On the relations between Smith operations and Steenrod powers

by

Wu Wen-tsün (Peking)

Let K be a complex, I_p the field of integers mod p, p being a prime. The so-called Steenrod powers [4]

$$\mathrm{St}^k_{(p)} = \mathrm{St}^k\colon\ H^r(K, I_p) \to H^{r+k}(K, I_p)$$

are deduced from the consideration of the pth power $K = \underbrace{K \times \ldots \times K}_{p}$ under the cyclic transformation [1] $t(a_1, \ldots, a_p) = (a_p, a_1, \ldots, a_{p-1})$, $a_i \in \overline{K}$. On the other hand, from K^p under the transformation t, we may introduce in a natural manner according to the theory of P. A. Smith [3], [2] a system of homomorphisms

$$\mathrm{Sm}_k^{(p)} = \mathrm{Sm}_k\colon\ H_r(K, I_p) \to H_{r-k}(K, I_p).$$

The question what relations exist between the Smith operations Sm_k and the Steenrod powers St^k naturally arises. The author discovered formerly [6] that these two systems of operations are actually equivalent, in the sense that one is determined by the other, and found the mode of their mutual determination. This furnishes a more natural and simpler definition of Steenrod powers and makes it directly connected with the theory of Smith. However, the original proof of the author depends on the intrinsic axiomatic theory of Steenrod powers of Thom [5], which is quite complicated. We need therefore a direct proof without the use of Thom's theory, which is the object of the present paper.

§ 1. The definition of Sm^r

Let K be a finite simplicial complex, $K^p = \underbrace{K \times \ldots \times K}_{p}$ its pth power subdivided as a product complex, $t\colon \overline{K}^p \to \overline{K}^p$ the transformation defined by $t(a_1, \ldots, a_p) = (a_p, a_1, \ldots, a_{p-1})$, $a_i \in \overline{K}$, where p is a fixed prime. Let $\Delta\colon \overline{K} \to \overline{K}^p$ be the diagonal map, then $\Delta(\overline{K})$ may be subdivided as K,

[1] For a complex K, $\overline{K}$ means the space of K.

and the complex isomorphic to K thus obtained will be denoted by $\Delta(K)$. Take a subdivision $\widetilde{K}^p$ of K^p such that $\Delta(K)$ is a subcomplex of $\widetilde{K}^p$ and that t is a cell map of $\widetilde{K}^p$. Let ω be the corresponding chain mapping. Then for any $x_i \in C_{r_i}(K)$ we have

$$t\omega(x_1 \otimes \ldots \otimes x_p) = (-1)^{r_p(r_1+\ldots+r_{p-1})}\omega(x_p \otimes x_1 \otimes \ldots \otimes x_{p-1}),$$

where t denotes also the chain mapping induced by t in $\widetilde{K}^p$. Put as usual $d = 1 - t$, $s = 1 + t + \ldots + t^{p-1}$: then for any $x \in Z_q(K, I_p)$, $\omega x^p = \omega(\underbrace{x \otimes \ldots \otimes x}_{p})$ is always a d-cycle (*i. e.*, a cycle z with $dz = 0$) and according to P. A. Smith [3], [2] we have a sequence (we shall call it the *sequence associated with* x):

$$\begin{aligned} \omega x^p &= s x_0 + x_0', \\ \partial x_{2i-1} &= s x_{2i} + x_{2i}', \qquad i > 0, \\ \partial x_{2i} &= d x_{2i+1} + x_{2i-1}', \qquad i \geqslant 0, \end{aligned} \tag{1}$$

in which $x_j' \subset \Delta(K)$, $x_j \subset \widetilde{K}^p - \Delta(K)$, and

$$\dim x_j = \dim x_j' = pq - j. \tag{2}$$

Put $\bar{d} = 1 + 2t + 3t^2 + \ldots + (p-1)t^{p-2}$, then for coefficients mod p we have for the operations $\bar{d}$, $d\bar{d} = s$. If we set

$$s_i = \begin{cases} s & \text{for} \quad i \text{ even}, \\ d & \text{for} \quad i \text{ odd}, \end{cases}$$

then (1) may also be written as

$$\begin{aligned} \omega x^p &= s x_0 + x_0', \\ \partial x_i &= s_{i+1} x_{i+1} + x_{i+1}', \qquad i \geqslant 0. \end{aligned} \tag{1'}$$

Definition. $S_k x = \Delta^{-1} x'_{k+(p-1)q} \in Z_{q-k}(K, I_p)$.

Remark 1. The cycle $S_k x$ is completely determined by x (as well as the subdivision $\widetilde{K}^p$) and lies in the smallest complex $|x|$ determined by x. It follows that $x_j' = 0$ for $j < (p-1)q$, and $S_k x$ have a meaning only for $0 \leqslant k \leqslant q$. In particular, we have $S_0 x = x$ for $q = 0$.

Lemma. *S_k: $Z_q(K, I_p) \to Z_{q-k}(K, I_p)$ is a homomorphism such that*

$$S_k[B_q(K, I_p)] \subset B_{q-k}(K, I_p), \qquad k < q, \tag{3}$$

$$S_q[Z_q(K, I_p)] \subset B_0(K, I_p), \qquad q > 0. \tag{4}$$

Proof. Consider any $x, y \in Z_q(K, I_p)$. We have

$$\omega(x+y)^p - \omega x^p - \omega y^p = sz,$$

where $z=\omega(x\otimes y\otimes\ldots\otimes y+\ldots)$ is a cycle $\subset\widetilde{K}^p-\Delta(K)$. Hence the sequence associated with $x+y$ will be obtained by adding the corresponding equations of the sequences associated with x and y. We have therefore $S_k(x+y)=S_k(x)+S_k(y)$. Similarly we have $S_k(-x)=(-1)^pS_kx=-S_kx$ (coefficient group I_p). Hence S_k is a homomorphism.

Let σ be a $(q+1)$-dimensional simplex of K: then $S_k(\partial\sigma)$ is a cycle in the complex determined by σ and hence for $k<q$, $S_k(\partial\sigma)\sim 0$ in this complex, *a fortiori* ~ 0 in K. As S_k is already known to be a homomorphism, we get (3).

Let $x\in Z_q(K,I_p)$, $q>0$. By (1') we get

$$S_qx=\Delta^{-1}x'_{pq}\in C_0(K,I_p)\quad\text{ and }\quad \partial x_{pq-1}=s_{pq}x_{pq}+x'_{pq}.$$

Hence $KI(x'_{pq})=KI(-s_{pq}x_{pq})=0 \bmod p$ and we have $\Delta^{-1}x'_{pq}\sim 0 \bmod p$ in K if K is connected. The general case follows then from the fact that S_k is a homomorphism. This proves (4), q. e. d.

From the lemma it follows that S_k induces a system of homomorphisms

$$\mathrm{Sm}_k\colon\ H_q(K,I_p)\to H_{q-k}(K,I_p),\qquad q\geqslant k\geqslant 0.$$

Furthermore we have

$$\mathrm{Sm}_q/H_q(K,I_p)=\begin{cases}0, & q>0,\\ 1, & q=0,\end{cases}$$

in which 1 means the identity.

Since I_p is a field, we have, dual to Sm_k, a system of homomorphisms

$$\mathrm{Sm}^k\colon\ H^{q-k}(K,I_p)\to H^q(K,I_p)$$

such that

$$\mathrm{Sm}^q/H^0(K,I_p)=\begin{cases}0, & q>0,\\ 1, & q=0,\end{cases}$$

where 1 is again the identity.

Remark 2. Let $\widetilde{K}$ be a complex, t a cell map of $\widetilde{K}$ with prime period p such that any face of a cell fixed under t is also a fixed cell of t. The set of all fixed cells under t then forms a (closed) subcomplex L. Let ϱ and $\bar{\varrho}$ denote either $d=1-t$ and $s=1+t+\ldots+t^{p-1}$ or s and d. Also, let H_k^ϱ be the special homology group of Smith determined from the k-dimensional cycles x satisfying $\varrho x=0$. Then by Smith we have some homomorphisms

$$\mu_\varrho\colon\ H_k^\varrho(\widetilde{K},I_p)\to H_{k-1}^{\bar{\varrho}}(\widetilde{K},I_p),$$

and

$$\nu_\varrho\colon\ H_k^\varrho(\widetilde{K},I_p)\to H_k(L,I_p).$$

In particular if $\widetilde{K}$ is a subdivision $\widetilde{K}^p$ of $K^p=K\times\ldots\times K$ and $t\colon \widetilde{K}^p\to\widetilde{K}^p$, as given at the beginning of this section, such that $L=\Delta(K)$, then for any cycle $x \bmod p$ of K, ωx^p may be considered as a d-cycle mod p and the d-homology class of ωx^p depends only on the homology class of x, whence it may be denoted by αX (however, α is not necessarily a homomorphism). Then Sm_k may be defined as

$$\mathrm{Sm}_k=\nu_\varrho\mu_\varrho\ldots\mu_d\mu_s\mu_d\alpha,$$

in which μ occur $(p-1)q-k$ times ($q=\dim X$) alternatively as μ_s and μ_d, and ϱ is d or s according as k is odd or even.

§ 2. Relations beetwen Sm^i and St^j

The Steenrod pth powers (p being a prime)

$$\mathrm{St}^j\colon\ H^r(K,I_p)\to H^{r+j}(K,I_p)$$

may be defined as follows: Form K^p and its subdivision $\widetilde{K}^p$ with $\Delta(K)$ as a subcomplex as in § 1. As in the original proof of Steenrod we may show that there exists a system of homomorphisms [2] (q,j are arbitrary integers)

$$D^j\colon\ C_q(\widetilde{K}^p)\to C_{q+j}(K^p)$$

satisfying the following conditions [3]:

1° $D^j=0$, $j<0$.

2° $\mathrm{In}\,D^0c=\mathrm{In}\,c$, $c\in C_0(\widetilde{K}^p)$.

3° If $\tau\in|\omega(\sigma_1\ldots\sigma_p)|$, $\sigma_i\in K$, then $D^j\tau\subset|\sigma_1\ldots\sigma_p|$.

4° $\partial D^{2j}=D^{2j}\partial+\sum\limits_{a=0}^{p-1}t^{-a}D^{2j-1}t^a$, $\partial D^{2j+1}=-D^{2j+1}\partial+(t^{-1}D^{2j}t-D^{2j})$.

From 1°-4° we may get the following property of D^0:

(1) $$D^0\omega(\sigma_1\ldots\sigma_p)=\sigma_1\ldots\sigma_p,\qquad \sigma_i\in K.$$

Proof. If $\sum\dim\sigma_i=0$, (1) is true by 2° and 3°. Suppose that (1) has been proved for $\sum\dim\sigma_i<k$. Let $\sigma_i\in K$, with $\dim\sigma_i=d_i$, $d_1+\ldots+d_i=r_i$, $\sum\dim\sigma_i=k$: then by 3° we must have $D^0\omega(\sigma_1\ldots\sigma_p)=\lambda\sigma_1\ldots\sigma_p$, where λ is a certain integer. From (1) and the first formula of 4°, we get

$$\begin{aligned}\lambda\partial(\sigma_1\ldots\sigma_p)&=\partial D^0\omega(\sigma_1\ldots\sigma_p)=D^0\partial\omega(\sigma_1\ldots\sigma_p)\\&=D^0\omega\partial(\sigma_1\ldots\sigma_p)=\sum(-1)^{r_{i-1}}D^0\omega(\sigma_1\ldots\partial\sigma_i\ldots\sigma_p),\end{aligned}$$

---

[2]) In the case of $p=2$, D^j have been introduced by R. Bott [1].

[3]) In what follows we write for simplicity $\sigma_1\ldots\sigma_p$ instead of $\sigma_1\otimes\ldots\otimes\sigma_p$, and thus for the others.

which, by the induction hypothesis, is

$$=\sum (-1)^{r_{i-1}}\sigma_1\ldots\partial\sigma_i\ldots\sigma_p=\partial(\sigma_1\ldots\sigma_p)\,.$$

As $\partial(\sigma_1\ldots\sigma_p)\neq 0$, we get $\lambda=1$ and (1) is proved.

Let

$$D_j\colon\ C^{q+j}(K^p,R)\to C^q(\widetilde{K}^p,R)$$

be the homomorphisms dual to D^j (R is a commutative ring with unit element). D_j must satisfy the following conditions corresponding to 1°-4°:

$\bar{1}$° $D_j=0,\ j<0$.

$\bar{2}$° If $u\in C^0(K^p,R)$ takes the same constant value $a\in R$ on all vertices of K^p, then $D_0u\in C^0(\widetilde{K}^p,R)$ takes the same value a on all vertices of $\widetilde{K}^p$.

$\bar{3}$° $D_jc\subset|\omega'c|$, where $c\in C^q(K^p,R)$, and ω' is the dual of ω.

$\bar{4}$° $D_{2j}\delta=\delta D_{2j}+\sum_{a=0}^{p-1}t^aD_{2j-1}t^{-a},\ D_{2j+1}\delta=-\delta D_{2j+1}+(tD_{2j}t^{-1}-D_{2j})$,

here t stands also for the cochain mapping induced by the cell mapping t in the complex K^p or $\widetilde{K}^p$.

In particular for c with $tc=c$ and $\delta c=0$, we have

(2) $$\delta D_ic=-s_iD_{i-1}c\,.$$

Let $U\in H^q(K,I_p)$: then by definition $\mathrm{St}^j_{(p)}U=\mathrm{St}^jU\in H^{q+j}(K,I_p)$ is the class uniquely determined by the cocycle $\Delta^{-1}D_{(p-1,q-j}u^p$ (u^p stands for $u\otimes\ldots\otimes u$), which is independent of the chosen cocycle $u\in U$. It is easy to see that this definition of St^j coincides with the original one of Steenrod.

In the above discussion the subdivision $\widetilde{K}^p$ of K^p is rather arbitrary, subject only to conditions already stated. Now take a *canonical subdivision* $\widetilde{K}^p$ of K^p as follows. The complex $\widetilde{K}^p$ will be formed by the following three sets of cells: (a) $\Delta(\sigma)$, $\sigma\in K$; (b) $\sigma_1\times\ldots\times\sigma_p$, where $\sigma_i\in K$ and $\sigma_1,\ldots,\sigma_p$ have no vertices common to all of them; (c) $\Delta(\sigma)\circ(\sigma_1\times\ldots\times\sigma_p)$, where $\sigma,\sigma_i\in K$, $\sigma_1\times\ldots\times\sigma_p$ a cell of type (b), and for each i, σ and σ_i span a simplex τ_i of K (σ,σ_i may have common vertices). The symbol $\circ$ means the join operation.

In the case (c) we have $\Delta(\sigma)\circ(\sigma_1\times\ldots\times\sigma_p)\subset|\omega(\tau_1\times\ldots\times\tau_p)|$. Let $\dim\sigma=d$, $\dim\sigma_i=d_i$, $\dim\Delta(\sigma)\circ(\sigma_1\times\ldots\times\sigma_p)=q$, $\dim(\tau_1\times\ldots\times\tau_p)=r$: then $q=d+d_1+\ldots+d_p+1$, $r\leqslant pd+d_1+\ldots+d_p+p$, and consequently $r\leqslant pq$. In view of 2°, 3°, it follows that for any cell $\xi\in\widetilde{K}^p$ of type (c) we have $D^j\xi=0$, $j>(p-1)q$. As the same is true for cells of type (a) or (b), with $j>(p-1)q$, we have for any $c\in C_q(\widetilde{K}^p)$,

$$D^jc=0,\quad j>(p-1)q\,,$$
$$D^0c=c,\quad q=0\,.$$

Dually, we have for any $c\in C^q(K^p,R)$,

(3) $$D_jc=0,\quad pj>(p-1)q\,,$$

(3′) $$D_0c=c,\quad q=0\,.$$

From (3) and $\bar{4}$° it follows that for any $u\in Z^q(K,I_p)$ we have for either $p>2$ or $p=2$,

$$-dD_{(p-1)q}u^p=(tD_{(p-1)q}t^{-1}-D_{(p-1)q})\,u^p=\delta D_{(p-1)q+1}u^p=0\,,$$

or

(4) $$dD_{(p-1)q}u^p=0,\quad u\in Z^q(K,I_p)\,.$$

THEOREM 1. *The two sets of operations* $\{\mathrm{Sm}^i\}$ *and* $\{\mathrm{St}^j\}$, *are equivalent to each other in the sense that either one may be determined by the other. The mode of mutual determination is given by the following system of equations*:

(5) $$A^k=\sum_{j=0}^{k}(-1)^j\,\mathrm{Sm}^{k-j}\,\mathrm{St}^j=\begin{cases}0, & k>0\\ 1, & k=0\end{cases}\quad (p=2\ \text{or}\ pk\ \text{odd})\,,$$

(6) $$B^k=\sum_{j=0}^{k}\mathrm{Sm}^{2k-2j}\,\mathrm{St}^{2j}=\begin{cases}0, & k>0\\ 1, & k=0\end{cases}\quad (p>2)\,,$$

in which 1 *means identity.*

For example let us prove (6) as follows:

Let $U\in H^q(K,I_p)$, $X\in H_r(K,I_p)$, $r=q+2k$. Take $u\in U$, $x\in X$ and form the sequence associated with x as given in § 1 (1). By $\bar{4}$° and (2), we then have [4]):

$$\begin{aligned}\mathrm{Sm}^{2k-2j}\,\mathrm{St}^{2j}U\cdot X&=\mathrm{St}^{2j}U\cdot\mathrm{Sm}_{2k-2j}X\\
&=D_{(p-1)q-2j}\,u^p\cdot x'_{(p-1)r+2k-2j}\\
&=\delta D_{(p-1)q-2j}\,u^p\cdot x_{(p-1)r+2k-2j-1}-sD_{(p-1)q-2j}\,u^p\cdot x_{(p-1)r+2k-2j}\\
&=-sD_{(p-1)q-2j-1}\,u^p\cdot x_{(p-1)r+2k-2j-1}-sD_{(p-1)q-2j}\,u^p\cdot x_{(p-1)r+2k-2j}\\
&=-\bar{d}\delta D_{(p-1)q-2j-1}\,u^p\cdot x_{(p-1)r+2k-2j-2}-sD_{(p-1)q-2j}\,u^p\cdot x_{(p-1)r+2k-2j}\\
&=+sD_{(p-1)q-2j-2}\,u^p\cdot x_{(p-1)r+2k-2j-2}-sD_{(p-1)q-2j}\,u^p\cdot x_{(p-1)r+2k-2j}\,.\end{aligned}$$

Adding together the above equations, we get

$$B^kU\cdot X=sD_{(p-1)q-2k-2}\,u^p\cdot x_{(p-1)r-2}-sD_{(p-1)q}\,u^p\cdot x_{(p-1)r+2k}\,.$$

Now take $\widetilde{K}^p$ as the canonical subdivision of K^p so that, if we apply (4), the last term in the above equation vanishes. Applying again $\bar{4}$° and

[4]) For a cohomology class U (or a cocycle u) and a homology class of the same dimension X (or a cycle x) on coefficient group I_p, $U\cdot X$ (or $u\cdot x$) will mean the value of U on X (or of u on x).

§ 1 (1) and noting that $x'_j = 0$, $j < (p-1)r$, we may successively reduce the resulting equations as follows:

$$B^k U \cdot X = D_{(p-1)q-2k-2} u^p \cdot s x_{(p-1)r-2} = D_{(p-1)q-2k-4} u^p \cdot s x_{(p-1)r-4} = \ldots = D_{-2kp} u^p \cdot s x_0 .$$

Hence for $k > 0$, $B^k U \cdot X = 0$. Since both X and U are arbitrary, we get the first equation of (6). If $k = 0$, then by (1) we get

$$B^0 U \cdot X = D_0 u^p \cdot s x_0 = D_0 u^p \cdot \omega x^p = u^p \cdot D^0(\omega x^p) = u^p \cdot x^p = (u \cdot x)^p = u \cdot x ,$$

since the coefficients considered are in the group I_p. It follows that $B^0 U \cdot X = U \cdot X$ for any U, X, and we get the second formula of (6).

The proof of (5) is similar and will thus be omitted.

THEOREM 2. *The sets of operations* $\{\mathrm{Sm}^i\}$ *and* $\{\mathrm{St}^j\}$ *may also be mutually determined by the following relations*:

$$(\bar{5}) \qquad \bar{A}^k = \sum_{j=0}^{k} (-1)^j \mathrm{St}^{k-j} \mathrm{Sm}^j = \begin{cases} 0, & k > 0 \\ 1, & k = 0 \end{cases} \qquad (p = 2 \text{ or } pk \text{ odd}),$$

$$(\bar{6}) \qquad \bar{B}^k = \sum_{j=0}^{k} \mathrm{St}^{2k-2j} \mathrm{Sm}^{2j} = \begin{cases} 0, & k > 0 \\ 1, & k = 0 \end{cases} \qquad (p > 2) .$$

Proof. For example let us prove $(\bar{5})$ in the case pk = odd. The proof of other cases is similar. Suppose that $(\bar{5})$ and $(\bar{6})$ have been proved in the case $\leqslant k-1$. Let δ_i denote the homomorphism 0 or 1 according as $i > 0$ or $= 0$: then by (5) and the induction hypothesis we have

$$\begin{aligned} \sum_{j>0} (-1)^j \mathrm{St}^{k-j} \mathrm{Sm}^j \mathrm{St}^0 = & -\sum_{j>0} \Big(\mathrm{St}^{k-2j} \sum_{i>0} \mathrm{Sm}^{2j-2i} \mathrm{St}^{2i}\Big) + \\ & + \sum_{j>0} \Big(\mathrm{St}^{k-2j+1} \sum_{i>0} (-1)^i \mathrm{Sm}^{2j-i-1} \mathrm{St}^i\Big) \\ = & -\sum_{i>0} \Big(\sum \mathrm{St}^{k-2j} \mathrm{Sm}^{2j-2i}\Big) \mathrm{St}^{2i} + \sum_{i>0} \Big(\sum (-1)^i \mathrm{St}^{k-2j+1} \mathrm{Sm}^{2j-i-1}\Big) \mathrm{St}^i \\ = & -\sum_{i>0} \Big(\sum (-1)^j \mathrm{St}^{k-j} \mathrm{Sm}^{j-2i}\Big) \mathrm{St}^{2i} - \sum_{i>0} \Big(\sum \mathrm{St}^{k-2j+1} \mathrm{Sm}^{2j-2i}\Big) \mathrm{St}^{2i-1} \\ = & -\sum_{i>0} \delta_{k-2i} \mathrm{St}^{2i} - \sum_{i>0} \delta_{k+1-2i} \mathrm{St}^{2i-1} = -\mathrm{St}^k . \end{aligned}$$

Hence $\bar{A}^k = 0$. As $\bar{A}^0 = \bar{B}^0 = 1$, $(\bar{5})$ is proved by induction on k.

Remark. Let $\beta = \big((1/p)\delta\big)_p$ be the homomorphism of Bockstein: then $\beta \mathrm{St}^{2k} = -\mathrm{St}^{2k+1}$. Hence by comparing $\beta \bar{B}^k = 0$ with $(\bar{5})$ we know that in case of pk being odd $(\bar{5})$ may also be written as

$$\bar{A}^k = \sum_{j=0}^{k} \mathrm{St}^{k-j} \mathrm{Sm}^j = 0 .$$

Similarly for pk odd (5) may also be written as

$$A^k = \sum_{j=0}^{k} \mathrm{Sm}^{k-j} \mathrm{St}^j = 0 .$$

References

[1] R. Bott, *On symmetric products and the Steenrod squares*, Ann. of Math. 57 (1953), p. 579-590.

[2] M. Richardson and P. A. Smith, *Periodic transformation of complexes*, Ann. of Math. 39 (1938), p. 611-633.

[3] P. A. Smith, *Fixed points of periodic transformations*, Appendix B, p. 351-373 to S. Lefschetz, Algebraic topology, New York 1942.

[4] N. E. Steenrod, *Reduced powers of cohomology classes*, Paris 1951.

[5] R. Thom, *Une théorie intrinsèque de puissances de Steenrod*, Colloque de Topologie de Strasbourg, 1951.

[6] Wu Wen-tsün, *Sur les puissances de Steenrod*, Colloque de Topologie de Strasbourg, 1951.

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA

Reçu par la Rédaction le 30.5.1956