On the Weierstrass-Stone approximation theorem

by

B. Banaschewski (Hamilton, Ont.)

1. The Weierstrass-Stone approximation theorem states: If a ring R of bounded continuous real functions on a compact space E contains (i) the constants and (ii) to any $x, y \neq x$ of E a function f for which $fx \neq fy$, then R is dense in the ring $\mathcal{C}(E)$ of all bounded continuous real functions on E with respect to the uniform topology [1]).

In the following, by saying the Weierstrass-Stone approximation theorem is true for a space E we shall mean that any ring of bounded continuous real functions on E for which (i) and (ii) hold is uniformly dense in $\mathcal{C}(E)$. For property (ii) of a ring R of functions we shall also use the expression: *R separates the points of E.*

A filter $\mathfrak{A}$ on a topological space E is called *completely regular* if it has a basis $\mathfrak{B}$ of open sets such that for any $B \in \mathfrak{B}$ there exists a $B' \subseteq B$ in $\mathfrak{B}$ and a function $f \in \mathcal{C}(E)$ which maps E into the real unit interval, vanishes on B' and equals 1 on the complement $\complement B$ of B (see [3], chap. IX).

In this note we restrict ourselves to spaces E for which $\mathcal{C}(E)$ separates the points of E. Concerning such spaces we are going to prove:

The Weierstrass-Stone approximation theorem holds for the space E if and only if for any completely regular filter $\mathfrak{A}$ on E the intersection $\bigcap A$, $A \in \mathfrak{A}$, is non-void.

2. Our first step will be to show that on any non-compact completely regular space E there exist subrings $R \neq \mathcal{C}(E)$ of $\mathcal{C}(E)$ which are uniformly closed, contain the constants and separate the points of E. This will prove that a completely regular space for which the Weierstrass-Stone approximation theorem holds is compact [2]).

If the space E is non-compact and completely regular, let a be any fixed point of $\beta E - E$, βE the Čech compactification of E (see [4]), and a_* a fixed point of E. The set of all $f \in \mathcal{C}(E)$ for which $fa_* = \tilde{f}a$, where $\tilde{f}$

[1]) See, for example, [3], chap. X. All topological concepts used here will be taken in the sense of N. Bourbaki.

[2]) As M. Henriksen has pointed out to me, this statement was first given in [5].

denotes the continuous extension of f to βE, is clearly a uniformly closed subring R of $\mathcal{C}(E)$ which contains the constants. As the bounded continuous real functions on βE separate the points of βE, there exists a $g \in \mathcal{C}(E)$ for which $ga_* \neq \tilde{g}a$, and one has, therefore, $R \neq \mathcal{C}(E)$. Finally, for any given, $x, y \neq x$ in E there are functions $h \in \mathcal{C}(E)$ such that $hx = 0$ and $hy = ha_* = \tilde{h}a = 1$ as one readily sees by applying Urysohn's lemma to βE. This shows R to be of the desired type.

Another proof, implicitly making similar use of βE, is as follows: If E is non-compact completely regular, then there exists a strictly coarser topology on E for which E is still completely regular [3]) and the ring R of all bounded continuous real functions on E which are continuous with respect to this coarser topology is a uniformly closed subring of $\mathcal{C}(E)$ which contains the constants and separates the points of E. Furthermore, since the topology of a completely regular space is determined by its bounded continuous real functions, one has $R \neq \mathcal{C}(E)$.

3. With any space E there is connected a certain completely regular space $\dot{E}$, the uniform space which has the same points as E [4]) and whose uniform structure is defined by the $f \in \mathcal{C}(E)$ in terms of the relations $|fx - fy| \leqslant \varepsilon$, $\varepsilon > 0$, $f \in \mathcal{C}(E)$. By definition, one has $\mathcal{C}(E) = \mathcal{C}(\dot{E})$. If $\dot{E}$ is compact and R a subring of $\mathcal{C}(E)$ containing the constants and separating the points of E — which at the same time means: of $\dot{E}$ — then, in virtue of the compactness of $\dot{E}$, one has $R = \mathcal{C}(E)$.

If, on the other hand, $\dot{E}$ is non-compact, there exist, as shown above uniformly closed rings $R \neq \mathcal{C}(\dot{E})$ in $\mathcal{C}(\dot{E})$ which contain the constants and separate the points of $\dot{E}$, and R has the same properties relative to E.

Therefore: The Weierstrass-Stone approximation theorem holds for E if and only if $\dot{E}$ is compact.

4. Let $\dot{E}$ be compact and $\mathfrak{A}$ any completely regular filter on E. Then, $\mathfrak{A}$ is also a completely regular filter on $\dot{E}$ [5]) and as $\bigcap A = \bigcap \bar{A}$, $A \in \mathfrak{A}$ (the bar denoting closure in $\dot{E}$), because $\mathfrak{A}$ is a completely regular

[3]) For this, see [2]. This is where implicit use is made of βE, by operating with maximal completely regular filters.

[4]) This is only true when $\mathcal{C}(E)$ separates the points of E. Otherwise, the statement must be modified. See [4].

[5]) $\mathfrak{A}$ has a basis $\mathfrak{B}$ of open sets in E such that to every $B \in \mathfrak{B}$ there exists a $B' \subseteq B$ in $\mathfrak{B}$ and an $f \in \mathcal{C}(E)$ mapping E into the real unit interval, vanishing on B' and equal to 1 on B. For the sets M_n of all x such that $fx < 1/n$ one then has $B' \subseteq M_n \subseteq B$. These M_n are open in $\dot{E}$, because f is continuous on $\dot{E}$, and contained in $\mathfrak{A}$ because of $B' \subseteq M_n$. Furthermore, one can find continuous mappings φ_n of the unit interval onto itself such that the function $\varphi_n \circ f$ maps $M_{1/(n+1)}$ onto 0 and $\complement M_{1/n}$ onto 1. Therefore, the collection of all these M_n, taken for any pair B', B and one of the corresponding functions f forms a basis of $\mathfrak{A}$ which shows that $\mathfrak{A}$ is completely regular in $\dot{E}$.

filter, one has $\bigcap A \neq \emptyset$ in view of the compactness of $\dot{E}$. On the other hand, let any completely regular filter on E have non-void intersection. Then, in particular, any maximal completely regular filter $\mathfrak{M}$ on $\dot{E}$, being also a completely regular filter on E, has a non-void intersection. This means, there is an $x \in E$ whose neighbourhood filter $\dot{\mathfrak{B}}(x)$ in $\dot{E}$ contains $\mathfrak{M}$: take $x \in \bigcap M$, $M \in \mathfrak{M}$. As $\dot{\mathfrak{B}}(x)$ is also completely regular in $\dot{E}$ [6]), one gets from this $\mathfrak{M} = \dot{\mathfrak{B}}(x)$. In other words: every maximal completely regular filter in $\dot{E}$ is convergent. This, however, implies that $\dot{E}$ is compact: If $\dot{E}$ were non-compact, each point u of its Čech compactification $\beta\dot{E}$ which does not belong to $\dot{E}$ would determine on $\dot{E}$ a completely regular filter by the intersections $U \cap E$, U any neighbourhood of u in $\beta\dot{E}$. It is known that this filter would be a maximal completely regular filter (see [1], also [3], chap. IX) and, of course, converge to u in $\beta\dot{E}$. Therefore, as every maximal completely regular filter on $\dot{E}$ has a limit in $\dot{E}$, $\dot{E}$ is compact.

5. In order to show that the proposition thus proved actually covers a wider class of spaces than just the compact spaces, we shall now give an example of non-compact Hausdorff spaces for which the Weierstrass-Stone approximation theorem also holds.

Let E be any non-compact completely regular space, βE its Čech compactification and $\mathfrak{M}$ its maximal completely regular filters. Then, a new extension space $\beta' E$ of E can be defined by taking the collection of sets $M \cup \{u\}$, $M \in \mathfrak{M}$, as new neighbourhood filter of $u = \lim \mathfrak{M} \in \beta E$ and the neighbourhood filter (in E) $\mathfrak{B}(x)$ of $x \in E$ as new neighbourhood filter of x in $\beta' E$. Obviously, $\beta' E$ contains E as a dense subspace. On the other hand, the topology of $\beta' E$ is a refinement of the topology of βE, for all sets open in βE are also open in $\beta' E$: If M is open in βE and $M = M_1 \cup M_2$ its decomposition into $M_1 \subseteq E$ and $M_2 \subseteq \beta E - E$, then M_1 is open in E and M_2 consists of such points u for which $M_1 \in \mathfrak{M}$ where $u = \lim \mathfrak{M}$. As $M_1 \cup \{u\}$ is open in $\beta' E$ for any such u, $M_1 \cup M_2$, being a union of sets of the latter type, is also open in $\beta' E$. In general, this $\beta' E$ will not be compact: Its compactness implies $\beta' E = \beta E$ and therefore discreteness of $\beta E - E$. However, for, say, a locally compact E one would then have finite $\beta E - E$, which, in turn, will not be true if E is also normal [7]). In the following, $\beta' E \neq \beta E$ will be assumed.

Now, let f be any bounded continuous real function on $\beta' E$. Then, the restriction f_* of f to E is a function of the same type on E. f_* has continuous extension $\tilde{f}_*$ to βE. The two functions f and $\tilde{f}_*$, however, are actually equal: If $\lim \mathfrak{M} = u$ in βE, then also $\lim \mathfrak{M} = u$ in $\beta' E$ and

[6]) A direct consequence of the complete regularity of $\dot{E}$.

[7]) See [4]. Actually, there is a (much more complicated) process which yields a modification of βE of the desired type for arbitrary completely regular E.

vice versa. $\lim \mathfrak{M} = u$ in βE gives $\widetilde{f_*}u = \lim f_*\mathfrak{M}$ and $\lim \mathfrak{M} = u$ in $\beta' E$ gives $fu = \lim f_*\mathfrak{M}$, hence $\widetilde{f_*} = f$. In other words: Any bounded continuous real function on $\beta' E$ is also continuous on βE. From this, one concludes that the Weierstrass-Stone approximation theorem holds for $\beta' E$, which can either be shown directly or by applying the result proved in the previous sections, using the fact that every completely regular filter in $\beta' E$ is also completely regular in βE.

References

[1] P. Alexandroff, *Bikompakte Erweiterung topologischer Räume*, Mat. Sborn. N. S. 5 (1939), p. 420-423.

[2] B. Banaschewski, *Über zwei Extremaleigenschaften topologischer Räume*, Math. Nachr. 13 (1955), p. 141-150.

[3] N. Bourbaki, *Topologie générale*, Paris 1948.

[4] E. Čech, *On bicompact spaces*, Ann. of Math. 38 (1937), p. 823-844.

[5] E. Hewitt, *Certain generalizations of the Weierstrass approximation theorem*, Duke Math. 14 (1947), p. 419-427.

MCMASTER UNIVERSITY, HAMILTON, ONTARIO

Reçu par la Rédaction le 19.4.1956

On the limiting probability distribution on a compact topological group

by

K. Urbanik (Wrocław)

I. Let G be a compact (not necessarily commutative) topological group. A regular completely additive measure μ defined on the class of all Borel subsets of G, with $\mu(G)=1$, will be called a *probability distribution*. A sequence of probability distributions $\mu_1, \mu_2, \ldots$ is said to be *weakly convergent to a probability distribution* μ if

$$\lim_{n\to\infty} \int_G f(x)\,\mu_n(dx) = \int_G f(x)\,\mu(dx)$$

for any complex-valued continuous function f defined on G.

Let $X_1, X_2, \ldots$ be a sequence of independent G-valued random variables with the probability distributions $\mu_1, \mu_2, \ldots$ Put

$$Y_n = X_1 \cdot X_2 \cdot \ldots \cdot X_n \quad (n=1,2,\ldots), \tag{1}$$

where the product is taken in the sense of group multiplication in G. Let us denote by ν_n the probability distribution of the random variable Y_n. It is well known that

$$\nu_n = \mu_1 * \mu_2 * \ldots * \mu_n \quad (n=1,2,\ldots),$$

where the convolution $*$ is defined by the formula:

$$\nu * \lambda(E) = \int_G \nu(E\cdot x^{-1})\,\lambda(dx).$$

The limiting distribution of the sequence of random variables Y_n is the weak limit of the sequence of the probability distributions $\mu_1 * \mu_2 * \ldots * \mu_n$.

We say that the sequence of probability distributions $\mu_1, \mu_2, \ldots$ is *normal* (by an analogy with normal numbers in the sense of Borel), if for every integer k exists a sequence of integers $n_1 < n_2 < \ldots$ such that

$$\mu_s = \mu_{n_j+s} \quad (s=1,2,\ldots,k;\ j=1,2,\ldots).$$

In particular, the sequence $\mu_1 = \mu_2 = \ldots$ is normal.