of the local coordinates of ξ — see [14], proof of Lemma 2). As in the proof of (3.3) we verify for every φ the existence of an element $\zeta \in M$ for which $\varphi(\zeta) \neq 1$. Then the sets $\varphi^{-1}(1) = \{\xi\colon \varphi(\xi) = 1,\ \xi \in G\}$ are analytic surfaces in the analytic manifold G (see [3]). Hence $\bigcup_{\varphi \in \Phi} \varphi^{-1}(1)$ is a border set in G (by [3], Theorem 1), which proves the existence of a ξ satisfying (4.2).

Clearly (4.1) and (4.2) imply Lemma 1.

LEMMA 2. *A connected, semi-simple Lie group $G \neq \{1\}$ contains a free subgroup of the rank $2^{\aleph_0}$.*

Proof. M. Kuranishi [10] proves the existence of a free group of rank 2 in such a group G. Therefore G is functionally free and so Lemma 2 follows from Lemma 1.

5. Free groups in compact and locally compact groups. Now we state our chief theorems.

THEOREM 2. *Every compact, connected, non-Abelian group G contains a free subgroup of the rank $2^{\aleph_0}$.*

Proof. By hypothesis $\xi\eta\xi^{-1}\eta^{-1} = \tau \neq 1$ for some $\xi, \eta \in G$. Since G is compact, every neighbourhood of unity in G contains such a closed normal subgroup N that $G^* = G/N$ is a Lie group (see *e. g.* [16], p. 326). Then we can suppose that $\tau \notin N$ and consequently G^* is not Abelian. Hence we have

$$G^* = (A \times L)/F$$

(see [16], p. 479) where A is a connected compact Abelian group, L is a semi-simple connected Lie group $\neq \{1\}$, and F a finite, central, normal subgroup of $A \times L$.

By Lemma 2 the group L contains a free subgroup W of the rank $2^{\aleph_0}$. The natural mapping of $A \times L$ on G^* is an isomorphism on W because $W \cap F = \{1\}$ (the only finite subgroup of a free group is the unity group). Then G^* contains a free subgroup W^* of the rank $2^{\aleph_0}$, and consequently G also, q. e. d.

By (2.3) and Theorem 1 every connected, locally compact, functionally free group contains a free subgroup of the rank $\aleph_1$; but we shall prove more over:

THEOREM 3. *Every locally compact, connected, non-solvable group G contains a free subgroup of the rank $2^{\aleph_0}$.*

Proof. By a theorem of Gleason ([4], p. 101) G has such a closed normal solvable subgroup N, that $G' = G/N$ is semi-simple. A fundamental theorem of H. Yamabe [18] states that every locally compact group is a generalized Lie group. For the connected, locally compact group G' this means that (see *e. g.* [11], p. 175 or [8], p. 541):

Every neighbourhood V of the unity of G' contains such a closed normal subgroup N' that G'/N' is a Lie group.

We suppose that V has a compact closure. Then N' is compact. If $N' = \{1\}$ then G' is a semi-simple Lie group and the theorem holds by Lemma 2. Suppose $N' \neq \{1\}$. If N' is totally disconnected, then N', being a normal subgroup of a connected group, is central and consequently Abelian, this is not possible because G' is semi-simple. Let N^* be the component of unity in N'. Then $N^* \neq \{1\}$ and is a copmact, connected, normal subgroup of G'. Hence, G' being semi-simple, N^* is non-Abelian and, by Theorem 2, N^* contains a free subgroup of the rank $2^{\aleph_0}$.

THEOREM 4. *Every locally compact, non 0-dimensional group G contains a free Abelian subgroup of the rank $2^{\aleph_0}$.*

Proof. It is proved in [15] that G contains an infinite, connected, closed Abelian subgroup G^*. There exists a homomorphism h of G^* into the group K of rotations of the circle such that $h(G^*) \neq \{1\}$ [7]; and, G^* being connected, $h(G^*) = K$. Or K contains a free Abelian subgroup of the rank $2^{\aleph_0}$, and then G^* and G also contains such a group.

6. Remarks and problems. 1. Concerning the hypothesis of Theorem 1 let us note that we do not know whether the supposition that G is not of the first category onto itself implies the same on the product $G \times G$ [8].

2. The hypothesis "G is locally compact" in theorems 3 and 4 is necessary because there exist connected infinite groups every element of which is of order 2, and connected functionally free groups every element of which is of finite order (see [7]).

3. Theorem 3 implies that if for a connected locally compact group G there exists such a word σ that

$$\sigma(\xi_1, \ldots, \xi_m) = 1 \quad \text{for every} \quad \xi_1, \ldots, \xi_m \in G,$$

i. e., G is not functionally free, then G is solvable, *i. e.*, one of the words φ_n in (2.1) vanishes identically on G. Can we evaluate n, *i. e.*, the length of the sequence of different consecutive commutants of G by the length of σ?

[7]) By a well known theorem of Van Kampen [9], see also [16], p. 278.

[8]) A related problem was stated by A. Alexiewicz and W. Orlicz [1].

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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Some problems of definability in the lower predicate calculus

by

A. Robinson (Toronto)

1. Introduction. The present paper[1]) arose out of the consideration of the following problem.

Let M be an ordered field such that every positive element of M can be represented as a sum of squares of elements of the field and such that there exists a uniform bound to the number of squares required for the purpose. Let M' be a finite algebraic extension of M. Is there a uniform bound to the number of squares required to express a totally positive element of M' as a sum of squares of elements of M'?

It will be shown in due course (section 5, below) that the answer to this question is in the affirmative. Its investigation led to another type of problem which can be introduced conveniently by means of the following example.

Let

$$p(x)=y_0+y_1x+\ldots+y_nx^n$$

be a polynomial of the variable x where $y_0,\ldots,y_n$ are parameters which take values in the field of rational numbers, R. Then the property of $p(x)$ of possessing (or not possessing) a real root may be regarded as a predicate of its coefficients, $Q^*(y_0,\ldots,y_n)$, say. We note that, as stated, this predicate is not formulated within the field of the coefficients, R, but with reference to the more comprehensive field of real (or real algebraic) numbers, R^*. However, Sturm's test shows that there exists a predicate $Q(y_0,\ldots,y_n)$, formulated within the language of the lower predicate calculus in terms of the relation of addition, multiplication, equality, and order, such that whenever $Q(y_0,\ldots,y_n)$ holds in R, for rational $y_0,\ldots,y_n$, $Q^*(y_0,\ldots,y_n)$ holds in R^*, and conversely, whenever $Q^*(y_0,\ldots,y_n)$ holds in R^*, for rational $y_0,\ldots,y_n$, $Q(y_0,\ldots,y_n)$ holds in R.

[1]) This paper was written while the author was a Fellow of the Summer Research Institute of the Canadian Mathematical Congress, Kingston, Ontario, 1956. The author is indebted to A. H. Lightstone for suggesting a number of improvements in the presentation.