On the method of category in analytic manifolds

by

S. Balcerzyk (Toruń) and Jan Mycielski (Wrocław)

1. The following hypothesis may be considered as classical (see W. Sierpiński [4] and F. Hausdorff [3]):

(H) *The sum of fewer than $2^{\aleph_0}$ nowhere dense subsets of a complete metric space is a border set in this space.*

Clearly, if the continuum hypothesis is supposed, a positive answer follows by the theorem of Baire on the sets of the first category. Then, by the results of Gödel, (H) is consistent with the present mathematical knowledge, but it has not been proved even in the case where the space is the real line.

It is the purpose of this paper to prove (H) for some special classes of nowhere dense sets in analytic manifolds.

Several applications of theorem 1 of this paper will be given in other works. This theorem is a refinement of a lemma of J. de Groot and T. Dekker [1].

2. Analytic surfaces in analytic manifolds. All manifolds considered are supposed to be connected, real and analytic.

For two manifolds M and A, a mapping $f\colon M\to A$ is called *analytic* if the local coordinates of $f(p)$ in A are analytic functions of the local coordinates of p in M.

LEMMA. *Let f_1 and f_2 be two analytic mappings of a manifold M into a manifold A. If the set $S=\{p\colon p\in M,\ f_1(p)=f_2(p)\}$ has an interior point, then $S=M$.*

Indeed it can easily be proved that then the set S is open and closed and non-empty. Hence $S=M$ since M is connected.

Definition. A set S is called an *analytic surface* in a manifold M if there exist an open connected set $C\subset M$, a manifold A, and two analytic mappings $f_1,f_2\colon C\to A$ such that $f_1(p_0)\neq f_2(p_0)$ for some $p_0\in C$, and

$$S\subset\{p\colon p\in C,\ f_1(p)=f_2(p)\}\,.$$

THEOREM 1. *The sum of fewer than $2^{\aleph_0}$ analytic surfaces in an analytic manifold M is a border set in M.*

Proof. Let $\{S_t\}_{t\in T}$, where $\overline{\overline{T}} < 2^{\aleph_0}$, be the family of analytic surfaces in M and

$$S_t = \{p:\ p \in K_t,\ f_t(p) = g_t(p)\}$$

where f_t and g_t are analytic mappings of a connected open set $V(K_t)$ $(K_t \subset V(K_t) \subset M)$ into some manifolds A_t, and

(1) $$f_t(p_t) \neq g_t(p_t) \quad \text{for some} \quad p_t \in V(K_t).$$

The proof consists of three steps. The first two give reductions of the problem.

1. We may suppose that M is the n-dimensional Euclidean space $\mathcal{E}^n$ and K_t are closed cubes in $\mathcal{E}^n$ with faces resp. parallel to the axes of a coordinate system.

In order to prove this let us consider any sufficiently small open neighbourhood U in M such that there exists a local system of analytic coordinates in which all the points of U are expressed. Then let

$$U \cap V(K_t) = K_{t1} \cup K_{t2} \cup \ldots$$

where K_{ti} are the required closed cubes in U (such a repressntation of $U \cap V(K_t)$ exists because M is separable). By the lemma and (1) K_{ti} contains such a point p_{ti} that $f_t(p_{ti}) \neq g_t(p_{ti})$. Then the problem reduces to the surfaces $S_{ti} = \{p:\ p \in K_{ti},\ f_t(p) = g_t(p)\}$ the number of which is $< 2^{\aleph_0}$.

2. We may suppose that A_t is the real line (for every $t \in T$).

This may be proved as follows:

We take a fixed $t \in T$. A_t being separable, there exists such a denumerable system of open sets $U_1, U_2, \ldots \subset A_t$ that

(2) $$f_t(S_t) = g_t(S_t) \subset \bigcup_{i=1}^{\infty} U_i$$

and in every U_i there exists a local system of analytic coordinates $\xi_i^1, \xi_i^2, \ldots, \xi_i^n$ in which all the points of U_i are expressed. The sets

(3) $$V_i = f_t^{-1}(U_i) \cap g_t^{-1}(U_i)$$

are open in M, and consequently every V_i may be decomposed:

(4) $$V_i = \bigcup_{j=1}^{\infty} V_{ij}$$

where

(5) V_{ij} are open connected subsets of M.

By the lemma and (1) for every non-empty V_{ij} there exists such a $p_{ij} \in V_{ij}$ that $f_t(p_{ij}) \neq g_t(p_{ij})$.

Put $f_t = [f_i^1, \ldots, f_i^n]$, $g_t = [g_i^1, \ldots, g_i^n]$ where f_i^k and g_i^k are the coordinates of f_t and g_t on the axis $\{\xi_i^k\}$ (for the partial functions $f_t|V_i$, $g_t|V_i$). Then for every p_{ij} there exists such a k_{ij} that

(6) $$f_i^{k_{ij}}(p_{ij}) \neq g_i^{k_{ij}}(p_{ij}).$$

We put

$$S_{tij} = \{p:\ p \in V_{ij},\ f_i^{k_{ij}}(p) = g_i^{k_{ij}}(p)\}$$

where V_{ij}, $f_i^{k_{ij}}$, $g_i^{k_{ij}}$ depend of course on t.

By (2), (3), (4) $S_t \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} S_{tij}$. By (5), (6) the sets S_{tij} are analytic surfaces in M and their number is $< 2^{\aleph_0}$. Then the problem reduces to the surfaces S_{tij}, and the functions $f_i^{k_{ij}}$, $g_i^{k_{ij}}$ are real valued.

3. On account of preceding steps we can suppose that $M = \mathcal{E}^n$, $A_t = \mathcal{E}$, K_t are n-dimensional closed cubes with faces parallel to the axes of coordinates in $\mathcal{E}^n$, and $h_t = f_t - g_t$ are real valued functions defined and analytic in K_t non-vanishing identically.

$n = 1$. An analytic function non-vanishing identically has at most denumerably many zeros; then $\overline{\overline{\bigcup_{t\in T} S_t}} < 2^{\aleph_0}$. Hence $\bigcup_{t\in T} S_t$ is a border set.

Let us suppose that the theorem is true for some $\mathcal{E}^m$ and put $n = m+1$.

Let $V \subset \mathcal{E}^n$ be an open non-empty set.

For any real x we put

(7) $$P_x = \{p:\ p \in \mathcal{E}^n,\ p = [\xi^1, \ldots, \xi^m, x]\};$$

then

(8) $P_x \cap K_x$ is a closed cube in P_x or is empty.

There exists an x_0 which fulfils the conditions:

(9) $$P_{x_0} \cap V \neq \emptyset.$$

(10) If $t \in T$ and $P_{x_0} \cap K_t \neq \emptyset$, there exists such a $p_t \in P_{x_0} \cap K_t$ that $h_t(p_t) \neq 0$.

In fact, the set

$$X_t = \{x:\ h_t(P_x \cap K_t) = (0)\}$$

is finite, because in otherwise it would have a limit point, and we should have $h_t(K_t) = (0)$, contrary to the assumption. Consequently $\overline{\overline{\bigcup_{t\in T} X_t}} < 2^{\aleph_0}$ and there exists a number

$$x_0 \in \{x:\ P_x \cap V \neq \emptyset\} \setminus \bigcup_{t\in T} X_t,$$

and this x_0 satisfies (9) and (10).

We put
$$S'_t = \{p:\ p \in P_{x_0} \cap K_t,\ h_t(p) = 0\}.$$

Then by (8) and (10) the set S'_t is an analytic surface in $P_{x_0} = \mathcal{E}^m$. Hence by the inductive hypothesis $\bigcup_{t \in T} S'_t$ is a border set in P_{x_0}. Then by (9)
$$P_{x_0} \cap V \not\subset \bigcup_{t \in T} S'_t = P_{x_0} \cap \bigcup_{t \in T} S_t,$$

i. e., $V \not\subset \bigcup_{t \in T} S_t$, which proves the theorem.

3. Convex surfaces in Euclidean spaces. The border of any convex set in the n-dimensional Euclidean space $\mathcal{E}^n$ is called a *convex surface.*

THEOREM 2. *The sum of fewer than $2^{\aleph_0}$ convex surfaces in $\mathcal{E}^n$ is a border set in $\mathcal{E}^n$.*

Proof. Let $\{S_t\}_{t \in T}$, where $\overline{\overline{T}} < 2^{\aleph_0}$, be the family of convex surfaces. It can easily be verified that for every non-empty open set $V \subset \mathcal{E}^n$ there exists such an x_0 that $P_{x_0} \cap V \neq \emptyset$ (see (7)) and $P_{x_0} \cap S_t$ is a convex surface in P_{x_0}, or is empty, for every $t \in T$. Then the proof follows by an easy induction with respect to n analogous to the step 3 of the proof of theorem 1.

4. Remarks and problems. 1. Analyzing the proof of theorems 1 and 2 it is easy to verify that *the sum of fewer than $2^{\aleph_0}$ surfaces in $\mathcal{E}^n$, each of them being convex or analytic, is a border set in $\mathcal{E}^n$.* A more natural generalization would be of great interest. *E. g.*, we do not know whether the sum of fewer than $2^{\aleph_0}$ simple closed curves in the plane must be a border set.

2. Our theorems concern only finite dimensional spaces. Generalizations for the infinite dimensional case are also needed. *E. g.*, the following problem may perhaps be treated by such a generalized category method:

Let Q be a set of continuous real functions and $\overline{\overline{Q}} < 2^{\aleph_0}$. Does there exist such a continuous function f that
$$\frac{df}{dg}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{g(x+h) - g(x)}$$
does *not* exist for every x and every $g \in Q$?

This problem remains open. A positive answer would imply (by means of the axiom of choice) the existence of a system of $2^{\aleph_0}$ mutually non-differentiable functions. This has been proved in another (constructive) way by J. de Groot [2].

References

[1] J. de Groot and T. Dekker, *Free subgroups of the orthogonal group*, Compositio Mathematica 12 (1954), p. 134-136.

[2] J. de Groot, *A system of continuous, mutually non-differentiable functions*, Math. Zeitschr. 64 (1956), p. 192-194.

[3] F. Hausdorff, *Summen von $\aleph_1$ Mengen*, Fund. Math. 26 (1936), p. 241-255.

[4] W. Sierpiński, *Probl. 6*, Fund. Math. 1 (1920), p. 224.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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