Recursive families of sets *)

by

S. Mrówka (Warszawa)

We recall that a set M of non-negative integers is called *recursive denumerable* if there exists a recursive function $f \in N^N$ (N denoting the set of all non-negative integers) such that $M=f(N)$. A set M is called *recursive* if there exists a recursive function $f \in N^N$ such that $m \in M$ if and only if $f(m)=0$.

It is known that countable summation, in general, leads out of the classes of recursive denumerable or recursive sets. The problem arise for which countable families $\{A_n\}$ of recursive denumerable (or recursive) sets the sum $\sum_n A_n$ or the intersection $\prod_n A_n$ is a recursive denumerable (or recursive) set. A similar problem may be raised with respect to recursive functions: for which countable families $\{f_n(m)\}$ of recursive functions the functions $\sup_n f_n(m)$ and $\inf_n f_n(m)$ are recursive. The present paper is devoted to these problems.

§ 1. Recursive denumerable families

Let us introduce the following definitions:

Definition 1. A countable family $\{f_n(m)\}$ of functions is called *recursive denumerable* if the function $f(n,m)=f_n(m)$ is recursive.

Definition 2. A countable family $\{A_n\}$ of sets is called *recursive denumerable* if there exists a recursive denumerable family $\{f_n(m)\}$ of functions such that $A_n=f_n(N)$.

THEOREM 1. *The sum of a recursive denumerable family of sets is a recursive denumerable set.*

Proof. Let $p_1(x)$ and $p_2(x)$ be "functions of pair", *i. e.*, recursive functions such that if x runs over N then $\langle p_1(x), p_2(x)\rangle$ runs over the set of all pairs $\langle n,m\rangle$ where $n,m \in N$. We have

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$$\sum_n A_n = \mathop{E}_z \sum_n (z \in A_n) = \mathop{E}_z \sum_n \sum_m \big(z=f(m)\big) = \mathop{E}_z \sum_{\langle n,m\rangle} \big(z=f(n,m)\big)$$
$$= \mathop{E}_z \sum_x \Big(z=f\big(p_1(x),p_2(x)\big)\Big).$$

Since $f\big(p_1(x),p_2(x)\big)$ is a recursive function, the set $\sum_n A_n$ is a recursive denumerable set.

Let us notice that the intersection of a recursive denumerable family $\{A_n\}$ of sets is not necessarily a recursive denumerable set. The example is following:

Example 1. Let A be a recursive denumerable set such that $0 \in A$ and $N-A$ is not recursive denumerable. Let us set

$$f(n,m)=m \operatorname{sign} \prod_{i=0}^{n} \big(m-f(i)\big), \qquad f_n(m)=f(n,m), \qquad A_n=f_n(N)$$

(f is a recursive function such that $f(N)=A$). Since $f(n,m)$ is a recursive function, $\{A_n\}$ is a recursive denumerable family of sets. But $\prod_n A_n$ is equal to $N-A$, whence it is not recursive denumerable.

§ 2. Recursive families of functions

Let us introduce the following notation:

If $\{f_n(m)\}$ is a family of functions, we set

$$E_m = \mathop{E}_z \sum_n \big(z=f(m)\big) \quad \text{and} \quad E_m^k = \mathop{E}_z \sum_{n\leqslant k} \big(z=f(m)\big).$$

Definition 3. A family $\{f_n(m)\}$ of functions is called *strongly recursive* if the function $f(n,m)=f_n(m)$ is recursive and if there exists a recursive function $\varphi(m)$ such that

$$E_m = E_m^{\varphi(m)}.$$

Definition 4. A family $\{f_n(m)\}$ of functions is called *recursive* if the function $f(n,m)=f_n(m)$ is recursive and if there exists a recursive function $\varphi(m,k)$ such that

$$(k \in E_m) \equiv (k \in E_m^{\varphi(m,k)}).$$

It is obvious that a strongly recursive family of functions is recursive, namely it is enough to set $\varphi(m)=\varphi(m,k)$. We also see that if $\{f_n(m)\}$ is a strongly recursive family of functions, then the set E_m is finite for every m. It follows at once that there exist recursive families of functions which are not strongly recursive (for example the family $\{f_n(m)\}$ where $f_n(m) = m+n$ and we set $\varphi(m,k) = k$). In the sequel we shall give an example of a recursive family of functions for which the set E_m is finite for every m, but which is not strongly recursive.

THEOREM 2. *If $\{f_n(m)\}$ is a recursive family of functions then the function $\inf_n f_n(m)$ is recursive.*

Proof. Evidently

$$\inf_n f_n(m) = \inf E_m .$$

Let us set

$$\psi(m,k) = 1 - \operatorname{sign} \prod_{i=0}^{\varphi(m,k)} \big(k - f_i(m)\big)^2 .$$

The function $\psi(m,k)$ is obviously recursive and $\psi(m,k)=1$ if and only if k belongs to $E_m^{\varphi(m,k)}$, and thus to E_m.

We have

$$\inf E_m =$$

$$0\cdot\psi(m,0) + 1\cdot\psi(m,1)[1-\psi(m,0)] + 2\psi(m,2)[1-\psi(m,0)][1-\psi(m,1)] +$$
$$+ \ldots + f_0(m)\psi\big(m, f_0(m)\big)[1-\psi(m,0)][1-\psi(m,1)]\ldots[1-\psi(m, f_0(m) \dot- 1)] .$$

Since $\psi(m,k)$ and $f_0(m)$ are recursive, $\inf E_m$ is also recursive.

THEOREM 3. *If $\{f_n(m)\}$ is a strongly recursive family of functions, then the function $\sup_n f_n(m)$ is recursive.*

Proof. Evidently

$$\sup_n f_n(m) = \sup E_m = \sup E_m^{\varphi(m)} .$$

Let us set

$$\psi(m,i) = \operatorname{sign} \prod_{k=0}^{\varphi(m)} \{[f(m)+1] \dot- f(m)\} .$$

The function $\psi(m,i)$ is obviously recursive and $\psi(m,i)=1$ if and only if $f_i(m) = \sup E_m^{\varphi(m)}$. Hence

$$\sup E_m^{\varphi(m)} = f_0(m)\psi(m,0) + f_1(m)\psi(m,1)[1-\psi(m,0)] +$$
$$+ f_2(m)\psi(m,2)[1-\psi(m,0)][1-\psi(m,1)] +$$
$$+ f_{\varphi(m)}(m)\psi\big(m,\varphi(m)\big)[1-\psi(m,0)][1-\psi(m,1)]\ldots[1-\psi(m,\varphi(m)-1)] .$$

Since the function $f(n,m) = f_n(m)$ is recursive, the function $\sup E_m^{\varphi(m)}$ is also recursive.

THEOREM 4. *A recursive family of functions $\{f_n(m)\}$ is strongly recursive if and only if there exists a recursive function $\psi(m)$ such that $f_n(m) \leqslant \psi(m)$ for $n,m = 0,1,2,\ldots$*

Proof. If a family $\{f_n(m)\}$ is strongly recursive, then $\sup_n f_n(m)$ is recursive. Conversely, if there exists a recursive function $\psi(m)$ such that $f_n(m) \leqslant \psi(m)$, then $E_m = E_m^{\varphi^*(m,\psi(m))}$, where $\varphi^*\big(m,\psi(m)\big) = \max_{0\leqslant i\leqslant \psi(m)} \varphi(m,i)$, whence the family is strongly recursive.

Example 2. Let $h(n,m)$ be a recursive function with the values 0 and 1 such that the function $\sup_n h(n,m)$ is not recursive. Let us set

$$f(0,m) = 0, \qquad f(n,m) = nh(n-1,m) \dot- \sum_{i=0}^{n-2} h(i,m) \qquad (n = 1,2,\ldots),$$

and

$$f_n(m) = f(n,m) .$$

We see that $f_n(m) = 0$ or n and if $f_n(m) \neq 0$, then $f_{n'}(m) = 0$ for every $n' > n$. It follows that the set E_m is finite for every m and $(k \in E_m) \equiv (k \in E_m^k)$. Hence the family $\{f_n(m)\}$ is recursive. On the other hand $\sup_n f_n(m) \neq 0$ if and only if $\sup_n h(n,m) = 1$, whence $\sup_n f_n(m)$ is not recursive. By Theorem 3 the family is not strongly recursive, although the set E_m is finite for every m.

The following example shows that if $\{f_n(m)\}$ is a strongly recursive family of functions, then the functions $\overline{\lim}_n f_n(m)$ and $\underline{\lim}_n f_n(m)$ are not necessarily recursive.

Example 3. Let $h(n,m)$ be a recursive function with the values 0 and 1 such that the functions $\sup_n h(n,m)$ and $\inf h(n,m)$ are not recursive. Let us set

$$\varphi(n) = n - [\sqrt{n}]^2 .$$

We see that the function $\varphi(n)$ is recursive and takes every non-negative integer value infinitely many times. In fact, if $n = (p+q)^2 + p$ $(q = 1,2,\ldots)$, then

$$(p+q)^2 \leqslant (p+q)^2 + p < (p+q+1)^2 = (p+q)^2 + 2(p+q) + 1 ,$$

whence

$$p+q \leqslant \sqrt{(p+q)^2+p} < p+q+1$$

and

$$[\sqrt{(p+q)^2+p}] = p+q$$

and

$$\varphi(n) = (p+q)^2 + p - (p+q)^2 = p .$$

Let us set

$$f(0,m)=0, \quad f(1,m)=1, \quad f(n,m)=h(\psi(n),m), \quad n=2,3,\ldots,$$

and

$$f_n(m)=f(n,m).$$

Since $E_m=E_m^1$, the family $\{f_n(m)\}$ is strongly recursive. But $\overline{\lim\limits_n} f_n(m) = \sup\limits_n h(n,m)$, $\underline{\lim\limits_n} f_n(m)=\inf\limits_n h(n,m)$, whence $\overline{\lim\limits_n} f_n(m)$ and $\underline{\lim\limits_n} f_n(m)$ are not recursive.

THEOREM 5. *If $\{f_n(m)\}$ and $\{g_n(m)\}$ are recursive (strongly recursive) families of functions, then the families*

$$\{\max[f_n(m),g_n(m)]\} \quad \textit{and} \quad \{\min[f_n(m),\, g_n(m)]\}$$

are recursive (strongly recursive).

The proof is evident.

Example 4. This example shows that if $\{f_n(m)\}$ and $\{g_n(m)\}$ are strongly recursive families of functions, then the family $\{f_n(m)+g_n(m)\}$ is not necessarily recursive.

Let $h(n,m)$ be a recursive function with the values 0 and 1 such that $\inf\limits_n h(n,m)$ is not recursive. Let us set

$$f(0,m)=0, \qquad g(0,m)=1$$
$$f(1,m)=1, \qquad g(1,m)=0$$
$$f(n,m)=h(n-2,m), \qquad g(n,m)=h(n-2,m) \quad \text{for} \quad n=2,3,\ldots$$

and

$$f(m)=f(n,m), \qquad g(m)=g(n,m).$$

The families $\{f_n(m)\}$ and $\{g_n(m)\}$ are evidently strongly recursive but $\inf\limits_n \big(f_n(m)+g_n(m)\big)=2\inf\limits_n h(n,m)$, whence $\inf\limits_n \big(f_n(m)+g_n(m)\big)$ is not recursive and the family is not even recursive. A similar example may be constructed for $\{f_n(m)\cdot g_n(m)\}$.

Example 5. For the family $\{f_n(m)\}$ mentioned in Example 2 the family $\{1\dot{-}f_n(m)\}$ is not recursive. In fact, we have

$$\inf_n \big(1\dot{-}f_n(m)\big)=1\dot{-}\sup_n f_n(m)=1\dot{-}\sup_n h(n,m),$$

whence $\inf\limits_n \big(1\dot{-}f_n(m)\big)$ is not recursive and the family $\{1\dot{-}f_n(m)\}$ is not recursive.

§ 3. Recursive families of sets

Definition 5. A family $\{A_n\}$ of sets is called *strongly recursive* if there exists a strongly recursive family $\{f_n(m)\}$ of functions such that $(m\in A_n)\equiv\big(f_n(m)=0\big)$.

Definition 6. A family $\{A_n\}$ of sets is called *recursive* if there exists a recursive family $\{f_n(m)\}$ of functions such that $(m\in A_n)\equiv\big(f_n(m)=0\big)$.

THEOREM 6. *If $\{A_n\}$ is a strongly recursive family of sets, then the family $\{N-A_n\}$ is also strongly recursive.*

Proof. Let $\{f_n(m)\}$ be a family of functions mentioned in Definition 5. Since the family $\{1\dot{-}f_n(m)\}$ is also strongly recursive and $m\in N-A_n$ if and only if $1\dot{-}f_n(m)=0$, the family $\{N-A_n\}$ is also strongly recursive.

THEOREM 7. *If $\{A_n\}$ is a strongly recursive family of sets, then the set $\prod\limits_n A_n$ is recursive.*

Proof. Let $\{f_n(m)\}$ be a family of functions mentioned in Definition 5. Since the function $\sup\limits_n f_n(m)$ is recursive and $m\in\prod\limits_n A_n$ if and only if $\sup\limits_n f_n(m)=0$, the set $\prod\limits_n A_n$ is recursive.

THEOREM 8. *If $\{A_n\}$ is a recursive family of sets, then the set $\sum\limits_n A_n$ is recursive.*

Proof. Let $\{f_n(m)\}$ be a family of functions mentioned in Definition 5. Since the function $\inf\limits_n f_n(m)$ is recursive and $m\in\sum\limits_n A_n$ if and only if $\inf\limits_n f_n(m)=0$, the set $\sum\limits_n A_n$ is recursive.

Example 6. Example 2 gives at once an example of a recursive family of sets which is not strongly recursive. It is, namely, the family $\{A_n\}$ where $A_n=\underset{n}{E}\big(f_n(m)=0\big)$ and $\{f_n(m)\}$ is the family mentioned in Example 2. The family is recursive but the set $\prod\limits_n A_n$ is not recursive, whence the family is not strongly recursive.

Example 7. For the family $\{A_n\}$ mentioned in Example 6 the family $\{N-A_n\}$ is not recursive. In fact, in the opposite case the set $\sum\limits_n (N-A_n)$ would be recursive, which contradicts $\prod\limits_n A_n=N-\sum\limits_n (N-A_n)$.

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