Lebesgue area and Hausdorff measure

by

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§ 1. Introduction

1.1. Given a continuous mapping

(1) $$T: x_1 = x_1(u,v),\ x_2 = x_2(u,v),\ x_3 = x_3(u,v),\ (u,v) \in Q,$$

from the unit square Q: $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$ into Euclidean 3-space R^3, we denote by $A(T)$ the Lebesgue area of the Fréchet surface determined by T. The treatise [8] will be used as a general reference for the theory of the Lebesgue area (numbers in square brackets refer to the Bibliography at the end of this paper). In recent years, important contributions were made to the problem of representing $A(T)$ in the form

(2) $$A(T) = \iint_{R^3} k(x,T)\, dH^2,$$

where dH^2 indicates integration with respect to two-dimensional Hausdorff measure H^2, x is a generic notation for a point in R^3, and $k(x,T)$ is a multiplicity function which describes in some reasonable manner the number of times the point x is covered by the image of Q under T. Far-reaching methods and results in this direction (as well as an extensive bibliography) are contained in the comprehensive paper [2] of Federer. Quite recently, a remarkable multiplicity function satisfying (2) has been discovered by Mickle [5]. Subsequent studies by the present writer [10] and by Mickle [6] revealed that the novel approach initiated by Mickle in [5] yields, after appropriate modifications, an infinite variety of multiplicity functions satisfying (2).

The present paper was motivated by the following observations. For continuous mappings from the square Q into a Euclidean plane R^2 there is available an *essential multiplicity function* (see [8], IV. 1) which plays a basic role in the theory of such mappings. The researches presented in [5], [10], [6] may be construed as efforts to construct some multiplicity function $k(x,T)$ which would play a similar role in the theory of continuous mappings from Q into Euclidean 3-space R^3. It is

then natural to require that such a multiplicity function should possess properties similar to those of the *essential multiplicity function* associated with plane mappings. From this point of view it seems natural to require, in particular, that the multiplicity function $k(x,T)$ (see (2)) should be Borel measurable, and that it should be of *local origin* (see 1.6 (iv)). The multiplicity functions exhibited in [5], [10], [6] fail, apparently, to meet these two requirements. Our main objective in the present paper is to construct a multiplicity function $k(x,T)$ which satisfies these two requirements and also possesses various other desirable properties already present in the cases studied in [5], [10], [6]. We proceed to state some definitions needed to describe our approach and our results.

1.2. Let $\tau: X \to Y$ be a continuous mapping from a compact metric space X into a metric space Y (this is the only case occurring in the sequel). For each point $y \in Y$ the components of the set $\tau^{-1}y$ are continua, each of which is termed a *maximal model continuum* (abbreviated to m. m. c.) for y under τ. The number (perhaps infinite) of those m. m. c.-s for y under τ which are contained in a subset S of X will be denoted by $N^*(y,\tau,S)$. If S is a subset of X and y is a point of Y, then the number (perhaps infinite) of the points of the set $S \cap \tau^{-1}y$ will be denoted by $N(y,\tau,S)$.

1.3. All the multiplicity functions $k(x,T)$ occurring in [5], [10], [6] were obtained by means of the following general scheme. For each point $x \in R^3$, some of the m. m. c.-s for x under T were designated as *significant* for x under T, and then $k(x,T)$ was defined as the number (perhaps zero or infinite) of the significant m. m. c.-s for x under T. Apparently, this scheme is not sufficiently flexible to produce a multiplicity function $k(x,T)$ which satisfies the additional requirements stated in 1.1 (Borel measurability and local origin). We shall describe presently a more general scheme which is adequate from this point of view, and which includes as special instances all the constructions used in [5], [10], [6]. The new feature of this scheme is that the *significant sets* are *not* required to be m. m. c.-s under T.

1.4. For each continuous mapping $T: Q \to R^3$ (see 1.1) and for each point $x \in R^3$ let there be assigned certain sets $S \subset Q$ as *significant* for x under T. On denoting by $\mathfrak{L}$ the law which is used to effect this assignment, the symbol $\mathfrak{L}$ may be thought of as a functional $\mathfrak{L}(T,x)$ whose arguments are T and x and whose value (for each choice of T and x) is a family of subsets of Q (namely, the family of those sets $S \subset Q$ which are significant for x under T in the sense of the law $\mathfrak{L}$). In this terminology, the statement that a set $S \subset Q$ is significant for x under T according to the law $\mathfrak{L}$ is equivalent to the inclusion $S \in \mathfrak{L}(T,x)$.

Given a law $\mathfrak{L}$ in the sense just explained, we shall use the symbol $\mathfrak{S}(x,T,\mathfrak{L})$ as a generic notation for a finite (perhaps empty) system of pair-wise disjoint sets of the family $\mathfrak{L}(T,x)$, and we shall denote by $N[\mathfrak{S}(x,T,\mathfrak{L})]$ the number of the sets of the system $\mathfrak{S}(x,T,\mathfrak{L})$. Thus $N[\mathfrak{S}(x,T,\mathfrak{L})]$ is always a non-negative (finite) integer. We define

$$k(x,T,\mathfrak{L}) = \underset{\mathfrak{S}}{\text{l. u. b.}}\, N[\mathfrak{S}(x,T,\mathfrak{L})], \tag{3}$$

where the letter $\mathfrak{S}$ under the l. u. b. symbol indicates that the least upper bound is taken with respect to all the systems $\mathfrak{S}(x,T,\mathfrak{L})$ for fixed T and x.

Note that once the law $\mathfrak{L}$ has been agreed upon, the preceding scheme yields for each continuous mapping $T\colon Q \to R^3$ a function $k(x,T,\mathfrak{L})$ of the point $x \in R^3$. This function k will be termed the *multiplicity function generated by the law* $\mathfrak{L}$. Clearly, each value of $k(x,T,\mathfrak{L})$ is either a non-negative integer or $+\infty$.

1.5. An important special case arises if the law $\mathfrak{L}$ is of *disjoint character* in the following sense: If S_1, S_2 are distinct sets in Q which are significant for x under T according to the law $\mathfrak{L}$, then $S_1 \cap S_2 = O$. All the laws $\mathfrak{L}$ used in [5], [10], [6] are of disjoint character, and all those laws are such that only m. m. c.-s for x under T can be significant for x under T. The new law to be exhibited in this paper will not be of disjoint character, and the significant sets will not generally be m. m. c.-s under T.

If a law $\mathfrak{L}$ happens to be of disjoint character (as in [5], [10], [6]), then it is clear that $k(x,T,\mathfrak{L})$, as defined in (3), is merely the number of those sets $S \subset Q$ which are significant for x under T according to the law $\mathfrak{L}$. However, if $\mathfrak{L}$ is not of disjoint character, then significant sets must be counted according to the scheme indicated by the formula (3).

1.6. Definition 1. Let τ_1, τ_2 be continuous mappings from the topological space X into the topological space Y, and let S be a subset of X. Then τ_1, τ_2 are said to *agree in the vicinity of* S if there exists an open subset O of X such that $O \supset S$ and $\tau_1 x = \tau_2 x$ for $x \in O$.

Definition 2. Let τ_1, τ_2 be continuous mappings from the topological spaces X_1, X_2 respectively into the metric space M. Then τ_1, τ_2 are termed *F-equivalent* (equivalent in the Fréchet sense) provided that for every $\varepsilon > 0$ there exists a homeomorphism h_ε from X_1 onto X_2 such that

$$d(\tau_1 x_1, \tau_2 h_\varepsilon x_1) < \varepsilon \quad \text{for} \quad x_1 \in X_1,$$

where d denotes distance in M (for a detailed study of this basic concept, see [8], part II).

We can now formulate our problem as follows: find a law $\mathfrak{L}$ (see 1.4) such that the following requirements are satisfied (see 1.1).

(i) For every continuous mapping $T\colon Q \to R^3$ (see 1.1) we should have

$$A(T) = \iint_{R^3} k(x,T,\mathfrak{L})\, dH^2.$$

Explicitly: for fixed T, the function $k(x,T,\mathfrak{L})$ should be H^2-measurable as a function of the point $x \in R^3$, and the preceding formula should hold. In particular, this requirement implies that $k(x,T,\mathfrak{L})$ is H^2-summable if and only if $A(T) < \infty$ (for the theory of measure and integration the reader may consult the treatise [11] of Saks).

(ii) The law $\mathfrak{L}$ should be F-invariant in the following sense: if the continuous mappings $T_1\colon Q \to R^3$, $T_2\colon Q \to R^3$ are F-equivalent (see Definition 2 above), then we should have

$$k(x, T_1, \mathfrak{L}) = k(x, T_2, \mathfrak{L})$$

for all points $x \in R^3$.

(iii) The law $\mathfrak{L}$ should be invariant under distance-preserving transformations in the following sense: if τ is any distance-preserving transformation in R^3, then we should have

$$k(x, T, \mathfrak{L}) = k(\tau x, \tau T, \mathfrak{L})$$

for every continuous mapping $T\colon Q \to R^3$ and for every point $x \in R^3$.

(iv) The law $\mathfrak{L}$ should be of *local character* in the following sense: if a set $S \subset Q$ is significant for x under T according to the law $\mathfrak{L}$, and if T^* agrees with T in the vicinity of S (see Definition 1 above), then S should also be significant for x under T^* according to the law $\mathfrak{L}$.

(v) $k(x,T,\mathfrak{L})$ should be (for each fixed T) a Borel measurable function of the point $x \in R^3$.

(vi) The law $\mathfrak{L}$ should be of *covering character* in the following sense: if a set $S \subset Q$ is significant for x under T according to the law $\mathfrak{L}$, then one should have $x \in TS$.

Let us recall that regardless of the choice of the law $\mathfrak{L}$, each value of $k(x,T,\mathfrak{L})$ is either a non-negative integer or $+\infty$ (see 1.4).

1.7. The requirement (i) in 1.6 is of course fundamental in the present context (see 1.1). Let us recall that if T_1, T_2 are two F-equivalent mappings from Q into R^3 then T_1, T_2 are considered as representations of the same surface (see [8], II. 3]). Accordingly, the requirement (ii) in 1.6 merely means that the multiplicity function $k(x,T,\mathfrak{L})$ should be independent of the particular representation T chosen for the surface. The requirement (iii) in 1.6 may be interpreted to mean that $k(x,T,\mathfrak{L})$

should be independent of the choice of the Cartesian coordinate system in R^3. Let us note that the original law $\mathfrak{L}$ used by Mickle in [5] satisfies the requirements (i) and (ii) but (apparently) fails to satisfy the invariance requirement (iii). Infinitely many of the laws $\mathfrak{L}$ exhibited in [10] satisfy, beyond (i) and (ii), the requirement (iii) in the following approximate sense:

If $A(T)<\infty$, then for fixed T and τ one has

$$k(x,T,\mathfrak{L})=k(\tau x,\tau T,\mathfrak{L})$$

with the possible exception of points x which constitute a set of H^2-measure zero. The significant advance achieved by Mickle in [6] was the discovery of a law $\mathfrak{L}$ which satisfies (beyond (i) and (ii)) also (iii) in the strict sense. All the laws $\mathfrak{L}$ studied in [5], [10], [6] satisfy (trivially) the requirement (vi) of being of *covering character*. As mentioned above, none of these laws seem to satisfy the requirements (iv) and (v).

1.8. We proceed to state the definitions needed to formulate the new law $\mathfrak{L}$ which satisfies all the requirements (i) through (vi) in 1.6.

Let τ be a continuous mapping from the unit square Q: $0\leqslant u\leqslant 1$, $0\leqslant v\leqslant 1$ into a plane. Let us use the complex variables $w=u+iv$, $z=x+iy$ to identify points in Q and in the image plane respectively. Let $\mathfrak{R}$ be a finitely connected Jordan region in Q. The exterior boundary curve of $\mathfrak{R}$ is oriented counter-clockwise, the interior boundary curves (if any) of $\mathfrak{R}$ are oriented clockwise. Let $C_1,\dots,C_m$ be the boundary curves of $\mathfrak{R}$, oriented in the manner just explained. Let z be a point in the image plane and let C_j be one of the (oriented) boundary curves of $\mathfrak{R}$. One introduces then an *index-function* $\mu(z,\tau,C_j)$ as follows. If $z\in\tau C_j$, one puts $\mu(z,\tau,C_j)=0$. If $z\notin\tau C_j$, then $\mu(z,\tau,C_j)$ is equal to the *topological index* of z with respect to τC_j (see [8], II. 4.34]). Finally one defines

$$\mu(z,\tau,\mathfrak{R})=\begin{cases}\sum_{j=1}^{m}\mu(z,\tau,C_j) & \text{if} \quad z\notin\tau(C_1\cup\dots\cup C_m),\\ 0 & \text{if} \quad z\in\tau(C_1\cup\dots\cup C_m).\end{cases}$$

If $\mu(z,\tau,\mathfrak{R})\neq 0$, then $\mathfrak{R}$ is termed an *indicator region* for z under τ. An m. m. c. γ for z under τ is termed an *essential maximal model continuum* (abbreviated to e. m. m. c.) for z under τ if the following holds. (i) $\gamma\subset Q^0$ (where Q^0 denotes the interior of Q). (ii) For every open set O such that $\gamma\subset O\subset Q^0$ there exists an indicator region $\mathfrak{R}$ for z under τ such that $\gamma\subset R^0$, $\mathfrak{R}\subset O$, where $\mathfrak{R}^0$ indicates the interior of $\mathfrak{R}$. For each point z of the image plane, the *essential multiplicity function* $\varkappa(z,\tau,Q)$ is defined as the number (perhaps infinite) of the e. m. m. c.-s for z under τ.

If D is a domain (connected open set) in Q^0, then $E(\tau,D)$ denotes the union of all the e. m. m. c.-s under τ contained in D. The set $E(\tau,D)$ is a Borel set (see [8], IV. 1.56, where this set is denoted by $\mathfrak{E}^*$). If O is an open set in Q^0, then similarly $E(\tau,O)$ denotes the union of all the e. m. m. c.-s under τ contained in O. Since the components of O constitute a countable family of domains, it follows from the preceding statement that $E(\tau,O)$ is a Borel set.

For a detailed study of the concepts introduced in this section, the reader may consult the treatise [8].

1.9. We shall describe presently the law $\mathfrak{L}$ used by Mickle in [6]. We shall denote this law by $\mathfrak{L}_M$. The following definitions are used in stating the law $\mathfrak{L}_M$. The class of H^2-measurable sets in Euclidean 3-space R^3 is denoted by Γ. For $E\in\Gamma$, H^2E is the two-dimensional Hausdorff measure of E. The unit sphere $x_1^2+x_2^2+x_3^2=1$ in R^3 is denoted by U. For each point $P\in U$, $R^2(P)$ denotes the plane through the origin which is perpendicular to the line through the origin and the point P. T_P denotes the orthogonal projection from R^3 onto $R^2(P)$. The class of those sets $E\in\Gamma$ for which $L_2T_PE=0$ is denoted by Γ_P (here and in the sequel, L_2 refers to two-dimensional Lebesgue measure in the plane indicated by the context). For $x\in R^3$ and $r>0$, $s(x,r)$ denotes the interior of the sphere with center x and radius r.

For sets $E\in\Gamma$, Mickle introduces an auxiliary measure H_P by means of the formula

$$H_PE=\text{gr. l. b.}\,H^2(E-S), \qquad S\in\Gamma_P.$$

For $E\in\Gamma$, $x\in R^3$, $P\in U$, Mickle defines:

$$h_P(x,E)=\overline{\lim}\,\frac{H_P[s(x,r)\cap E]}{r^2\pi}, \qquad D_PE=\{x|\ h_P(x,E)>0\}.$$

For positive integers n,m, for $S\in\Gamma$ and $P\in U$, let us denote by $G(n,m,S,P)$ the set of those points $x\in R^3$ for which

$$\frac{H_P[s(x,r)\cap S]}{r^2\pi}>\frac{1}{n}$$

for some r such that $0<r<1/m$.

Let there be given now a continuous mapping T: $Q\to R^3$ (see 1.1). For each open set $O\subset Q$, Mickle defines a corresponding set $D^*(T,O)\subset R^3$ by the formula (see 1.8)

$$D^*(T,O)=\bigcup_n\bigcap_m\bigcup_P G\{n,m,T[O\cap E(T_PT,Q^0)],P\},$$

where $n=1,2,\dots,m=1,2,\dots,P\in U$. The law $\mathfrak{L}_M$ used by Mickle in [6] may now be described as follows. A set $S\subset Q$ is significant for a point

$x \in R^3$ under T (according to the law $\mathfrak{L}_M$) provided that the following holds:

(i) S is an m. m. c. for x under T.

(ii) For every open set O such that $S \subset O \subset Q$, one has the inclusion $x \in D^*(T,O)$.

In view of (i), this law $\mathfrak{L}_M$ is clearly of disjoint character (in the sense of 1.5). Accordingly, the corresponding multiplicity function $k(x,T,\mathfrak{L}_M)$ is equal to the number of those m. m. c.-s γ for x under T which are significant for x in the sense just explained. Mickle proves in [6] that $k(x,T,\mathfrak{L}_M)$ is H^2-measurable and

$$A(T) = \iint_{R^3} k(x,T,\mathfrak{L}_M)\, dH^2 .$$

Mickle also proves that the law $\mathfrak{L}_M$ is F-invariant. The invariance of $\mathfrak{L}_M$ under distance-preserving transformations in R^3 is readily deduced from the observation that no point $P \in U$ plays a privileged role in the formula for the set $D^*(T,O)$. Thus the law $\mathfrak{L}_M$ satisfies the requirements (i), (ii), (iii) in 1.6 and (trivially) also the requirement (vi). As regards the requirement (iv), note that the defining formula for the set $D^*(T,O)$ involved the sets $E(T_P T, Q^0)$ which depend upon the behavior of T in all of Q, and thus the law $\mathfrak{L}_M$ cannot be expected to be of local character (in any case, the present writer was unable to prove that $\mathfrak{L}_M$ is of local character).

1.10. In trying to enforce the requirement (iv) in 1.6, a plausible idea is to replace, in the defining formula for $D^*(T,O)$, the set $O \cap E(T_P T, Q^0)$ by the set $E(T_P T, O)$ which depends only upon the behavior of T in O itself (see 1.8). One obtains in this manner the set

$$\Delta_*(T,O) = \bigcup_n \bigcap_m \bigcup_P G[n, m, TE(T_P T, O), P],$$

where again $n = 1, 2, \ldots, m = 1, 2, \ldots, P \in U$. By a general remark in [6], the sets G occurring in the preceding formula are open sets, and hence $\Delta_*(T,O)$ is a Borel set. On replacing in the definition of the law $\mathfrak{L}_M$ (see 1.9) the set $D^*(T,O)$ by the set $\Delta_*(T,O)$, one obtains a law which we denote by $\mathfrak{L}'_M$. An argument entirely similar to that used by Mickle in [6] shows that the corresponding multiplicity function $k(x,T,\mathfrak{L}'_M)$ satisfies the requirements (ii), (iii), (iv), (vi) in 1.6, and that

$$A(T) = \iint_{R^3} k(x,T,\mathfrak{L}'_M)\, dH^2 \quad \text{if} \quad A(T) < \infty .$$

However, there is no evidence to indicate that the preceding formula holds if $A(T) = \infty$.

1.11. We shall define presently a new law $\mathfrak{L}$, to be denoted by $\mathfrak{L}_0$, which satisfies all the requirements (i) through (vi) stated in 1.6.

The law $\mathfrak{L}_0$. A set $S \subset Q$ is *significant for x under T according to the law* $\mathfrak{L}_0$ provided that the following holds:

(i) $S \neq \emptyset$, $S \subset Q^0$ (where Q^0 is the interior of Q).

(ii) S is compact.

(iii) S is a union of m. m. c.-s under T (however, these m. m. c.-s are *not* required to be m. m. c.-s for the point x under consideration).

(iv) For every open set O such that $S \subset O \subset Q^0$, one has $x \in \Delta_*(T,O)$ (see 1.10).

The fact that the requirements (i), (ii), (v) stated in 1.6 are satisfied by this law $\mathfrak{L}_0$ will be verified in §§ 7, 4, 3 respectively. Noting that the defining formula for $\Delta_*(T,O)$ (see 1.10) involves only the sets $E(T_P T, O)$, which depend solely upon the behavior of T in O itself, it is immediate that the law $\mathfrak{L}_0$ is of local character (and thus the requirement (iv) in 1.6 is satisfied). Noting further that no point $P \in U$ plays a privileged role in the defining formula for $\Delta_*(T,O)$, it follows easily that the law $\mathfrak{L}_0$ is invariant under distance-preserving transformations in R^3, in the sense of the requirement (iii) in 1.6. We proceed to check the requirement (vi) in 1.6. Consider a non-empty compact subset S of Q^0, and let $x \in R^3$ be a point such that $x \notin TS$. Then there exists an open subset G of R^3 such that $TS \subset G$, $x \notin \bar{G}$ (where $\bar{G}$ is the closure of G). Hence there exists a positive integer m_0 such that $s(x,r) \cap G = \emptyset$ for $0 < r < 1/m_0$. Now since T is continuous, we have an open set O such that $S \subset O \subset Q^0$ and $TQ \subset G$. If we select now any point $P \in U$, then (see 1.8) $E(T_P T, O) \subset O$, and hence $TE(T_P T, O) \subset G$. Consequently

$$s(x,r) \cap TE(T_P T, O) = \emptyset \quad \text{for} \quad 0 < r < 1/m_0$$

and thus *a fortiori*

$$H_P[s(x,r) \cap TE(T_P T, O)] = 0 \quad \text{for} \quad 0 < r < 1/m_0 .$$

It follows (see 1.10) that $x \notin \Delta_*(T,O)$, and hence S cannot be significant for x. It is thus shown that the law $\mathfrak{L}_0$ satisfies the requirement (vi) in 1.6.

1.12. The basic pattern of the argument used in the present paper is analogous to that in the initial paper of Mickle [5], and full advantage has been taken of the improved methods developed in the subsequent papers [10] and [6]. However, due to the general character of the definition of the new law $\mathfrak{L}_0$, few of the results of the papers [5], [10], [6] could be used in the present paper without more or less substantial modifications, and several issues (including measurability questions and

F-invariance) required more elaborate treatment. Also, for various technical reasons, the notations in this paper differ occasionally from those used in the previous papers [5], [10], [6]. As we shall see in the concluding § 8, various intriguing problems still await solution in the line of thought initiated by Mickle, and accordingly the exposition in this paper is designed to facilitate the task of the reader who may wish to enter this field of research.

§ 2. Preliminaries

2.1. For convenient reference, we first summarize various simple facts about mappings. Let X, Y be arbitrary sets, and let $\varphi: X \to Y$ be an arbitrary mapping (single-valued transformation) from X into Y. If S is any subset of X, then φS denotes the set of those points $y \in Y$ for which there exists some point $x \in X$ such that $y = \varphi x$. If U is a subset of Y, then $\varphi^{-1}U$ denotes the set of those points $x \in X$ for which $\varphi x \in U$. A set $S \subset X$ is termed an *inverse set* under φ if there exists a set $U \subset Y$ such that $S = \varphi^{-1}U$. The mapping φ is said to be *onto* Y if $\varphi X = Y$.

LEMMA 1. *If S is any subset of X, then $S \subset \varphi^{-1}\varphi S$. The relation $S = \varphi^{-1}\varphi S$ holds if and only if S is an inverse set under φ.*

LEMMA 2. *If U is any subset of Y, then $U \supset \varphi\varphi^{-1}U$. If φ is onto, then $U = \varphi\varphi^{-1}U$ for every subset U of Y.*

LEMMA 3. *If A, B are any two subsets of X, then*

$$\varphi(A \cap B) \subset \varphi A \cap \varphi B .$$

However, if at least one of the sets A, B is an inverse set under φ, then

$$\varphi(A \cap B) = \varphi A \cap \varphi B .$$

LEMMA 4. *Let X_1, X_2, Y be arbitrary sets, and let $\varphi_1: X_1 \to Y$, $\varphi_2: X_2 \to Y$ be arbitrary mappings from X_1, X_2 respectively onto Y. Let A_1', A_1'' be subsets of X_1 which are inverse sets under φ_1. Put*

$$A_2' = \varphi_2^{-1}\varphi_1 A_1', \quad A_2'' = \varphi_2^{-1}\varphi_1 A_1'' .$$

Then

$$A_2' \cap A_2'' = \varphi_2^{-1}\varphi_1(A_1' \cap A_1'') .$$

LEMMA 5. *Let φ_1, φ_2 have the same meaning as in Lemma 4. Let A_1 be a subset of X_1 which is an inverse set under φ_1. Put $A_2 = \varphi_2^{-1}\varphi_1 A_1$. Then the following holds:*

(i) *A_2 is an inverse set under φ_2.*

(ii) $\varphi_1 A_1 = \varphi_2 A_2$.

(iii) $A_1 = \varphi_1^{-1}\varphi_2 A_2$.

The simple proofs of the preceding lemmas are left to the reader.

2.2. Let X, Y be compact metric spaces, and let $\varphi: X \to Y$ be a continuous mapping from X onto Y.

LEMMA 1. *A subset U of Y is open if and only if $\varphi^{-1}U$ is open.*

LEMMA 2. *Let S be a subset of X which is an inverse set under φ. Then S is open if and only if φS is open.*

LEMMA 3. *Let $\varphi_1: X_1 \to Y$, $\varphi_2: X_2 \to Y$ be continuous mappings from the compact metric spaces X_1, X_2 respectively onto the compact metric space Y. Let A_1 be a subset of X_1 which is an inverse set under φ_1. Put*

$$A_2 = \varphi_2^{-1}\varphi_1 A_1 .$$

Then the following holds:

(i) *A_2 is an inverse set under φ_2.*

(ii) $\varphi_1 A_1 = \varphi_2 A_2$.

(iii) $A_1 = \varphi_1^{-1}\varphi_2 A_2$.

(iv) *A_2 is closed if and only if A_1 is closed.*

(v) *A_2 is open if and only if A_1 is open.*

LEMMA 4. *Let Z and Y be metric spaces, let X be a compact metric space, and let $\varphi: X \to Y$, $\psi: Y \to Z$ be continuous mappings from X into Y and from Y into Z respectively. Put $\tau = \psi\varphi$. Let C be an m. m. c. under τ* (see 1.2). *Then C is a union of m. m. c.-s under φ.*

The simple proofs of these lemmas are left to the reader.

2.3. A continuous mapping $m: X \to Y$ from a compact metric space X *onto* a compact metric space Y is termed *monotone* if for every point $y \in Y$ the set $m^{-1}y$ is connected (and hence is a continuum). As an obvious consequence of the definition, a subset A of X is an inverse set under m if and only if A is a union of m. m. c.-s under m.

2.4. LEMMA. *Let $m_1: X_1 \to \mathfrak{M}$, $m_2: X_2 \to \mathfrak{M}$ be monotone mappings from the compact Hausdorff spaces X_1, X_2 respectively onto the compact Hausdorff space $\mathfrak{M}$. Let A_1 be a subset of X_1 which is a union of m. m. c.-s under m_1. Put $A_2 = m_2^{-1}m_1 A_1$. Then the following holds:*

(i) *A_2 is a union of m. m. c.-s under m_2.*

(ii) $m_1 A_1 = m_2 A_2$.

(iii) $A_1 = m_1^{-1}m_2 A_2$.

(iv) *A_2 is closed if and only if A_1 is closed.*

(v) *A_2 is open if and only if A_1 is open.*

(vi) *A_2 is connected if and only if A_1 is connected.*

Proof. By 2.3, A_1 is an inverse set under m_1. Thus (ii), (iii), (iv), (v) follow directly from 2.2, lemma 3. Since A_2 is clearly an inverse set under m_2, by 2.3 it is a union of m.m.c.-s under m_2, and (i) follows. As regard (vi), note that if A_1 is connected, then m_1A_1 is connected (since m_1 is continuous). Since the inverse of a connected set under a monotone mapping is connected (see [8], II. 1.2), it follows that $A_2=m_2^{-1}(m_1A_1)$ is connected. Conversely, if A_2 is connected, then it follows by the same argument that $A_1=m_1^{-1}m_2A_2$ is connected.

2.5. Lemma 1. *Let $m_1\colon X_1\to\mathfrak{M}$, $m_2\colon X_2\to\mathfrak{M}$ be monotone mappings from the compact metric spaces X_1, X_2 respectively onto the compact metric space $\mathfrak{M}$, and let ψ be an arbitrary continuous mapping from $\mathfrak{M}$ into a metric space Y. Put $\tau_1=\psi m_1$, $\tau_2=\psi m_2$. Let C_1 be an m.m.c. under τ_1. Then the set $C_2=m_2^{-1}m_1C_1$ is an m.m.c. under τ_2.*

Proof. By 2.2, lemma 4, C_1 is a union of m.m.c.-s under m_1. Hence, by 2.4, we have the following facts at our disposal:

(a) $m_2C_2=m_1C_1$.

(b) $C_1=m_1^{-1}m_2C_2$.

(c) C_2 is a continuum.

Now since C_1 is an m.m.c. under τ_1, we have

$$\tau_1C_1=y\,, \tag{1}$$

where y is a point in Y. On applying ψ to the relation (a), we obtain $\tau_2C_2=\tau_1C_1$. Hence, by (1), $\tau_2C_2=y$, and consequently $C_2\subset\tau_2^{-1}y$. In view of (c) it follows that C_2 is a connected subset of $\tau_2^{-1}y$, and thus C_2 is contained in a (unique) m.m.c. C_2' for y under τ_2. We have then

$$C_2\subset C_2'\subset\tau_2^{-1}y\,. \tag{2}$$

Let us put

$$C_1'=m_1^{-1}m_2C_2'\,. \tag{3}$$

The argument used above in connection with the m.m.c. C_1 yields, if applied to the m.m.c. C_2', the relation

$$C_2'=m_2^{-1}m_1C_1'\,, \tag{4}$$

as well as the fact that C_1' is a continuum. From (b), (2), (3) we infer that

$$C_1\subset C_1'\subset m_1^{-1}m_2\tau_2^{-1}y\,. \tag{5}$$

Now since $\tau_2=\psi m_2$, we have

$$m_1^{-1}m_2\tau_2^{-1}y=m_1^{-1}m_2m_2^{-1}\psi^{-1}y=m_1^{-1}\psi^{-1}y=\tau_1^{-1}y\,,$$

where we used the fact that

$$m_2m_2^{-1}\psi^{-1}y=\psi^{-1}y\,,$$

since m_2 is onto (see 2.1, lemma 2). In view of (5) it follows that

$$C_1\subset C_1'\subset\tau_1^{-1}y\,. \tag{6}$$

Now since C_1 is a component of $\tau_1^{-1}y$ and C_1' is connected, (6) implies that $C_1=C_1'$. By (4) we conclude (on applying $m_2^{-1}m_1$) that $C_2=C_2'$, an m.m.c. under τ_2, and the lemma is proved.

Lemma 2. *Under the assumptions of lemma 1, consider a subset S_1 of X_1 which is a union of m.m.c.-s under τ_1. Put $S_2=m_2^{-1}m_1S_1$. Then the following holds:*

(i) *S_2 is a union of m.m.c.-s under τ_2.*

(ii) $m_1S_1=m_2S_2$.

(iii) $S_1=m_1^{-1}m_2S_2$.

(iv) *S_2 is closed if and only if S_1 is closed.*

(v) *S_2 is open if and only if S_1 is open.*

(vi) *S_2 is connected if and only if S_1 is connected.*

Proof. Let F_1 be the class of those m.m.c.-s under τ_1 whose union is S_1. Thus

$$S_1=\bigcup C_1\,,\qquad C_1\in F_1\,. \tag{7}$$

It follows that

$$S_2=\bigcup m_2^{-1}m_1C_1\,,\qquad C_1\in F_1\,.$$

By the preceding lemma 1, each one of the sets $m_2^{-1}m_1C_1$ is an m.m.c. under τ_2, and thus (i) is established. Note now that by 2.2, lemma 4, each one of the sets $C_1\in F_1$ is a union of m.m.c.-s under m_1. Hence, by (7), S_1 is also a union of m.m.c.-s under m_1, and thus (ii), (iii), (iv), (v), (vi) follow directly from 2.4.

2.6. Let $m\colon X\to\mathfrak{M}$ be a monotone mapping from the compact metric space X onto the compact metric space $\mathfrak{M}$, and let l be a *light mapping* from $\mathfrak{M}$ into a metric space Y (the assumption that l is light means that for every point $y\in Y$ the set $l^{-1}y$ is either empty or totally disconnected). Then (see [8], II. 1.19) a set $S\subset X$ is an m.m.c. under the mapping $\tau=lm$ if and only if it is an m.m.c. under m.

2.7. Consider now two continuous F-equivalent mappings $\varphi_1\colon X_1\to Y$, $\varphi_2\colon X_2\to Y$ from the compact metric spaces X_1, X_2 into the metric space Y (see 1.6). The following two lemmas are immediate consequences of the definition of F-equivalent mappings.

Lemma 1. *There exists a sequence h_n of homeomorphisms from X_1 onto X_2 such that the mappings φ_2h_n, $\varphi_1h_n^{-1}$ converge uniformly to the mappings φ_1, φ_2 respectively.*

For each positive integer n, let O_n be the $(1/n)$-neighborhood of S (the set of those points (u,v) whose distance from S is less than $1/n$). Since S is compact and lies in Q^0, there exists a positive integer N such that the closure $\bar{O}_n$ of O_n is contained in Q^0 for $n \geqslant N$. Also, since S is compact by (2), we have

$$S = \bigcap_{n=N}^{\infty} \bar{O}_n . \tag{3}$$

We assert that

$$\lambda(S, T, \mathfrak{L}_0') = \bigcap_{n=N}^{\infty} \Delta_*(T, O_n) . \tag{4}$$

Indeed, consider any point $x \in \lambda(S, T, \mathfrak{L}_0')$. Then $S \in \mathfrak{L}_0'(T, x)$, and since $S \subset O_n \subset Q^0$ for $n \geqslant N$, it follows that $x \in \Delta_*(T, O_n)$ for $n \geqslant N$. Hence

$$x \in \lambda(S, T, \mathfrak{L}_0') \quad \text{implies} \quad x \in \bigcap_{n=N}^{\infty} \Delta_* T, O_n) . \tag{5}$$

Assume, conversely, that

$$x \in \bigcap_{n=N}^{\infty} \Delta_*(T, O_n) . \tag{6}$$

Take any open set O such that $S \subset O \subset Q^0$. From (3) we conclude that $O_n \subset O$ for $n \geqslant N'$, where N' is a properly chosen positive integer. Thus clearly (see 1.10)

$$\Delta_*(T, O_n) \subset \Delta_*(T, O) \quad \text{for} \quad n \geqslant N' .$$

On taking $n > N + N'$, there follows in view of (6) the inclusion $x \in \Delta_*(T, O)$. Since $S \subset K_*$ by (2), we conclude that S is significant for x under T according to the law $\mathfrak{L}_0'$, and hence $x \in \lambda(S, T, \mathfrak{L}_0')$. Thus (6) implies that $x \in \lambda(S, T, \mathfrak{L}_0')$, and in view of (5) the relation (4) is proved. Now since $\Delta_*(T, O_n)$ is a Borel set (see 1.10), we conclude from (4) that $\lambda(S, T, \mathfrak{L}_0')$ is a Borel set, and the lemma is proved.

3.4. For $A \subset R^3$, let $c(x, A)$ be the characteristic function of A (that is, $c(x, A) = 1$ if $x \in A$ and $c(x, A) = 0$ if $x \notin A$). Let the continuous mapping T be fixed. Let F be a generic notation for a finite (perhaps empty) system of sets of the class K^* (see 3.1). We put (see 3.3)

$$\varphi(x, F) = \sum c[x, \lambda(S, T, \mathfrak{L}_0')], \qquad S \in F , \tag{7}$$

$$\Phi(x) = \underset{F}{\text{l.u.b.}}\, \varphi(x, F) . \tag{8}$$

LEMMA 4. *$\Phi(x)$ is a Borel measurable function of the point $x \in R^3$.*

Proof. By lemma 3 the sets $\lambda(S, T, \mathfrak{L}_0')$ are Borel measurable, and hence by (7) the functions $\varphi(x, F)$ are Borel measurable. Since the class K^* is countable (see 3.1) the class of all the systems F is also countable. Thus $\Phi(x)$, as the least upper bound of a countable class of Borel measurable functions, is also Borel measurable.

LEMMA 5. $\Phi(x) = k(x, T, \mathfrak{L}_0')$.

Proof. Consider a system F and a point $x \in R^3$. Those sets $S \in F$ which are significant for x under T according to the law $\mathfrak{L}_0'$ constitute a system $\mathfrak{S}(x, T, \mathfrak{L}_0')$, and clearly

$$\varphi(x, F) = N[\mathfrak{S}(x, T, \mathfrak{L}_0')] .$$

Since $N[\mathfrak{S}(x, T, \mathfrak{L}_0')] \leqslant k(x, T, \mathfrak{L}_0')$, and since the system F was arbitrary, we conclude (see (8)) that

$$\Phi(x) \leqslant k(x, T, \mathfrak{L}_0') . \tag{9}$$

Consider now any system $\mathfrak{S}(x, T, \mathfrak{L}_0')$. In view of 3.2 (b), this is also a system F, and clearly

$$N[S(x, T, \mathfrak{L}_0')] = \varphi(x, F) .$$

Since $\varphi(x, F) \leqslant \Phi(x)$, and since the system $\mathfrak{S}(x, T, \mathfrak{L}_0')$ was arbitrary, it follows that

$$k(x, T, \mathfrak{L}_0') \leqslant \Phi(x) ,$$

and thus in view of (9) the lemma is proved.

3.5. THEOREM. *$k(x, T, \mathfrak{L}_0)$ is a Borel measurable function of the point $x \in R^3$.*

This is a direct consequence of the lemmas 5, 4, and 2.

§ 4. *F*-invariance

4.1. The following proof for the F-invariance of the law $\mathfrak{L}_0$ (see 1.6, 1.11) is similar to the corresponding proof of Mickle [6] for the F-invariance of his law $\mathfrak{L}_M$ (see 1.9). The heart of the proof is the following fundamental lemma of Mickle (for the proof of this lemma, the reader is referred to Mickle [5]):

LEMMA 1 (of Mickle). *Let μ_1, μ_2 be F-equivalent monotone mappings from the square Q* (see 1.1) *onto a Peano space $\mathfrak{M}$, and let l be a light mapping from $\mathfrak{M}$ into a plane R^2. Put $\tau_1 = l\mu_1$, $\tau_2 = l\mu_2$. Let C_1 be an e.m.m.c. under τ_1* (see 1.8). *Then the set $C_2 = \mu_2^{-1}\mu_1 C_1$ is an e.m.m.c. under τ_2.*

4.2. In view of the general character of our scheme for defining the law $\mathfrak{L}_0$ (see 1.4, 1.5, 1.11), it will be a matter of convenience to use a generalization of the lemma 1 of Mickle (the generalization consists in replacing the light mapping l occurring in lemma 1 by an arbitrary continuous mapping).

LEMMA 2. *Let m_1, m_2 be F-equivalent monotone mappings from the square Q onto a Peano space $\mathfrak{M}$, and let ψ be a continuous mapping from $\mathfrak{M}$ into a plane R^2. Put $\tau_1=\psi m_1$, $\tau_2=\psi m_2$. Let C_1 be an e. m. m. c. under τ_1. Then the set $C_2=m_2^{-1}m_1C_1$ is an e. m. m. c. under τ_2.*

Proof. It is convenient to divide the proof into three steps.

Step 1. We note that C_2 is an m.m.c. under τ_2 by 2.5, lemma 1.

Step 2. We take a monotone-light factorization $\psi=lm$ (see [8], II. 1.17) of ψ, where m is a monotone mapping from $\mathfrak{M}$ onto a Peano space $\mathfrak{M}'$ and l is a light mapping from $\mathfrak{M}'$ into the plane R^2. On setting

$$\mu_1=mm_1, \quad \mu_2=mm_2,$$

we have then

(1) $$\tau_1=l\mu_1, \quad \tau_2=l\mu_2.$$

Since m_1, m_2 are F-equivalent, clearly μ_1, μ_2 are also F-equivalent. Furthermore, μ_1 and μ_2 are monotone (see [8], II. 1.4). Hence by lemma 1 the set

(2) $$C_2'=\mu_2^{-1}\mu_1C_1$$

is an e.m.m.c. under τ_2. By (1) and (2) we have

(3) $$C_2'=m_2^{-1}m^{-1}mm_1C_1.$$

Now by 2.1, lemma 1, we have the inclusion

$$m^{-1}mm_1C_1\supset m_1C_1,$$

and hence (see (3))

(4) $$C_2'\supset m_2^{-1}m_1C_1=C_2.$$

Step 3. We note that C_2' is an m.m.c. under τ_2 (see Step 2), and C_2 is also an m.m.c. under τ_2 (see Step 1). Since $C_2'\supset C_2$ by (4), it follows that $C_2=C_2'$. Since C_2' is an e.m.m.c. under τ_2 (see Step 2), the lemma follows.

4.3. Let m_1, m_2 be F-equivalent monotone mappings from the square Q onto a Peano space $\mathfrak{M}$, and let ψ be a continuous mapping from $\mathfrak{M}$ into a plane R^2. Put $\tau_1=\psi m_1$, $\tau_2=\psi m_2$.

LEMMA 3. *Let O_2 be an open set in the interior Q^0 of Q which is a union of m. m. c.-s under m_2. Put $O_1=m_1^{-1}m_2O_2$. Then the following holds:*

(i) *O_1 is open.*

(ii) *$O_1\subset Q^0$.*

(iii) *O_1 is a union of m. m. c.-s under m_1.*

(iv) *$O_2=m_2^{-1}m_1O_1$.*

(v) *$m_1E(\tau_1,O_1)=m_2E(\tau_2,O_2)$* (see 1.8).

Proof. (i), (iii), (iv) follow directly from 2.4, and (ii) follows directly from 2.10. In view of (i), (ii), (iii), (iv) the relation between O_1 and O_2 is symmetric, and hence in verifying (v) it is sufficient to prove the inclusion

(5) $$m_1E(\tau_1,O_1)\subset m_2E(\tau_2,O_2).$$

Consider any e.m.m.c. C_1 under τ_1 such that $C_1\subset O_1$. Clearly (5) will be established if we show that

(6) $$m_1C_1\subset m_2E(\tau_2,O_2).$$

Let us put

(7) $$C_2=m_2^{-1}m_1C_1.$$

Since m_1, m_2 are *onto*, by 2.1, lemma 2, we have then

(8) $$m_1C_1=m_2C_2.$$

Now since $C_1\subset O_1$, by (iv) above (which is already verified) we conclude that

(9) $$C_2=m_2^{-1}m_1C_1\subset m_2^{-1}m_1O_1=O_2.$$

By lemma 2, C_2 is an e.m.m.c. under τ_2. Hence we infer from (9) that

(10) $$C_2\subset E(\tau_2,O_2).$$

Clearly (8) and (10) imply (6), and the lemma is proved.

4.4. LEMMA 4. *Let m_1, m_2 be F-equivalent monotone mappings from the square Q onto a Peano space $\mathfrak{M}$, and let l be a light mapping from $\mathfrak{M}$ into Euclidean 3-space R^3. Put*

(11) $$T_1=lm_1, \quad T_2=lm_2.$$

Let $O_2\subset Q^0$ be an open set which is a union of m. m. c.-s under T_2. Put $O_1=m_1^{-1}m_2O_2$. Then O_1 is an open subset of Q^0, and

(12) $$T_1E(T_PT_1,O_1)=T_2E(T_PT_2,O_2)$$

for every point $P\in U$ (see 1.9).

Proof. By 2.6, a subset of Q is an m. m. c. under T_2 if and only if it is an m.m.c. under m_2. Hence lemma 3 applies (with $\psi=T_Pl$). We conclude from lemma 3 that O_1 is an open subset of Q^0 and

$$m_1E(T_PT_1,O_1)=m_2E(T_PT_2,O_2).$$

On applying the mapping l to the preceding relation, we obtain (12), and the lemma is proved.

Lemma 5. *Under the same assumptions as in lemma 4, we have the relation*

(13) $$\Delta_*(T_1, O_1) = \Delta_*(T_2, O_2).$$

Proof. In view of the definition of Δ_* (see 1.10), the formula (13) is an immediate consequence of (12).

4.5. Again, let m_1, m_2 be F-equivalent monotone mappings from the square Q onto a Peano space $\mathfrak{M}$, and let l be a light mapping from $\mathfrak{M}$ into Euclidean 3-space R^3. Put $T_1 = lm_1$, $T_2 = lm_2$.

Lemma 6. *Let S_1 be a subset of Q which is significant for a point $x \in R^3$ under T_1 according to the law $\mathfrak{L}_0$* (see 1.11). *Put*

$$S_2 = m_2^{-1} m_1 S_1.$$

Then S_2 is significant for the point x under T_2 according to the law $\mathfrak{L}_0$.

Proof. The assumptions imply (see 1.11) that S_1 is a non-empty compact subset of Q^0 which is a union of m.m.c.-s under T_1. By 2.5, lemma 2, it follows that S_2 is non-empty and compact and is the union of m.m.c.-s under T_2. By 2.2, lemma 4, S_1 is also a union of m.m.c.-s under m_1, and since $S_1 \subset Q^0$, we infer from 2.10 that $S_2 \subset Q^0$.

Now consider any open set O such that

$$S_2 \subset O \subset Q^0.$$

In view of the facts already established, the lemma will be proved (see 1.11) if we show that

(14) $$x \in \Delta_*(T_2, O).$$

Now let O_2 be the union of all those m.m.c.-s under m_2 which lie in O. Since S_2 is a union of m.m.c.-s under T_2 and hence also under m_2 (see 2.2, lemma 4), it follows that

$$S_2 \subset Q_2 \subset O \subset Q^0.$$

Furthermore, O_2 is open (see [8], II. 1.12, II. 1.13). Hence on setting

$$O_1 = m_1^{-1} m_2 O_2,$$

we infer from 2.4 and 2.10 that O_1 is open and $O_1 \subset Q^0$. Also, it follows by 2.4 that

$$S_1 = m_1^{-1} m_2 S_2.$$

Since $S_2 \subset O_2$, we conclude that

$$S_1 \subset m_1^{-1} m_2 O_2 = O_1.$$

Since S_1 is significant for x under T_1 according to the law $\mathfrak{L}_0$, it follows (see 1.11) that

(15) $$x \in \Delta_*(T_1, O_1).$$

Note that O_2 is a union of m.m.c.-s under m_2 and hence (by 2.6) also a union of m.m.c.-s under T_2. Hence, by lemma 5,

(16) $$\Delta_*(T_1, O_1) = \Delta_*(T_2, O_2).$$

Since $O_2 \subset O$, we infer from (15) and (16) that

$$x \in \Delta_*(T_2, O_2) \subset \Delta_*(T_2, O).$$

Thus (14) is verified, and the lemma is proved.

Theorem. *If $T_1: Q \to R^3$, $T_2: Q \to R^3$ are F-equivalent continuous mappings, then*

$$k(x, T_1, \mathfrak{L}_0) = k(x, T_2, \mathfrak{L}_0).$$

Proof. Since T_1, T_2 are F-equivalent, there exist (see [12]) monotone-light factorizations of the form

$$T_1 = lm_1, \quad T_2 = lm_2,$$

where m_1, m_2 are F-equivalent monotone mappings from Q onto the same Peano space $\mathfrak{M}$ and l is a light mapping from $\mathfrak{M}$ into R^3. Consider now any system $\mathfrak{S}(x, T_1, \mathfrak{L}_0)$, consisting of sets $S_1^1, \ldots, S_1^n$ (see 1.11, 1.4). Put

$$S_2^j = m_2^{-1} m_1 S_1^j, \quad j = 1, \ldots, n.$$

By lemma 6, the sets $S_2^1, \ldots, S_2^n$ are significant for x under T_2 according to the law $\mathfrak{L}_0$. We proceed to verify that

(17) $$S_2^j \frown S_2^k = \emptyset \quad \text{for} \quad j \neq k.$$

Note that S_2^j, S_2^k are unions of m.m.c.-s under T_1 and hence (by 2.2, lemma 4) also under m_1. Since m_1 is monotone, it follows that S_2^j, S_2^k are inverse sets under m_1. Thus, by 2.1, lemma 4, we have

$$S_2^j \frown S_2^k = m_2^{-1} m_1 (S_1^j \frown S_1^k).$$

Since the sets $S_1^1, \ldots, S_1^n$ are pair-wise disjoint, (17) follows, and we conclude that $S_2^1, \ldots, S_2^n$ constitute a system $S(x, T_2, \mathfrak{L}_0)$ such that

$$N[\mathfrak{S}(x, T_1, \mathfrak{L}_0)] = N[\mathfrak{S}(x, T_2, \mathfrak{L}_0)].$$

Since $N[\mathfrak{S}(x, T_2, \mathfrak{L}_0)] \leqslant k(x, T_2, \mathfrak{L}_0)$, it follows that

$$N[\mathfrak{S}(x, T_1, \mathfrak{L}_0)] \leqslant k(x, T_2, \mathfrak{L}_0).$$

As the system $\mathfrak{S}(x, T_1, \mathfrak{L}_0)$ was arbitrary, we conclude (see 1.4) that

$$k(x, T_1, \mathfrak{L}_0) \leqslant k(x, T_2, \mathfrak{L}_0).$$

Exchanging the roles of the subscripts 1 and 2, we obtain by the same argument the complementary inequality

$$k(x,T_2,\mathfrak{L}_0)\leqslant k(x,T_1,\mathfrak{L}_0)\,,$$

and the theorem follows.

§ 5. An inequality

Consider a continuous mapping $T\colon Q\to R^3$ (see 1.1). Denote by Ω' the family of those open circular discs d^0 in the (u,v)-plane which satisfy the following conditions:

(i) The coordinates of the center of d^0 are rational numbers. (ii) The radius of d^0 is a positive rational number. (iii) d^0 is contained in the interior Q^0 of Q.

Let Ω'' be the family of those sets which are finite unions of open circular discs $d^0\in\Omega'$. Clearly, Ω'' is a countable family.

LEMMA 1. *For every point* $P\in U$ *we have the inclusion* (see 1.9, 1.10, 1.8)

(1) $$D_PTE(T_PT,O)\subset\Delta_*(T,O)\,,$$

where O *is any open subset of* Q^0.

Proof. The set $E(T_PT,O)$ is a Borel set, and hence $TE(T_PT,O)$ is an analytic set. Thus $TE(T_PT,O)$ is H^2-measurable (see for example [4]). Accordingly, (1) follows directly from 2.11, lemma 4 (applied with $S=TE(T_PT,O)$), in view of the definition of the set $\Delta_*(T,O)$.

LEMMA 2. *For every point* $P\in U$, *put*

(2) $$e_P=\bigcup[TE(T_PT,O'')-D_PTE(T_PT,O'')]\,,\qquad O''\in\Omega''\,.$$

Then $H_PE_P=0$.

Proof. If S is an H^2-measurable subset of R^3, then (see 2.11, lemma 5) we have

$$H_P(S-D_PS)=0\,.$$

As we noted in the course of the proof of lemma 1, each one of the sets $TE(T_PT,O'')$ is H^2-measurable (note that the sets $O''\in\Omega''$ are clearly open). Hence

$$H_P[TE(T_PT,O'')-D_PTE(T_PT,O'')]=0\quad\text{for}\quad O''\in\Omega''\,,$$

and since Ω'' is a countable family, the relation $H_Pe_P=0$ follows.

LEMMA 3. *Let* C *be an e.m.m.c. under* T_PT (see 1.8), *and let* $x\in R^3$ *be a point such that* (see (2))

(3) $x\in TC$,

(4) $x\notin e_P$.

Then C *is significant for* x *under* T *according to the law* $\mathfrak{L}_0$.

Proof. By the definition of an e.m.m.c., C is non-empty and compact, and $C\subset Q^0$. Also, C is a union of m.m.c.-s under T by 2.2, lemma 4. Hence (see 1.11) there remains to show that if O is any open set such that

$$C\subset O\subset Q^0\,,$$

then $x\in\Delta_*(T,O)$. Now observe that since C is a compact subset of the open set $O\subset Q^0$, clearly we can select an open set $O''\in\Omega''$ such that

$$C\subset O''\subset O\,.$$

As C is an e.m.m.c. under T_PT, it follows that

$$C\subset E(T_PT,O'')\,.$$

Hence, in view of (3), we have

$$x\in TE(T_PT,O'')\,.$$

By (4) and (2) we conclude that

(5) $$x\in D_PTE(T_PT,O'')\,.$$

Now $D_PTE(T_PT,O'')\subset\Delta_*(T,O'')$ by lemma 2, and

$$\Delta_*(T,O'')\subset\Delta_*(T,O)\,,$$

since $O''\subset O$. Hence (5) implies that $x\in\Delta_*(T,O)$, and the lemma is proved.

THEOREM 1. *If* P *is any point of* U, *then* (see 1.9, 1.8, 1.11, 1.4)

(6) $$\varkappa(x_P,T_PT,Q)\leqslant\sum k(x,T,\mathfrak{L}_0),\qquad T_Px=x_P\,,$$

almost everywhere in the plane $R^2(P)$ (*where* x_P *is a generic notation for a point in* $R^2(P)$).

Proof. Since $H_Pe_P=0$ by lemma 2, we have $L_2T_Pe_P=0$ (see 2.11, lemma 6). Hence it is sufficient to establish (6) for the case when

(7) $$x_P\notin T_Pe_P\,.$$

As (6) is obvious if $\varkappa(x_P,T_PT,Q)=0$, we can also assume that $\varkappa(x_P,T_PT,Q)\geqslant 1$. Let m be any integer such that

(8) $$1\leqslant m\leqslant\varkappa(x_P,T_PT,Q)\,.$$

Then we can select m distinct e.m.m.c.-s $C_1,\ldots,C_m$ for x_P under T_PT. Let us put

$$g=T_P^{-1}x_P\,.$$

Then g is a straight line which passes through x_P and is perpendicular to the plane $R^2(P)$. Clearly

(9) $$\sum_{T_Px=x_P}k(x,T,\mathfrak{L}_0)=\sum_{x\in g}k(x,T,\mathfrak{L}_0)\,.$$

Now $T_P T C_j = x_P$ by assumption, and hence

(10) $$TC_j \subset g, \quad j=1,\dots,m.$$

Also, by (7),

(11) $$g \cap e_P = \emptyset.$$

For each point $x \in g$, let $n(x)$ denote the number (perhaps zero) of those of the integers $j=1,\dots,m$ for which $x \in TC_j$. Note that in view of (10) and (11) the inclusion $x \in TC_j$ implies, by lemma 3, that C_j is significant for x under T according to the law $\mathfrak{L}_0$. Hence

$$k(x,T,\mathfrak{L}_0) \geqslant n(x) \quad \text{for} \quad x \in g.$$

Since obviously

$$\sum_{x \in g} n(x) \geqslant m,$$

it follows that

(12) $$m \leqslant \sum k(x,T,\mathfrak{L}_0), \quad x \in g.$$

Since m was an arbitrary integer satisfying (8), clearly (12) and (9) imply (6).

Remark. In 1.10 we noted that the law $\mathfrak{L}'_M$ seems to be inadequate in the case when $A(T)=\infty$. The reason is that the preceding theorem does not seem to hold for the multiplicity function $k(x,T,\mathfrak{L}'_M)$.

LEMMA 4. *For every point $P \in U$, we have the inequality*

$$\iint_{R^2(P)} \varkappa(x_P, T_P T, Q)\, dL_2 \leqslant \iint_{R^3} k(x,T,\mathfrak{L}_0)\, dH^2.$$

This follows directly from theorem 1, by integrating the inequality (6) (see the references in [5], [10], [6] for the corresponding inequalities relating to the laws $\mathfrak{L}$ used there).

§ 6. Differentiable mappings

6.1. We assume throughout the present § 6 that the continuous mapping $T: Q \to R^3$ (see 1.1) satisfies the following conditions.

(i) The first partial derivatives x_{iu}, x_{iv}, $i=1,2,3$, exist almost everywhere in Q.

Accordingly the Jacobians

$$J_1 = x_{2u}x_{3v} - x_{2v}x_{3u}, \quad J_2 = x_{3u}x_{1v} - x_{3v}x_{1u}, \quad J_3 = x_{1u}x_{2v} - x_{1v}x_{2u},$$

as well as the function

$$W = (J_1^2 + J_2^2 + J_3^2)^{1/2}$$

are defined almost everywhere in Q.

(ii) W is summable in Q. Thus

$$\iint_Q W\, du\, dv < \infty, \quad \iint_Q |J_i|\, du\, dv < \infty, \quad i=1,2,3.$$

(iii) $A(T) = \iint_Q W\, du\, dv.$

6.2. Given T as in 6.1, select a point $P \in U$ (see 1.9) with coordinates x_1^P, x_2^P, x_3^P. Let us introduce a new Cartesian coordinate system x'_1, x'_2, x'_3 (with the same origin) such that the positive direction of the x'_3-axis coincides with the direction from the origin to P. Then T appears in the form

(1) $$T: x'_i = x'_i(u,v), \quad i=1,2,3, \quad (u,v) \in Q,$$

where x'_1, x'_2, x'_3 are related to x_1, x_2, x_3 by an orthogonal transformation. Hence the partial derivatives x'_{iu}, x'_{iv} exist almost everywhere in Q, and the function W' corresponding to the representation (1) of T is equal to W almost everywhere in Q. Since $A(T)$ is independent of the choice of the coordinate system, in view of 6.1 (iii) it follows that

(2) $$A(T) = \iint_Q W'\, du\, dv.$$

The mapping $T_P T$ (see 1.9) is now given by the formulas

(3) $$T_P T: \ x'_1 = x'_1(u,v), \quad x'_2 = x'_2(u,v), \quad (u,v) \in Q,$$

and the Jacobian $J_P = x'_{1u}x'_{2v} - x'_{1v}x'_{2u}$ of $T_P T$ is given by the formula

(4) $$J_P = x_1^P J_1 + x_2^P J_2 + x_3^P J_3.$$

By [8], V. 2.64 we infer from (2) that the mapping $T_P T$ is eAC *(essentially absolutely continuous)* in Q. For our present purposes, it is convenient to use the characterization of eAC mappings given in [9], according to which the eAC mapping $T_P T$ has the following properties:

(i) The essential multiplicity function $\varkappa(x_P, T_P T, Q)$ (see 1.8), corresponding to the mapping $T_P T$, is summable in $R^2(P)$. Thus

$$\iint_{R^2(P)} \varkappa(x_P, T_P T, Q)\, dL_2 < \infty.$$

In the preceding formula, x_P is a generic notation for a point in $R^2(P)$.

(ii) If S is any subset of $E(T_P T, Q^0)$ (see 1.8) such that $L_2 S = 0$, then $L_2 T_P T S = 0$.

(iii) $\iint_Q |J_P|\, du\, dv = \iint_{R^2(P)} \varkappa(x_P, T_P T, Q)\, dL_2.$

6.3. Given T as in 6.1, let (u_0, v_0) be a point in the interior Q^0 of Q such that the partial derivatives x_{iu}, x_{iv}, $i=1,2,3$, exist at (u_0, v_0). For $(u,v) \in Q$, $i=1,2,3$, we put

$$\xi_i(u,v,u_0,v_0) = x_i(u,v) - x_i(u_0,v_0) - x_{iu}(u_0,v_0)(u-u_0) - x_{iv}(u_0,v_0)(v-v_0).$$

For $h>0$, let $s(h,u_0,v_0)$ be the perimeter of the square with center (u_0,v_0), side-length $2h$, and sides parallel to the u and v axes respectively. Since $(u_0,v_0) \in Q^0$, we shall have $s(h,u_0,v_0) \subset Q^0$ for h sufficiently small. Assuming that $s(h,u_0,v_0) \subset Q^0$, we denote by $M(h,u_0,v_0)$ the maximum of the function

$$f(u,v,u_0,v_0) = \frac{\left[\sum_{i=1}^{3} \xi_i(u,v,u_0,v_0)^2\right]^{1/2}}{[(u-u_0)^2+(v-v_0)^2]^{1/2}}$$

for $(u,v) \in s(h,u_0,v_0)$. We shall say that *the mapping T possesses a weak total differential at the point (u_0,v_0)* if there exists a sequence of numbers h_n such that

(5) $$h_n > 0, \quad h_n \to 0, \quad M(h_n,u_0,v_0) \to 0.$$

As a consequence of the assumption 6.1 (i) we have then (see [7], 10.8 through 10.12) the following statement:

LEMMA 1. *The mapping T (given as in 6.1) possesses a weak total differential almost everywhere in Q^0.*

6.4. The mapping T of 6.1 is given (see 1.1) by equations of the form

(6) $$T: x_i = x_i(u,v), \quad i=1,2,3, \quad (u,v) \in Q.$$

Let us introduce, in terms of these functions $x_i(u,v)$, the auxiliary mappings

$$T_1: x_2 = x_2(u,v), \ x_3 = x_3(u,v), \ (u,v) \in Q,$$
$$T_2: x_3 = x_3(u,v), \ x_1 = x_1(u,v), \ (u,v) \in Q,$$
$$T_3: x_1 = x_1(u,v), \ x_2 = x_2(u,v), \ (u,v) \in Q,$$

from Q into the three coordinate planes respectively. Consider the mapping T_1, for example. Let $(u_0,v_0) \in Q^0$ be a point where T possesses a weak total differential. We have then a sequence of numbers h_n satisfying 6.3 (5). Let us denote by $M_1(h_n,u_0,v_0)$ the maximum of the function

$$f_1(u,v,u_0,v_0) = \frac{\left[\sum_{i=2}^{3} \xi_i(u,v,u_0,v_0)^2\right]^{1/2}}{[(u-u_0)^2+(v-v_0)^2]^{1/2}}$$

for $(u,v) \in s(h_n,u_0,v_0)$ (see 6.3). Since clearly

$$0 \leq M_1(h_n,u_0,v_0) \leq M(h_n,u_0,v_0)$$

and $M(h_n,u_0,v_0) \to 0$, we have $M_1(h_n,u_0,v_0) \to 0$. Let us now assume that $J_1(u_0,v_0) \neq 0$ (see 6.1). On denoting by $Q(h_n,u_0,v_0)$ the square whose perimeter is $s(h_n,u_0,v_0)$, we have then by [8], IV. 3.35, the relation (see 1.8)

$$\mu[T_1(u_0,v_0), T_1, Q(h_n,u_0,v_0)] = \operatorname{sgn} J_1(u_0,v_0) \neq 0$$

for n sufficiently large. Thus, for n sufficiently large, $Q(h_n,u_0,v_0)$ is an indicator region for $T_1(u_0,v_0)$ under T_1. Since $h_n \to 0$, we conclude that the point (u_0,v_0) is an e.m.m.c. under T_1. Similar considerations apply of course to the mappings T_2, T_3. Furthermore, it is clear that if T possesses a weak total differential at a point $(u_0,v_0) \in Q^0$, then this property is preserved if T is represented in the form 6.2 (1) in terms of a new coordinate system x_1', x_2', x_3'. The mapping $T_P T$ takes then the place of the mapping T_1, and we obtain thus the following statement:

LEMMA 2. *Let $(u_0,v_0) \in Q^0$ be a point where T (given as in 6.1) possesses a weak total differential, and let P be any point of U. If $J_P(u_0,v_0) \neq 0$* (see 6.2), *then the point (u_0,v_0) is an e.m.m.c. under $T_P T$.*

Consider now a point $(u_0,v_0) \in Q^0$ where T possesses a weak total differential. Assume that $W(u_0,v_0) > 0$ (see 6.1). Then $J_i(u_0,v_0) \neq 0$ for at least one of the integers $i=1,2,3$, say for $i=2$. By lemma 2 it follows that (u_0,v_0) is an e.m.m.c. under $T_P T$ for $P=(0,1,0)$, and hence by 2.2, lemma 4, the point (u_0,v_0) is an m.m.c. under T. We obtain thus the following statement:

LEMMA 3. *If $(u_0,v_0) \in Q^0$ is a point where $W(u_0,v_0) > 0$ and T possesses a weak total differential, then (u_0,v_0) is an m.m.c. under T.*

Combining the lemmas 1, 2, 3, we obtain the following statement:

LEMMA 4. *Given T as in 6.1, the square Q can be represented as the union of three pair-wise disjoint Borel sets B_0, B_*, B such that the following holds:*

(i) $L_2 B_0 = 0$, $B_* \subset Q^0$, $B \subset Q^0$.

(ii) *W exists and is equal to zero on B_*.*

(iii) *W exists and is positive on B.*

(iv) *T possesses a weak total differential at every point of $B_* \cup B$.*

(v) *Every point of B is an m.m.c. under T.*

(vi) *If $(u_0,v_0) \in B$, $P \in U$, and $J_P(u_0,v_0) \neq 0$, then (u_0,v_0) is an e.m.m.c. under $T_P T$.*

6.5. LEMMA 5. *Given T, B_0, B_*, B as in lemma 4, let X_∞ be the set of those points $x \in R^3$ where $N(x,T,B) = \infty$* (see 1.2). *Then $H^2 X_\infty = 0$.*

Proof. Since W exists on B and is summable on B (see 6.1 (ii)), we have (see [2], p. 309)

$$\iint_{R^3} N(x,T,B)\,dH^2 = \iint_B W\,du\,dv < \infty .$$

Thus $N(x,T,B)$ is H^2-summable, and hence $H^2X_\infty = 0$.

Lemma 6. *Given T,B_0,B_*,B as in lemma 4, let S be a Borel subset of B such that $L_2S=0$. Then $H^2TS=0$.*

Proof. By [2], p. 309, we have

$$(7)\qquad \iint_{R^3} N(x,T,S)\,dH^2 = \iint_S W\,du\,dv .$$

Now since $L_2S=0$, the integral on the right in (7) is equal to zero. As $N(x,T,S)\geqslant 1$ for $x\in TS$, it follows that $H^2TS=0$.

Lemma 7. *Given T,B_0,B_*,B as in lemma 4, let S be a Borel subset of B such that* (see 1.9, 6.2)

(i) $H_PTS=0$,

(ii) $J_P\neq 0$ *on* S.

Then $H^2TS=0$.

Proof. (i) implies (see 2.11, lemma 6) that

$$(8)\qquad L_2T_PTS=0 .$$

By [2], p. 309, we have

$$(9)\qquad \iint_S |J_P|\,du\,dv = \iint_{R^2(P)} N(x_P,T_PT,S)\,dL_2 ,$$

where x_P is a generic notation for a point in the plane $R^2(P)$ (see 1.9). From (8) we infer that the integral on the right in (9) is equal to zero. Hence, by (9),

$$\iint_S |J_P|\,du\,dv = 0 .$$

In view of (ii) it follows that $L_2S=0$, and thus $H^2TS=0$ by lemma 6.

Lemma 8. *Given T,B_0,B_*,B as in lemma 4, denote by S_P the set of those points $(u,v)\in B$ where $J_P\neq 0$. Let s be an arbitrary subset of S_P such that $H_PTs=0$. Then $H^2Ts=0$.*

Proof. The assumption $H_PTs=0$ implies (see 2.11, lemma 6) that

$$(10)\qquad L_2T_PTs=0 .$$

Hence we have a Borel set G such that

$$G\supset T_PTs,\qquad L_2G=0 .$$

Let us put

$$S=S_P\cap (T_PT)^{-1}G .$$

Then S is a Borel set, and clearly

$$s\subset S,\qquad T_PTS\subset G .$$

Hence $L_2T_PTS=0$, and thus (see 2.11, lemma 2) $H_PTS=0$. Since $S\subset S_P$, we have $J_P\neq 0$ on S. By lemma 7 we conclude that $H^2TS=0$. Since $s\subset S$, it follows that $H^2Ts=0$.

Lemma 9. *Given T,B_0,B_*,B as in lemma 4, denote again by S_P the set of those points $(u_0,v_0)\in B$ where $J_P\neq 0$. Let (u_0,v_0) be a point in S_P, and let O be an open set such that $(u_0,v_0)\in O\subset Q^0$. Then $(u_0,v_0)\in E(T_PT,O)$.*

Proof. Since (u_0,v_0) is an e.m.m.c. under T_PT by part (vi) of lemma 4, this statement is a direct consequence of the definition of the set $E(T_PT,O)$ (see 1.8).

Lemma 10. *Given T,B_0,B_*,B,S_P as in lemma 9, let s_P be the set consisting of those points (u,v) which satisfy the following conditions:*

(i) $(u,v)\in S_P$.

(ii) *(u,v) is not significant for $T(u,v)$ under T according to the law $\mathfrak{L}_0$* (see 1.11).

Then $H^2Ts_P=0$.

Proof. Note that each point $(u,v)\in s_P$ is an e.m.m.c. under T_PT by part (vi) of lemma 4. Since, by (ii), (u,v) is not significant for $T(u,v)$ under T according to the law $\mathfrak{L}_0$, we conclude from 5.1, lemma 3, that $T(u,v)\in e_P$. Since this inclusion holds for every point $(u,v)\in s_P$, it follows that $Ts_P\subset e_P$, and hence $H_PTs_P=0$ by 5.1, lemma 2. Since $s_P\subset S_P$, by lemma 8 it follows that $H^2Ts_P=0$.

Lemma 11. *Given T,B_0,B_*,B as in lemma 4, let s_0 be the set of those points (u,v) which satisfy the following conditions:*

(i) $(u,v)\in B$.

(ii) *(u,v) is not significant for $T(u,v)$ under T according to the law $\mathfrak{L}_0$.*

Then $H^2Ts_0=0$.

Proof. Let s_i, $i=1,2,3$, be the set of those points (u,v) which satisfy the following conditions: (a) $(u,v)\in B$; (b) $J_i(u,v)\neq 0$ (see 6.1); (c) (u,v) is not significant for $T(u,v)$ according to the law $\mathfrak{L}_0$. We have then

$$(11)\qquad H^2Ts_i=0,\quad i=1,2,3,$$

by lemma 10, applied successively with $P=(0,0,1)$, $P=(0,1,0)$, $P=(1,0,0)$. Since W exists and is positive on B, clearly

$$(12)\qquad s_0=s_1\cup s_2\cup s_3 .$$

From (11) and (12) it follows that $H^2Ts_0=0$.

Lemma 12. *Given* T, B_0, B_*, B *as in lemma* 4, *put* (see lemma 5 and lemma 11)

$$X = X_\infty \cup Ts_0. \tag{13}$$

Then (see 1.11, 1.4)

$$k(x, T, \mathfrak{L}_0) \geqslant N(x, T, B) \quad \text{for} \quad x \in X. \tag{14}$$

Proof. Since (14) is obvious if $N(x,T,B)=0$, we can assume that $N(x,T,B) \geqslant 1$. The assumption $x \in X$ implies (see (13)) that $x \notin X_\infty$, and hence $N(x,T,B) < \infty$. On setting

$$m = N(x, T, B), \tag{15}$$

we have therefore $1 \leqslant m < \infty$, and there exist m distinct points $(u_1, v_1), \ldots, (u_m, v_m)$ in B such that $T(u_j, v_j) = x$, $j = 1, \ldots, m$. Since $x \notin Ts_0$ by (13), we conclude from lemma 11 that (u_j, v_j) is significant for x under T according to the law $\mathfrak{L}_0$, $j = 1, \ldots, m$. Hence clearly $k(x, T, \mathfrak{L}_0) \geqslant m$, and (14) follows in view of (15).

6.6. As we noted in 1.10, in our approach we replace the sets $O \cap E(T_P T, Q^0)$ used by Mickle in [6] by the sets $E(T_P T, O)$. The lemmas in the present section 6.6 constitute corresponding modifications of certain fundamental portions of the argument employed by Mickle [6].

Lemma 13. *Given* T, B_0, B_*, B *as in lemma* 4, *let* S *be a Borel subset of* Q^0 *such that*

$$H_P T(S - B) = 0, \tag{16}$$

and let A *be a Borel set in* R^3. *Then*

$$H_P(A \cap TS) = H_P[A \cap T(B \cap S)]. \tag{17}$$

Proof. We have the decomposition

$$S = (S - B) \cup (B \cap S),$$

and thus

$$A \cap TS = [A \cap T(S - B)] \cup [A \cap T(B \cap S)]. \tag{18}$$

By (16) we conclude that

$$H_P[A \cap T(S - B)] = 0,$$

and thus (17) follows from (18).

Lemma 14. *Given* T, B_0, B_*, B *as in lemma* 4, *let* O *be an open subset of* Q^0 *and let* A *be a Borel subset of* R^3. *Then, for every point* $P \in U$,

$$H_P[A \cap TE(T_P T, O)] \leqslant H^2[A \cap T(B \cap O)].$$

Proof. We first verify that

$$H_P T[E(T_P T, O) - B] = 0. \tag{19}$$

Indeed (see lemma 4), we have the decomposition

$$T[E(T_P T, O) - B] = T[E(T_P T, O) \cap B_0] \cup T[E(T_P T, O) \cap B_*]. \tag{20}$$

Now since $L_2 B_0 = 0$, we have also

$$L_2[E(T_P T, O) \cap B_0] = 0,$$

and hence, by 6.2 (ii),

$$L_2 T_P T[E(T_P T, O) \cap B_0] = 0.$$

By 2.11, lemma 2, we conclude that

$$H_P T[E(T_P T, O) \cap B_0] = 0. \tag{21}$$

Let us put

$$G = E(T_P T, O) \cap B_*.$$

By [2], p. 309, we have

$$\iint_{R^3} N(x, T, G)\, dH^2 = \iint_G W\, du\, dv.$$

Since $W = 0$ on B_* and hence on G, and since $N(x,T,G) \geqslant 1$ on the set TG, it follows that $H^2 TG = 0$. Thus

$$H^2 T[E(T_P T, O) \cap B_*] = 0,$$

and hence also (see 2.11, lemma 1)

$$H_P T[E(T_P T, O) \cap B_*] = 0,$$

and (19) follows in view of (20) and (21). Thus we can apply lemma 13 with $S = E(T_P T, O)$, obtaining the relation

$$H_P[A \cap TE(T_P T, O)] = H_P\{A \cap T[B \cap E(T_P T, O)]\}.$$

Since always $H_P \leqslant H^2$ (see 2.11, lemma 1) and $E(T_P T, O) \subset O$, the inequality asserted in the lemma follows.

Lemma 15. *Given* T, B_0, B_*, B *as in lemma* 4, *let* O *be any open subset of* Q^0. *Then* (see 1.10 and 2.11, lemma 3)

$$\Delta_*(T, O) \subset DT(B \cap O). \tag{22}$$

Proof. We have

$$\Delta_*(T, O) = \bigcup_n \bigcap_m \bigcup_P G[n, m, TE(T_P T, O), P], \tag{23}$$

where $n = 1, 2, \ldots$, $m = 1, 2, \ldots$, $P \in U$. By lemma 14 we have

$$\frac{H^2[s(x,r) \cap T(B \cap O)]}{r^2 \pi} \geqslant \frac{H_P[s(x,r) \cap TE(T_P T, O)]}{r^2 \pi}$$

for $r>0$, and hence (see 1.9)

(24) $$G[n,m,TE(T_PT,O),P]\subset G[n,m,T(B\cap O)],$$

where $G[n,m,T(B\cap O)]$ denotes the set of those points $x\in R^3$ where

$$\frac{H^2[s(x,r)\cap T(B\cap O)]}{r^2\pi}>\frac{1}{n}$$

for some r such that $0<r<1/m$. Since clearly

$$DT(B\cap O)=\bigcup_n\bigcap_m G[n,m,T(B\cap O)],\qquad n,m=1,2,\ldots,$$

the inclusion (22) follows from (23) and (24).

6.7. Using the symbols Ω', Ω'' in the sense of 5.1, we put (see lemma 4)

(25) $$e=\bigcup[DT(B\cap O'')-T(B\cap O'')],\qquad O''\in\Omega''.$$

LEMMA 16. *Given T,B_0,B_*,B as in lemma 4, we have $H^2e=0$.*

Proof. By [2], p. 309, we have, in view of 6.1 (ii),

$$\iint_{R^3}N(x,T,B)\,dH^2=\iint_B W\,du\,dv<\infty.$$

As $N(x,T,B)\geqslant 1$ on the set TB, it follows that $H^2TB<\infty$, and hence *a fortiori* $H^2T(B\cap O'')<\infty$. By 2.11, lemma 3, we conclude that

$$H^2[DT(B\cap O'')-T(B\cap O'')]=0.$$

Since Ω'' is countable, the relation $H^2e=0$ follows in view of (25).

LEMMA 17. *Given T,B_0,B_*,B as in lemma 4, let S be a subset of Q^0 and x a point of R^3 such that*

(i) $x\notin e$ (see (25)),

(ii) *S is significant for x under T according to the law $\mathfrak{L}_0$* (see 1.11).

We have then for every set $O''\in\Omega''$ such that $S\subset O''$ the inclusion

(26) $$x\in T(B\cap O'').$$

Proof. From (ii) it follows that $x\in\Delta_*(T,O'')$, and hence $x\in DT(B\cap O'')$ by lemma 15. Since $x\notin e$, the inclusion (26) follows.

LEMMA 18. *Given T,B_0,B_*,B as in lemma 4, we have*

(27) $$k(x,T,\mathfrak{L}_0)\leqslant N(x,T,B)\quad\text{for}\quad x\notin e.$$

Proof. Since (27) is obvious if $k(x,T,\mathfrak{L}_0)=0$, we can assume that $k(x,T,\mathfrak{L}_0)\geqslant 1$. Choose any integer m such that

(28) $$1\leqslant m\leqslant k(x,T,\mathfrak{L}_0).$$

We can then select (see 1.11, 1.4) m pair-wise disjoint sets $S_1,\ldots,S_m$ in Q^0 which are significant for x under T according to the law $\mathfrak{L}_0$. Since these sets are compact, we can further select pair-wise disjoint sets $O_1'',\ldots,O_m''$ from the class Ω'' such that $S_j\subset O_j''$, $j=1,\ldots,m$. By lemma 17 we have then, since $x\notin e$, the inclusions

$$x\in T(B\cap O_j''),\qquad j=1,\ldots,m.$$

We have therefore points (u_j,v_j) such that

$$(u_j,v_j)\in B,\quad (u_j,v_j)\in O_j'',\quad T(u_j,v_j)=x,\quad j=1,\ldots,m.$$

Since the sets $O_1'',\ldots,O_m''$ are pair-wise disjoint, it follows that the points $(u_1,v_1),\ldots,(u_m,v_m)$ are distinct, and hence

$$m\leqslant N(x,T,B).$$

As m was any integer satisfying (28), the inequality (27) follows.

THEOREM. *Given T as in 6.1, we have*

(29) $$A(T)=\iint_{R^3}k(x,T,\mathfrak{L}_0)\,dH^2.$$

Proof. From the lemmas 18 and 14 we infer that

(30) $$k(x,T,\mathfrak{L}_0)=N(x,T,B)\quad\text{for}\quad x\notin X\cup e.$$

Now $H^2X=0$, $H^2e=0$ by the lemmas 5, 11, 16 (see (13)). Hence (30) yields

(31) $$\iint_{R^3}k(x,T,\mathfrak{L}_0)\,dH^2=\iint_{R^3}N(x,T,B)\,dH^2.$$

By [2], p. 309, we have

(32) $$\iint_{R^3}N(x,T,B)\,dH^2=\iint_B W\,du\,dv.$$

Now (see lemma 4) recall that

$$Q=B_0\cup B_*\cup B,\quad L_2B_0=0,\quad W=0\ \text{on}\ B_*.$$

Thus

(33) $$\iint_B W\,du\,dv=\iint_Q W\,du\,dv.$$

Finally, by 6.1 (iii),

$$\iint_Q W\,du\,dv=A(T),$$

and (29) follows in view of (33), (32), (31).

§ 7. The integral formula for $A(T)$

THEOREM. *For every continuous mapping $T\colon Q\to R^3$ (see 1.1) we have the relation*

$$A(T)=\iint_{R^3} k(x,T,\mathfrak{L}_0)\,dH^2. \tag{1}$$

Proof. By lemma 4 in § 5 we have the inequality

$$\iint_{R^2(P)} \varkappa(x_P,T_PT,Q)\,dL_2 \leqslant \iint_{R^3} k(x,T,\mathfrak{L}_0)\,dH^2. \tag{2}$$

Let us put

$$a(P)=\iint_{R^2(P)} \varkappa(x_P,T_PT,Q)\,dL_2. \tag{3}$$

By [3], 0.3, we have

$$A(T)=\frac{1}{2\pi}\iint_U a(P)\,d\sigma_P, \tag{4}$$

where $d\sigma_P$ is the area-element on the unit sphere $U\colon x_1^2+x_2^2+x_3^2=1$. From (2), (3), (4) we conclude that

$$\iint_{R^3} k(x,T,\mathfrak{L}_0)\,dH^2 \geqslant \tfrac{1}{2}A(T).$$

It follows that (1) holds if $A(T)=\infty$. Hence we can assume that $A(T)<\infty$. But then, by a fundamental result of Cesari [1], there exists a continuous mapping $T^*\colon Q\to R^3$ which is F-equivalent to T and satisfies the conditions stated in 6.1 (and which possesses further properties not needed in the present context). Since T^* is F-equivalent to T, by the theorem in 4.5 we have

$$k(x,T,\mathfrak{L}_0)=k(x,T^*,\mathfrak{L}_0),$$

and hence *a fortiori*

$$\iint_{R^3} k(x,T,\mathfrak{L}_0)\,dH^2=\iint_{R^3} k(x,T^*,\mathfrak{L}_0)\,dH^2. \tag{5}$$

By the theorem in 6.7 (applied to T^*) we have

$$A(T^*)=\iint_{R^3} k(x,T^*,\mathfrak{L}_0)\,dH^2. \tag{6}$$

Since T and T^* are F-equivalent, we have (see [8], V. 2.3)

$$A(T)=A(T^*). \tag{7}$$

Since (7), (6), (5) yield (1), the theorem is proved.

§ 8. Conclusion

8.1. Since the Hausdorff measure H^2 was designed to evaluate area (two-dimensional extent) of figures in R^3, a very natural approach in surface area theory is to consider the quantity (see 1.2)

$$A_1(T)=\iint_{R^3} N(x,T,Q)\,dH^2$$

as the area of the surface represented by T (see 1.1). Due to various disadvantages of this definition, $A_1(T)$ has been replaced by various authors by the quantities (see 1.2)

$$A_2(T)=\iint_{R^3} N^*(x,T,Q)\,dH^2, \qquad A_3(T)=\iint_{R^3} N^*(x,T,Q^0)\,dH^2$$

respectively, where in the last formula Q^0 denotes the interior of Q. Noting that if a set $S\subset Q$ is significant for a point x under T according to the law $\mathfrak{L}_0$, then (see 1.1) $S\subset Q^0$, S is the union of m.m.c.-s under T, and $x\in TS$, it is obvious that we have the inequalities

$$k(x,T,\mathfrak{L}_0)\leqslant N^*(x,T,Q^0)\leqslant N^*(x,T,Q)\leqslant N(x,T,Q), \tag{1}$$

and hence also (see 1.1)

$$A(T)\leqslant A_3(T)\leqslant A_2(T)\leqslant A_1(T). \tag{2}$$

Simple examples show that the sign of equality generally fails to hold in (2). Thus it appears that the most plausible choices for a multiplicity function $k(x,T)$ (see 1.1), namely $N(x,T,Q)$, $N^*(x,T,Q)$, $N^*(x,T,Q^0)$, generally fail to satisfy 1.1 (2). Actually, the construction of $k(x,T,\mathfrak{L}_0)$, as well as of the multiplicity functions occurring in [5], [10], [6], is based on the idea of reducing the plausible multiplicity functions N, N^* to the precise extent necessary to satisfy the formula 1.1 (2). In view of the rather involved character of these reduction processes, it may be of interest to point out that the problem of finding a multiplicity function satisfying 1.1 (2) admits of an altogether trivial solution. Indeed, given T as in 1.1, let us define a multiplicity function $k_t(x,T)$ (where the subscript t stands for trivial) in the following manner:

(a) If $A(T)=\infty$, then let $k_t(x,T)=1$ for every point $x\in R^3$.

(b) If $A(T)=0$, then let $k_t(x,T)=0$ for every point $x\in R^3$.

(c) If $0<A(T)<\infty$, then put

$$\varrho=\left[\frac{A(T)}{4\pi}\right]^{1/2},$$

and denote by U_ϱ the sphere $x_1^2+x_2^2+x_3^2=\varrho^2$. Define $k_t(x,T)=1$ if $x\in U_\varrho$ and $k_t(x,T)=0$ if $x\notin U_\varrho$.

We have then obviously

$$A(T)=\iint_{R^3} k_t(x,T)\,dH^2,$$

and furthermore, the reader will readily discover the *trivial law* $\mathfrak{L}_t$ that will produce this trivial multiplicity function, in the sense of 1.4.

8.2. It is now apparent that if trivialities are to be excluded, then the laws $\mathfrak{L}$ (see 1.4) should be subjected to some set $\{R\}$ of reasonable requirements R. This observation leads to an interesting problem which we proceed to formulate.

Definition 1. A set $\{R\}$ of requirements R, relating to the laws $\mathfrak{L}$ (see 1.4), is termed *categorical* if we have

$$k(x,T,\mathfrak{L}')=k(x,T,\mathfrak{L}'')$$

identically in x and T for any two laws $\mathfrak{L}'$, $\mathfrak{L}''$ which satisfy all the requirements of the set $\{R\}$.

Definition 2. Two laws $\mathfrak{L}_1$, $\mathfrak{L}_2$ (in the sense of 1.4) are termed *equivalent* if

$$k(x,T,\mathfrak{L}_1)=k(x,T,\mathfrak{L}_2)$$

identically in x and T.

In terms of Definition 1, the problem is to *find a categorical set $\{R\}$ of requirements.*

8.3. There arises the question whether the set of requirements (i) through (vi), stated in 1.6, is categorical in the sense just explained. In this connection, the following remark may be of interest. On replacing, in the definition of the law $\mathfrak{L}_0$ (see 1.11), the condition (ii) by the condition that S should be a continuum, we obtain a law to be denoted by $\mathfrak{L}_0^*$. An argument entirely analogous to that employed for $\mathfrak{L}_0$ reveals that $\mathfrak{L}_0^*$ also satisfies all the requirements (i) through (vi) in 1.6. However, this fact alone does not imply that the set of requirements (i) through (vi) in 1.6 is not categorical. As a matter of fact, a simple argument (based on Zorn's lemma) shows that $\mathfrak{L}_0^*$ and $\mathfrak{L}_0$ are equivalent (in the sense of 8.2, Definition 2). Thus the problem of discovering a categorical set $\{R\}$ of requirements is still open.

8.4. In view of the role played by the set $\Delta_*(T,O)$ (see 1.10) it probably occurred to the reader to consider the following law $\mathfrak{L}$: a set $S\subset Q$ is significant for x under T provided that

(i) $S\neq\emptyset$, $S\subset Q^0$.

(ii) S is open.

(iii) S is the union of m.m.c.-s under T.

(iv) $x\in\Delta_*(T,S)$.

However, the present writer was unable to show that the corresponding multiplicity function $k(x,T,\mathfrak{L})$ is H^2-measurable, and the F-invariance of $\mathfrak{L}$ is also a doubtful issue. An interesting situation arises, on the other hand, if one modifies the construction described in 1.4 by requiring that the elements of the systems $\mathfrak{S}(x,T,\mathfrak{L})$ should be *T-disjoint* in the following sense: two subsets S',S'' of Q are T-disjoint if no m.m.c. under T intersects the closure of both S', S''. After this modification, one sees readily that the requirements stated in 1.6 are met with the following qualifications:

(a) The requirement (vi) is satisfied in the weaker form that if S is significant for x under T, then x is contained in the closure of TS.

(b) The requirement (iv) is satisfied in the stronger form that if S is significant for x under T and if T^* agrees with T on S itself, then S is significant for x under T^* also.

Thus it would seem that this modified construction yields a satisfactory situation. However, it is clear that the concept of T-disjoint sets involves the behavior of T in all of Q, and hence the resulting multiplicity function cannot be properly said to be of local origin (even though the above law $\mathfrak{L}$ itself is *strongly of local character*, in the sense of (b) above).

Thus there arises the question whether there exists a law $\mathfrak{L}$ which is strongly of local character in the sense of (b) above and satisfies the requirements (i), (ii), (iii), (v), (vi) in the sense of the *original* scheme described in 1.4. In fact, it may be satisfactory to replace (vi) by the weaker version stated under (a) above.

8.5. We noted above (see 1.5) that in the initial paper [5] of Mickle, as well as in the subsequent papers [10] and [6], only m.m.c.-s under T were admitted as significant sets, while in defining our law $\mathfrak{L}_0$ (see 1.11) we admitted also sets which were unions of m.m.c.-s under T. Thus there arises the question whether there exists a law $\mathfrak{L}$ (see 1.4) which satisfies the requirements (i) through (vi) stated in 1.6, plus the additional requirement that if S is significant for x under T, then S should be an m.m.c. under T.

8.6. The Lebesgue area $A(T)$ is *cyclicly additive* (see [8], V. 2.55). Accordingly, one may want to add (to the list of requirements stated in 1.6), the further requirement that the multiplicity function $k(x,T,\mathfrak{L})$

should be, for each point $x \in R^3$, a cyclicly additive functional of T, in the sense of [8], II. 2.104. A rather trivial approach would be to define a multiplicity function k as the sum of multiplicity functions corresponding to the cyclic partial mappings associated with T (see [8], II. 2.94). A non-trivial answer to the problem so raised would have significant applications.

8.7. In view of the infinite variety of multiplicity functions studied in [5], [10], [6] and in the present paper, there arises the question whether there exist simple relationships between these multiplicity functions. To formulate a partial answer, we need the following

Definition. Two functions $f_1(x)$, $f_2(x)$ of the point $x \in R^3$ are termed *H^2-equivalent* if the set of those points x where $f_1(x) \neq f_2(x)$ is of H^2-measure zero.

Inspection of the proofs in [5], [10], [6] and in the present paper reveals that in the case $A(T) < \infty$ all the multiplicity functions $k(x, T, \mathfrak{L})$ in question are H^2-equivalent to $N(x, T, B)$, where B is defined as in 6.4, lemma 4. Accordingly, it follows that any two of these multiplicity functions are H^2-equivalent if $A(T) < \infty$. This observation yields two further problems. First, of course, there arises the problem of clarifying the situation in the case $A(T) = \infty$. Second, it is now natural to relax the problem stated in 8.2 as follows. Let us say that a set of requirements $\{R\}$ is *H^2-categorical* if for any two laws $\mathfrak{L}_1, \mathfrak{L}_2$ satisfying all the requirements of the set $\{R\}$ it is true that the corresponding multiplicity function $k(x, T, \mathfrak{L}_1)$, $k(x, T, \mathfrak{L}_2)$ are H^2-equivalent. The problem is then to discover an H^2-categorical set $\{R\}$ of requirements.

8.8. While we operated with H^2-measure, it would be quite natural to use integral-geometric measure (see [2]) in this line of thought. The approach suggested by this remark will be studied on another occasion.

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