On homotopically stable points and product spaces

by

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§ 1. Introduction

Let X be a topological space. A point x_0 of X is called *homotopically labile* in X whenever for every neighbourhood U of x_0 there exists a continuous transformation $f(x,t)$ which is defined in the Cartesian product $X \times I$ of X and of the closed interval $I = \langle 0,1 \rangle$ and which satisfies the following conditions:

(1) $\quad f(x,t) \in X \quad$ for every $\quad (x,t) \in X \times I$,

(2) $\quad f(x,0) = x \quad$ for every $\quad x \in X$,

(3) $\quad f(x,t) = x \quad$ for every $\quad (x,t) \in (X-U) \times I$,

(4) $\quad f(x,t) \in U \quad$ for every $\quad (x,t) \in U \times I$,

(5) $\quad f(x,1) \neq x_0 \quad$ for every $\quad x \in X$.

A point x_0 of X is called *homotopically stable* if it is not homotopically labile. K. Borsuk and J. W. Jaworowski [5] introduced this notion and studied the various properties of labile and stable points.

In this paper, we shall study first a certain characteristic property of homotopically labile points in ANR's for metric spaces. This shows that "homotopical stability" is equivalent to "n-homotopical stability for some integer n [1])". The main theorem, which states that the homotopical lability or stability of a point in a product space is determined by the local connectivity groups at that point [1]), is proved in § 4. This theorem gives a generalization of H. Noguchi's theorem [21] to the case of ANR.

Let X and Y be two topological spaces. The equality $\dim X \times Y = \dim X + \dim Y$ does not generally hold; for example, K. Borsuk [4] has proved that there exist 2-dimensional Cantor manifolds whose Cartesian product has dimension three. In § 5 we shall show that this equality holds in the following two cases:

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[1]) For these definitions, see §§ 2 and 4.

1. X is a locally compact fully normal space and Y is a 2-dimensional locally compact ANR for metric spaces.
2. X and Y are locally compact ANR's for metric spaces satisfying certain conditions.

§ 2. Some characterizations of homotopically labile points

A topological space X is called an ANR *for metric spaces* if whenever X is a closed subset of a metric space Y, there exists a continuous transformation from some neighbourhood of X in Y onto X which keeps X point-wise fixed (cf. [14], Definition 2.2).

We introduce the following definitions:

A point x_0 of a topological space X is called *homotopically n-labile* in X, $n=0,1,2,\ldots$, when, for every neighbourhood U of x_0, there exists a neighbourhood V of x_0 which is contained in U and satisfies the following condition: Let E^{n+1} be an $(n+1)$-cell whose boundary is an n-sphere S^n. Then every continuous mapping $f\colon S^n\to V-x_0$ is extended to a continuous mapping $f'\colon E^{n+1}\to U-x_0$. A point x_0 of X is called *homotopically n-stable* in X if it is not homotopically n-labile in X.

For convenience, we shall use the following abbreviations:

ANR = ANR for metric spaces,
HL = homotopically labile,
n-HL = homotopically n-labile,
HL^n = homotopically i-labile for each integer $i=0,1,2,\ldots,n$,
HS = homotopically stable,
n-HS = homotopically n-stable.

Moreover, we shall understand by "mapping" a continuous transformation and denote by "dimension" the covering dimension of Lebesgue.

We shall establish the following theorem:

THEOREM 1. *Let X be an m-dimensional* ANR. *Then a point x_0 of X is* HL *in X if and only if x_0 is HL^{m-1} in X.*

To prove this theorem, it is convenient to state the following lemmas:

LEMMA 1. *Let X be an* ANR *and x_0 a point of X and let U be a neighbourhood of x_0. Then there exists a neighbourhood U_0 of x_0 contained in U with the following property: If f,g are two mappings of a metric space Y into X such that*

$$f(y)=g(y) \quad \text{for} \quad y\in Y-f^{-1}(U_0),$$
$$g(y)\in U_0 \quad \text{for} \quad y\in f^{-1}(U_0),$$

then there exists a mapping $F\colon Y\times I\to X$ such that

$$F(y,0)=f(y) \quad \text{and} \quad F(y,1)=g(y) \quad \text{for} \quad y\in Y,$$
$$F(y,t)=f(y) \quad \text{for} \quad y\in Y-f^{-1}(U_0),$$
$$F(y,t)\in U \quad \text{for} \quad y\in f^{-1}(U_0).$$

Proof (cf. [12], p. 40). According to a theorem of Wojdysławski ([25], p. 186), X can be imbedded as a closed set of a convex subset D of a normed vector space B. Since X is an ANR, there exist a neighbourhood W of X in D and a retraction $h\colon W\to X$. Let U be a neighbourhood of x_0 in X. We can find a spherical neighbourhood V of x_0 in D such that $V\subset\overline{V}\subset h^{-1}(U)$. Put $U_0=V\cap X$. Let Y be a metric space and let f,g be two mappings of Y into X satisfying conditions of Lemma 1. Since D is a convex set, V is a convex set. Hence, there exists a homotopy $k_t\colon Y\to X\cup V$ such that $k_t(y)=f(y)$ for $y\in Y-f^{-1}(U_0)$, $k_0(y)=f(y)$, $k_1(y)=g(y)$ for $y\in Y$. Then the homotopy $F\colon Y\times I\to X$ defined by $F(y,t)=hk_t(y)$ is the required one.

LEMMA 2. *Let X be an* ANR. *If U is an open subset of X, then U is an* ANR ([10], Lemma 3.1).

Proof of Theorem 1. 1) Sufficiency. Let x_0 be a point of X which is HL^{m-1}. There exist two neighbourhoods U, U_0 of x_0 which satisfy the condition of Lemma 1. Since x_0 is HL^{m-1}, we can construct a decreasing sequence of neighbourhoods U_i of x_0 such that $x_0\in U_i\subset\overline{U}_i\subset U_{i-1}$ and every mapping $f\colon S^j\to U_i-x_0$ has an extension $f'\colon E^{j+1}U_{i-1}-x_0$ for $i=1,2,\ldots,m+1$ and $j=0,1,2,\ldots,m-1$. Put $M=\overline{U}_{m-1}$, $N=\overline{U}_{m+1}-U_{m+1}$. Since M is a metric space and N is a closed subset of M, we can construct a space Y and a continuous mapping $h\colon M\to Y$ ([8], Theorem 3.1) such that

1° $h|N$ is a homeomorphism and $h(N)$ is closed in Y,

2° $P=Y-h(N)$ is an m-dimensional infinite complex with the weak topology and $h(M-N)\subset P$.

Moreover, by [8], p. 357, there exists a continuous extension $g_0\colon P^0\cup h(N)\to N$ of a mapping $g=h^{-1}\colon h(N)\to N$, where P^i is the i-skeleton of the complex P. Consider g_0 as a mapping $P^0\cup h(N)$ into U_m-x_0. Since U_m-x_0 is an ANR by Lemma 2, we can find a mapping g_0' and a neighbourhood V of $h(N)$ in Y such that $g_0'\colon P^0\cup V\cup h(N)\to U_m-x_0$ and $g_0'|P^0\cup h(N)=g_0$ (cf. for example, [12], p. 40). Let Q be a subcomplex consisting of closed simplexes of P contained in V. Then $Q\cup h(N)$ forms a closed neighbourhood of $h(N)$ in Y. Consider the mapping $g_0''=g_0'|Q\cup h(N)\colon Q\cup h(N)\to U_m-x_0$. By the constructions of U_i, we can find a continuous extension $g_1\colon P^1\cup Q\cup h(N)\to U_{m-1}-x_0$ of g_0'', since P has the weak topology. By a repeated application of this process, we can see that there exists a mapping g_m of Y into U_0-x_0 such that $g_m|h(N)=h^{-1}$. Define a mapping f of X into $X-x_0$ as follows:

$$f(x)=x \quad \text{for} \quad x\in X-U_{m+1},$$
$$f(x)=g_m h(x) \quad \text{for} \quad x\in \overline{U}_{m+1}.$$

If l is an identity mapping of X into X, then we can find a homotopy F between f and l whose existence is proved by Lemma 1. This homotopy means the homotopical lability of x_0.

2) Necessity. It is sufficient to prove that, if a point x_0 of X is n-HS for some n such that $0\leqslant n\leqslant m-1$, then x_0 is HS. Let x_0 be n-HS. By the definition of n-HS, there exists a neighbourhood U of x_0 satisfying the condition that, if V is a neighbourhood of x_0 contained in U, there exists at least one mapping $f\colon S^n\to V-x_0$ such that f has no extension $f'\colon E^{n+1}\to U-x_0$. Since X is an ANR, we may suppose that V is contractible in U[2]). Therefore we have an extension $g\colon E^{n+1}\to U$ such that $g|S^n=f$. Since $f(S^n)$ is compact, we can find a positive number ε such that $0<\varepsilon<\varrho(f(S^n),x_0)$, where ϱ is a metric in X. Assume that x_0 is HL in X. Then there exists a mapping $F\colon X\times I\to X$ such that $F(x,0)=x$ for $x\in X$, $F(x,t)=x$ for $x\in X-S(x_0,\varepsilon)$, $F(x,t)\in S(x_0,\varepsilon)$ for $x\in S(x_0,\varepsilon)$ and $F(x,1)\neq x_0$ for $x\in X$, where $S(x_0,\varepsilon)$ is the spherical ε-nei[illegible]rhood of x_0 in X. Put $f'(s)=F(g(s),1)$ for $s\in E^{n+1}$. We have $f'(s)=f($[illegible] $s\in S^n$ and $f'(E^{n+1})\subset U-x_0$. This contradicts our hypothesis that f has no extension $f'\colon E^{n+1}\to U-x_0$. This completes the proof.

It follows from Theorem 1 that, in an m-dimensional ANR, a point x_0 is HL if and only if x_0 is HL^k for $k\geqslant m-1$. Moreover, in the same way as in the proof of the sufficiency of Theorem 1, the condition "x_0 is HL^k for $k\geqslant m$" is equivalent to the condition "for every neighbourhood U of x_0 there is a neighbourhood V of x_0 contained in U such that $V-x_0$ is contractible in $U-x_0$[2])". Therefore we have the following theorem:

THEOREM 2. *Let X be a finitely dimensional* ANR *and x_0 a point of X. Then x_0 is* HL *in X if and only if, for every neighbourhood U of x_0, there exists a neighbourhood V of x_0 contained in U such that $V-x_0$ is contractible in $U-x_0$.*

Theorems (3.1)-(3.4) of [20] are consequences of Theorem 2.

Remark 1. We can replace the condition "X is finitely dimensional" by the condition "X is finitely dimensional at the point x_0 in

[2]) We say that a subset A of a topological space X is *contractible* in a subspace B of X, if there exists a homotopy f such that $f_t\colon A\to B$, $t\in\langle 0,1\rangle$, and f_0 = the inclusion mapping: $A\subset B$ and $f_1(A)$ is a point of B.

the sense of C. H. Dowker (cf. [7], p. 103)" in Theorems 1 and 2. For the homotopical lability and stability are local properties in ANR's and if $\dim\overline{V}\leqslant n$, then $\dim V\leqslant n$ by [19], Theorems 5.1 and 8.6.

§ 3. Künneth's theorem

Let (X,A) and (Y,B) be pairs of topological spaces and let J be a commutative field. The following homomorphism h is naturally defined:

$$h\colon \sum_{p+q=n} H_p(X,A:J)\otimes H_q(Y,B:J)\to H_n\big((X,A)\times(Y,B):J\big),\qquad n=0,1,2,\dots,$$

where $H_p(X,A:J)$ means the p-dimensional Čech homology group of (X,A) with coefficients J, Σ means the direct sum of the groups, $\otimes$ means the tensor products of the groups and $(X,A)\times(Y,B)=(X\times Y, X\times B\cup A\times Y)$. The Künneth's theorem ([1], p. 308) shows that h is an isomorphism if (X,A) and (Y,B) are pairs of finite complexes. K. Borsuk ([3], p. 293) proved that h is an isomorphism if X and Y are compact ANR's and $A=B=\Phi$.

We shall state the following generalization of the theorems quoted above, but omit its proof, since it is proved by a straightforward computation:

THEOREM 3. *We have the following isomorphism:*

$$\sum_{p+q=n} H_p(X,A:J)\otimes H_q(Y,B:J)\approx H_n\big((X,A)\times(Y,B):J\big),\qquad n=0,1,2,\dots,$$

if

(i) *(X,A) and (Y,B) are pairs of compact Hausdorff spaces*[3]),
(ii) *(X,A) is a pair of compact Hausdorff spaces and (Y,B) is a pair of (finite or infinite) complexes,*
(iii) *(X,A) is a pair of S-spaces*[4]) *and (Y,B) is a pair of finite complexes.*

§ 4. Homotopical stability in product spaces

Let X be a topological space and x_0 a point of X. Let V and U be two neighbourhoods of x_0 such that $\overline{V}\subset U$. If we denote by Π_V^U the inclusion mapping $(X,X-U)\subset(X,X-V)$, we have the homomorphism $(\Pi_V^U)_*\colon H_n(X,X-U:R)\to H_n(X,X-V:R)$ induced by Π_V^U, $n=0,1,2,\dots$

[3]) Professor K. Morita proved the case (i) in his lecture at the Tokyo University of Education.

[4]) A topological space X is called an *S-space* if every open covering has a star finite open refinement. Cf. [16] and [2].

The system $\{H_n(X, X-U:R); (\Pi^U_V)_*|V$ and U range over all neighbourhoods of x_0 such that $\overline{V}\subset U\}$ forms the direct system of groups. Put $H_n(x_0:R) = \varprojlim H_n(X, X-U:R)$. We shall call this group an n-dimensional local connectivity Čech group at x_0 with coefficients R. If R is a commutative field, the rank of the group $H_n(x_0:R)$ is the n-dimensional local Betti number (over R) at x_0 (cf. [24], p. 191). If we replace the Čech group by the singular group, we have the n-dimensional local connectivity singular group at x_0 with coefficients. We denote this group by $\mathfrak{H}_n(x_0:R)$.

THEOREM 4. *Let X be a locally compact Hausdorff space and x_0 be a point of X. If x_0 is* HL *in X, then we have $H_n(x_0:R) = \mathfrak{H}_n(x_0:R) = 0$ for each integer n and any abelian group R.*

Proof. Let x_0 be HL in X. Take a neighbourhood U of x_0 with compact closure. We can find a homotopy $f_t: X\to X$ such that $f_0(x)=x$ for $x\in X$, $f_t(x)=x$ for $x\in X-U$, $f_t(x)\in U$ for $x\in U$, and $f_1(x)\neq x_0$ for $x\in X$. Since U has compact closure, $f_1(X)$ is closed in X and does not contain x_0. Since X is regular, there exists a neighbourhood V of x_0 such that $\overline{V}\subset U$ and $f_1(X)\cap V=\Phi$. Let us denote the inclusion mapping: $(X-U, X-V)\subset(X, X-V)$ by j. Consider a mapping $jf_1: (X, X-U)\to(X, X-V)$. We easily see that $\Pi^U_V\sim jf_1: (X, X-U)\to(X, X-V)$ [5]. Therefore they induce the same homomorphisms $(jf_1)_* = (\Pi^U_V)_*: H_n(X, X-U:R)\to H_n(X, X-V:R)$. Therefore $(\Pi^U_V)_*=0$. This shows $H_n(x_0:R)=0$. In the same way we can prove $\mathfrak{H}_n(x_0:R)=0$.

Since the Čech homology theory satisfies the excision axiom (cf. for example, Eilenberg and Steenrod [9], p. 243) by Theorem 3, we can easily prove the following theorem:

THEOREM 5. *Let X and Y be locally compact Hausdorff spaces. Let x_0, y_0 be points of X and Y respectively. If there exist a commutative field J and integers n, n' such that $H_n(x_0:J)\neq 0$ and $H_{n'}(y_0:J)\neq 0$, then the point (x_0, y_0) is* HS *in $X\times Y$.*

Corollaires 2 and 3 of [5], p. 175, are consequences of Theorem 5.

THEOREM 6. *Let X and Y be finite dimensional locally compact ANR's, and let x_0 and y_0 be points of X and Y respectively. Moreover, assume that X and Y are arc-wise connected and non-degenerate. Then the point (x_0, y_0) is* HS *in $X\times Y$ if and only if there exists a non-negative integer n such that $H_n((x_0, y_0):Z)\neq 0$ where Z is an additive group of integers.*

[5]) Let g_1 and g_2 be two mappings of (X, A) into (Y, B). Then "$g_1\sim g_2: (X, A)\to(Y, B)$" means that there exists a homotopy h_t such that $h_0=g_1$, $h_1=g_2$ and $h_t: (X, A)\to(Y, B)$ for each $t\in\langle 0, 1\rangle$.

Since the sufficiency is a consequence of Theorem 4, we have only to prove the necessity. Therefore, by Theorem 1, it is sufficient to prove that if $H_n((x_0, y_0):Z)=0$ for $n=0,1,2,\ldots$, then (x_0, y_0) is i-HL in $X\times Y$ for $i=0,1,2,\ldots$ We shall prove this statement in the following three stages:

I. (x_0, y_0) is 0-HL.

II. (x_0, y_0) is 1-HL.

III. (x_0, y_0) is k-HL for $k>1$.

At first, we need the following lemmas:

LEMMA 3. *Let X be an* ANR *and (X_i, A_i), $i=1,2$, be two pairs of closed subsets of X such that X_2 and A_2 are closed neighbourhoods of X_1 and A_1, respectively. Then there exist a pair of complexes (K, L) and mappings $\varphi: (X_1, A_1)\to(K, L)$ and $\psi: (K, L)\to(X_2, A_2)$ such that $i\sim\psi\varphi: (X_1, A_1)\to(X_2, A_2)$, where i means the inclusion mapping $(X_1, A_1)\subset(X_2, A_2)$.*

Proof (cf. [13], Theorem 2). Let us imbed X as a closed subset of a convex subset D of a normed vector space B as in Lemma 1. Let h be a retraction of some neighbourhood W of X in D to X and let ϱ be a metric function in B. For each point x of A_1, let $\varepsilon_0(x)$ be a positive number such that $\varepsilon_0(x)<\min\{\varrho(x, X-A_2), \varrho(x, D-W)\}$. For each point x of X_1-A_1, let $\varepsilon_0(x)$ be a positive number such that $\varepsilon_0(x)<\min\{\varrho(x, X-X_2), \varrho(x, D-W), \varrho(x, A_1)\}$. Take positive numbers $\varepsilon_1(x)$ and $\varepsilon_2(x)$ such that $S(x, \varepsilon_1(x))\subset h^{-1}(S(x, \varepsilon_0(x))\cap X)$ and $S(x, \varepsilon_2(x))\subset h^{-1}(S(x, \varepsilon_1(x))\cap X)$ for each point x of X_1, where $S(x, \varepsilon)$ means the spherical neighbourhood of x with the radius ε in D. Consider a covering $\mathfrak{A}=\{S(x, \varepsilon_2(x)); x\in X_1\}$ of X_1. According to a theorem of A. H. Stone [16], we have a locally finite collection of open sets $\mathfrak{Y}=\{V_\alpha; \alpha\in\Omega\}$ which covers X and is a star refinement of $\mathfrak{A}$, that is, $\mathfrak{Y}^*=\{V^*_\alpha=\bigcup_{V_\beta\cap V_\alpha\neq\Phi} V_\beta; \alpha\in\Omega\}$ is a refinement of $\mathfrak{A}$. Let (K, L) be a pair of nerves of the covering $\mathfrak{Y}\cap(X_1, A_1)$ with the weak topology. Since $\mathfrak{Y}$ is a star refinement of $\mathfrak{A}$, for each element V_α of $\mathfrak{Y}$, we can select a point x_α of X_1 such that $V^*_\alpha\subset S(x_\alpha, \varepsilon_2(x_\alpha))$. By the construction of $\varepsilon_0(x)$, if $V_\alpha\cap A_1\neq\Phi$, then $x_\alpha\in A_1$. Define $\psi_0: (K^0, L^0)\to(X_2, A_2)$ such that $\psi_0(v_\alpha)=x_\beta$, where K^i means the i-skeleton of K and v_α the vertex of K corresponding to an element V_α of Y. If $v_{\alpha_0}, \ldots, v_{\alpha_n}$ forms a simplex of K, $\mathfrak{Y}$ being a star refinement of $\mathfrak{A}$, $\bigcup_{i=0}^{n}\psi_0(v_{\alpha_i})\subset S(x_{\alpha_0}, \varepsilon_2(x_{\alpha_0}))$ by the definition of ψ_0. Since $S(x_{\alpha_0}, \varepsilon_0(x_{\alpha_0}))$ is a convex set and K is a complex with the weak topology, the mapping ψ_0 has an extension ψ' over K such that $\psi'(\mathrm{ClSt}(v_\alpha))\subset S(x_{\alpha_0}, \varepsilon_2(x_{\alpha_0}))$ for each vertex v of K, where $\mathrm{ClSt}(v_\alpha)$ means the union of all closed simplexes of K with v_α as a vertex. Define $\psi: K\to X_2$ by $\psi=h\psi'$. Obviously $\psi(L)\subset A_2$. Let φ be a canonical mapping (cf. [6], p. 202) of (X_1, A_1) into (K, L). We shall prove that $i \sim \psi\varphi$: $(X_1, A_1) \to (X_2, A_2)$. Let x be a point of X_1 and let $V_{a_0}, \ldots, V_{a_n}$ be all elements of $\mathfrak{Y}$ containing x. Then the point $\varphi(x)$ is contained in the closed simplex $\overline{(v_{a_0} \ldots v_{a_n})}$ (cf. [6], p. 202). Therefore, by the definitions of ψ and $\varepsilon_1(x)$, we have $\psi\varphi(x) \in S(x_{a_0}, \varepsilon_1(x_{a_0})) \cap X$. Since $S(x_{a_0}, \varepsilon_1(x_{a_0}))$ is a convex set and $x \cup \psi\varphi(x) \subset S(x_{a_0}, \varepsilon_1(x_{a_0}))$, there exists a homotopy f_t: $K \to W$ such that $f_0 = i$ and $f_1 = \psi\varphi$. Put F: $X_1 \times I \to X_2$ such that $F(s, t) = hf(s)$ for $(s, t) \in K \times I$. For each x of X_1, we have $F(x \times I) \subset S(x_{a_0}, \varepsilon_0(x_{a_0}))$. If x is a point of A_1, we can select x_{a_0} such that $x_{a_0} \in A_1$. Then, since $S(x_{a_0}, \varepsilon_0(x_{a_0})) \cap X \subset A_2$, we have $F(x \times I) \subset A_2$. This shows that $i \sim \psi\varphi$: $(X_1, A_1) \to (X_2, A_2)$.

LEMMA 4. *Let X be an ANR and x_0 be a point of X. Then we have $H_n(x_0 : R) \approx \mathfrak{H}_n(x_0 : R)$ for each integer n and any abelian group R.*

Proof. Since X is a metric space, there exists a countable sequence $\{U_i\}$ of a complete family of neighbourhoods of x_0 such that $\overline{U}_i \subset U_{-1}$, $i = 0, 1, 2, \ldots$; it is sufficient to use only $\{U_i\}$ in the definition of the local connectivity group at x_0. Apply Lemma 3 to the pairs $(X, X - U_i)$ and $(X, X - U_{i+1})$, $i = 1, 2, \ldots$ We then get a pair (K_i, L_i) of complexes and mappings φ_i: $(X, X - U_i) \to (K_i, L_i)$ and ψ_i: $(K_i, L_i) \to (X, X - U_{i+1})$ for $i = 1, 2, \ldots$ Consider the direct system $\{H_n(K_i, L_i : R);\ \varphi_{i+1}\psi_i,\ i = 1, 2, \ldots\}$. We have $H_n(x_0 : R) = \varprojlim H_n(K_i, L_i : R)$. Since the Čech homology theory and the singular homology theory are consistent in a pair of complexes (cf. for example, [13], Theorem 2), we have $H_n(x_0 : R) = \mathfrak{H}_n(x_0 : R)$.

Proof of I. Let W be a neighbourhood of (x_0, y_0) in $X \times Y$. Take a neighbourhood U_1 of x_0 in X and a neighbourhood U_2 of y_0 in Y such that $U_1 \times U_2 \subset W$. Since X and Y are ANR's, there exist neighbourhoods V_1 and V_2 of x_0 and y_0 such that V_i is contractible in U_i, $i = 1, 2$. Let a and b be any two points of $V_1 \times V_2 - (x_0, y_0)$. It is sufficient to prove that a and b are connected by an arc in $U_1 \times U_2 - (x_0, y_0)$. Let us denote by p_1 and p_2 the projections $X \times Y \to X$ and $X \times Y \to Y$ respectively. Assume that $p_1(a) = x_0$. Since X is a non-degenerate ANR, we can find a point x_1 of $V_1 - x_0$ such that x_0 and x_1 are connected by an arc in V_1. Put $a' = (x_1, p_2(a))$. Then $p_1(a') \neq x_0$ and $p_2(a') \neq y_0$. Therefore, we may assume that $p_1(a) \neq x_0 \neq p_1(b)$ and $p_2(a) \neq y_0 \neq p_2(b)$. Since V_1 is contractible in U_1, we can connect two points a and $(p_1(b), p_2(a))$ by an arc in $U_1 \times p_2(a) \subset U_1 \times U_2 - (x_0, y_0)$. Since V_2 is contractible in U_2, we can connect two points $(p_1(b), p_2(a))$ and b by an arc in $p_1(b) \times U_2 \subset U_1 \times U_2 - (x_0, y_0)$. Therefore, two points a and b are connected by an arc in $U_1 \times U_2 - (x_0, y_0)$. This shows that (x_0, y_0) is 0-HL in $X \times Y$.

Proof of II. It is sufficient to prove the following two lemmas.

Let X and Y be ANR's and x_0 and y_0 be points of X and Y.

LEMMA 5. *If either x_0 is 0-HL in X or y_0 is 0-HL in Y, then (x_0, y_0) is 1-HL in $X \times Y$.*

LEMMA 6. *If x_0 and y_0 are 0-HS in X and Y respectively, then $H_2((x_0, y_0) : Z) \neq 0$.*

Proof of Lemma 5. Assume that x_0 is 0-HL in X. Let W be a neighbourhood of (x_0, y_0) and U_i, $i = 1, 2$, be neighbourhoods of x_0 and y_0 in X and Y respectively such that $U_1 \times U_2 \subset W$. Take a neighbourhood V_i of x_0 and y_0 such that V_i is contractible in U_i, $i = 1, 2$. Let f be a mapping of 1-sphere S^1 into $V_1 \times V_2 - (x_0, y_0)$. We shall prove that f has an extension f': $E^2 \to U_1 \times U_2 - (x_0, y_0)$. There exists a positive number ε such that $0 < \varepsilon < \varrho((x_0, y_0), f(S^1))$, where ϱ is a metric in $X \times Y$. Put $W_0 = S(x_0, \varepsilon)$. Define f_i: $S^1 \to V_i$ such that $f_i = p_i f$, $i = 1, 2$, where p_1 and p_2 are projections $X \times Y \to X$ and $X \times Y \to Y$ respectively. Take neighbourhoods W_1, W_2, W_3 of x_0 such that

1° W_1 is contained in W_0 and contractible in W_0.

2° W_2 is contained in W_1 and any mapping g: $S^0 \to W_2 - x_0$ has an extension g': $E^1 \to W_1 - x_0$.

3° W_3 is contained in W_2 and contractible in W_2.

Put $N = f_1^{-1}(x_0)$. Let N_i, $i = 1, 2, \ldots$, be components of N. Put $G = f_1^{-1}(W_3)$. Then G is an open set containing N. Let $\{G_a\}$ be all components of G intersecting with N. Since S^1 is locally connected, each G_a is an open set (cf. [24], Chap. I, (14.1)) in S^1. Hence, $\{G_a\}$ is an open covering of the compact set N. Therefore $\{G_a\}$ consists of a finite number of sets. Let us denote them by $G_1, \ldots, G_n$. Put $M_j = \bigcup\{N_k;\ N_k \subset G_j\}$, $j = 1, 2, \ldots, n$. Let I_j be the minimal closed interval in S^1 containing M_j, $j = 1, 2, \ldots, n$. Then $I_i \cap I_j = \Phi$ for $i \neq j$. Define a mapping g_1: $S^1 \to V_1$ such that

$$g_1(s) = f_1(s) \quad \text{for} \quad s \in S^1 - \bigcup_{j=1}^{n} I_j,$$

$$g_1(s) = x_0 \quad \text{for} \quad s \in \bigcup_{j=1}^{n} I_j.$$

Obviously, g_1 is continuous. Moreover, if we define a mapping G_1: $S^1 \to V_1 \times V_2 - (x_0, y_0)$ such that $G_1(s) = (g_1(s), f_2(s))$ for $s \in S^1$, we have by the construction 3° of W_2 and W_3

$$G_1 \sim f:\ S^1 \to V_1 \times V_2 - (x_0, y_0). \tag{a}$$

Since $\bigcup_{j=1}^{n} I_j$ is contained in the open set $g_1^{-1}(W_2)$, there exists an open interval $H_j = (a_j, b_j)$ in S^1 containing I_j such that $\overline{H}_j \cap (\bigcup_{j \neq i} \overline{H}_i) = \Phi$ and

$g_1(\bar{H}_j) \subset W_2$, $j=1,2,\ldots,n$. Define a mapping $g_1'\colon S^1 - \bigcup_{j=1}^{n} H_j \to V_1 - x_0$ such that $g_1' = g_1 | S^1 - \bigcup_{j=1}^{n} H_j$. By the construction 2° of W_2, we find that a mapping $g_2'|a_j \cup b_j\colon a_j \cup b_j \to W_2 - x_0$ is extended $t_j\colon \bar{H}_j \to W_1 - x_0$ for $j=1,2,\ldots,n$. Put $g_2\colon S^1 \to V_1 - x_0$ such that

$$g_2(s) = g_1(s) \quad \text{for} \quad s \in S^1 - \bigcup_{j=1}^{n} H_j,$$

$$g_2(s) = t_j(s) \quad \text{for} \quad s \in \bar{H}_j, \quad j=1,2,\ldots,n.$$

If we define a mapping $G_2\colon S^1 \to V_1 \times V_2 - (x_0, y_0)$ such that $G_2(s) = (g_2(s), f_2(s))$ for $s \in S^1$, by the construction 1° of W_0 and W_1, we have

(b) $$G_2 \sim G_1\colon S^1 \to V_1 \times V_2 - (x_0, y_0).$$

Since V_2 is contractible in U_2 and Y is non-degenerate, there exists a homotopy $k_t\colon V_2 \to U_2$ such that $k_0(y) = y$ and $k_1(y) = y_1 \neq y_0$ for $y \in V_2$. Define $G_3\colon S^1 \to V_1 \times y_1$ by $G_3(s) = (g_2(s), y_1)$ for $s \in S^1$. Put $H_t\colon S^1 \to V_1 \times U_2 - (x_0, y_0)$ such that $H_t(s) = (g_2(s), k_t f_2(s))$ for $t \in \langle 0,1 \rangle$ and $s \in S^1$. Then $H_0 = G_2$ and $H_1 = G_3$. Therefore we have

(c) $$G_2 \sim G_3\colon S^1 \to V_1 \times U_2 - (x_0, y_0).$$

Since V_1 is contractible in U_1, there exists a homotopy $i_t\colon V_1 \to U_1$ such that $i_0(x) = x$ and $i_1(x) = x_1$ for $x \in V_1$. If we denote by G_4 the constant mapping $S^1 \to (x_1, y_1)$, we have, in the same way as in (c),

d) $$G_3 \sim G_4\colon S^1 \to U_1 \times U_2 - (x_0, y_0).$$

(a)-(d) completes the proof of Lemma 5.

Proof of Lemma 6. Since x_0 is 0-HS, there exists a neighbourhood U of x_0 such that, whenever V is a neighbourhood of x_0 and contained in U, there exists a mapping $f\colon S^0 \to V - x_0$ which has no extension $f'\colon E^1 \to U - x_0$. We can assume that V is contractible in U. Therefore, we have an extension $g\colon E^1 \to U$ of f. Take a neighbourhood W of x_0 such that $W \subset V$ and $W \cap g(S^0) = W \cap f(S^0) = \Phi$. Then g determines an element a of $\mathfrak{G}_1(U, U - W\colon Z)$, where $\mathfrak{H}_n(X, A\colon Z)$ means the n-dimensional singular homology group of (X, A) with coefficients Z. Let ∂ be the boundary homomorphism $\mathfrak{H}_1(U, U-W\colon Z) \to \mathfrak{H}_0(U - W\colon Z)$. Since ∂a is an element represented by f with the infinite order, the order of a is infinite. Moreover, let W' be a neighbourhood of x_0 contained in W. The homomorphism $j_*\colon \mathfrak{H}_1(U, U-W\colon Z) \to \mathfrak{H}_1(U, U-W'\colon Z)$ induced by the inclusion mapping $j\colon (U, U-W) \subset (U, U-W')$ maps a into an element $j_*(a)$ of $\mathfrak{H}_1(U, U-W'\colon Z)$ with the infinite order. Let $\{W_i\}$ be a complete family of neighbourhoods of x_0 such that $W_{i+1} \subset W_i \subset W$, $i=1,2,\ldots$ Apply Lemma 3 to pairs $(U, U-W_i)$ and $(U, U-W_{i+1})$. We have a pair of complexes (K_i, L_i) mappings $\varphi_i\colon (U, U-W_i) \to (K_i, L_i)$, $\psi_i\colon (K_i, L_i) \to (U, U-W_{i+1})$ and a homotopy $l \sim \psi_i\varphi_i\colon (U, U-W_i) \to (U, U-W_{i+1})$, where l is the inclusion mapping $(U, U-W_i) \subset (U, U-W_{i+1})$. There exists an element a_i of $\mathfrak{H}_1(K_i, L_i\colon Z)$ with the infinite order such that $(\varphi_{i+1}\psi_i)_* a_i = a_{i+1}$, $i=1,2,\ldots$ Since y_0 is 0-HS, we can find a complete family $\{W_i'\}$ of neighbourhoods of y_0, a sequence of pairs of complexes (M_i, N_i), mappings $\mu_i\colon (U', U'-W_i') \to (M_i, N_i)$, $\lambda_i\colon (M_i, N_i) \to (U', U'-W_{i+1}')$ and $l' \sim \lambda_i\mu_i\colon (U', U'-W_i') \to (U', U'-W_{i+1}')$, where l' is the inclusion mapping $(U', U'-W_i') \subset (U', U'-W_{i+1}')$. Moreover, there exists an element b_i of $\mathfrak{H}_1(M_i, N_i\colon Z)$ with the infinite order such that $(\mu_{i+1}\lambda_i)^* b_i = b_{i+1}$ for $i=1,2,\ldots$ Define $\pi_{i+1}^i\colon (K_i \times M_i, K_i \times N_i \cup L_i \times M_i) \to (K_{i+1} \times M_{i+1}, K_{i+1} \times N_{i+1} \cup L_{i+1} \times M_{i+1})$ such that $\pi_{i+1}^i(s, s') = (\varphi_{i+1}\psi_i(s), \mu_{i+1}\lambda_i(s'))$ for $s \in K_i$ and $s' \in M_i$, $i=1,2,\ldots$ If $(\pi_{i+1}^i)_*$ is the homomorphism induced by π_{i+1}^i, then the limit group of the direct system $\{\mathfrak{H}_2(K_i \times M_i, K_i \times N_i \cup L_i \times M_i\colon Z); (\pi_{i+1}^i)_*\}$ is equal to $\mathfrak{H}_2((x_0, y_0)\colon Z)$, because the singular theory satisfies the excision axiom (cf. [9], Chap. 7). By Künneth's theorem ([1], p. 308), we have $a_i \otimes b_i \in \mathfrak{H}_1(K_i, L_i\colon Z) \otimes \mathfrak{H}_1(M_i, N_i\colon Z) \subset \mathfrak{H}_2(K_i \times M_i, K_i \times N_i \cup L_i \times M_i\colon Z)$. Since the orders of a_i and b_i are both infinite, the order of $a_i \otimes b_i$ is infinite. Moreover, $(\pi_{i+1}^i)_*(a_i \otimes b_i) = a_{i+1} \otimes b_{i+1}$. This shows $\mathfrak{H}_2((x_0, y_0)\colon Z) \approx H_2((x_0, y_0)\colon Z) \neq 0$ by Lemma 4.

Proof of III. It is sufficient to prove that if X is an ANR, and x_0 is a point of X such that it is HL^{k-1} for $k > 1$ and $H_{k+1}(x_0\colon Z) = 0$, then x_0 is k-HL in X.

Let U be a neighbourhood of x_0. There exists a sequence V_i of neighbourhoods of x_0 such that

1° $x_0 \in V_0 \subset \ldots \subset V_i \subset \bar{V}_i \subset V_{i+1} \subset \ldots \subset \bar{V}_{k+1} \subset V_{k+2} = U$,

2° V_0 is contractible in V_1,

3° if $f\colon S^j\colon V_i - x_0$, there exists an extension $f'\colon E^{j+1} \to V_{i+1} - x_0$ of f for $i = 1,2,\ldots,k$ and $j = 0,1,2,\ldots,k-1$.

Let f be any mapping of S^k into $V_0 - x_0$. Fix a point s_0 of S^k. There exists a mapping $\varphi\colon (E^k, \dot{E}^k) \to (S^k, s_0)$ such that $\varphi | E^k - \dot{E}^k$ is a homeomorphism onto $S^k - s_0$, where $\dot{E}^k$ means the boundary of E^k. Since $H_{k+1}(x_0\colon Z) = 0$ and the singular homology theory satisfies the excision axiom (cf. [9], Chap. 7), we can find a sufficiently small neighbourhood W of x_0 such that the element of $\mathfrak{H}_k(V_1 - W, x_1\colon Z)$ determined by $f_0 = f\varphi$ is zero, where $x_1 = f_0(\dot{E}^k) = f(s_0)$. Therefore, by the definition of singular homology group, there exists a $(k+1)$-dimensional finite complex P^{k+1} containing E^k and a mapping $h\colon P^{k+1} \to V_1 - W$ such that

4° $\dot{P}^{k+1}=E^k+\partial^k$,

5° $h|E^k=f_0$ and $h(\partial^k)=x_1$ [6]).

Define a homotopy H: $P^{k+1}\times I\to U-x_0$ as follows: At first, put $H(p,0)=h(p)$ for $p\in P^{k+1}$ and $H(p,t)=f_0(p)$ for $p\in \dot{P}^{k+1}$ and $t\in I$. Next, put $H(p,1)=x_1$ for $p\in P_0^{k+1}$, where P_i^{k+1} is the i-skeleton of P^{k+1}. By the construction 3° of V_1 and V_2, we can extend H to a mapping of $(P^{k+1}\times 0)\cup\big((P^{k+1}\cup \dot{P}_0^{k+1})\times I\big)$ into V_2-x_0. By a repeated application of this process we can have a mapping $H:(P^{k+1}\times 0)\cup\big((\dot{P}^{k+1}\cup P_{k-1}^{k+1})\times I\big)\to U-x_0$ such that

$$H(p,0)=h(p) \quad \text{for} \quad p\in P^{k+1},$$
$$H(p,t)=(p)=f_0(p) \quad \text{for} \quad p\in \dot{P}^{k+1},$$
$$H(p,1)=x_1 \quad \text{for} \quad p\in P_{k-1}^{k+1}.$$

Since $U-x_0$ is an ANR by Lemma 2, we can extend H to a mapping of $P^{k+1}\times I$ into $U-x_0$. Consider a mapping $H|P^{k+1}\times 1$: $P^{k+1}\times 1\to U-x_0$. In the same way as in Theorem 12.6 of [11], we define the following homomorphism a of the k-dimensional chain group $C_k(P^{k+1}:Z)$ of P^{k+1} into the k-dimensional homotopy group $\pi_k(U-x_0,x_1)$ of $(U-x_0,x_1)$. Let T_i^k, $i=1,2,\dots,q$, be the n-simplexes of P^{k+1}, each in a definite orientation. Since the restricted mapping $H|P^{k+1}\times 1: P^{k+1}\times 1\to U-x_0$ maps the $(k-1)$-skeleton of $P^{k+1}\times 1$ into the point x_1, the mapping $H|T_i^k\times 1: T_i^k\times 1\to U-x_0$ determines the element $a(T_i^k)$ of $\pi_k(U-x_0,x_1)$. To the integral k-chain $C_k=\sum_i a_iT_i^k\in C_k(P^{k+1}:Z)$, let us assign the element $a(C_k)=\sum_i a_i a(T_i^k)$. Since $k>1$, a is a homomorphism. Moreover, if T^{k+1} is a $(k+1)$-simplex of P^{k+1}, $a(T^{k+1})=0$. Hence, since $H|P^{k+1}\times 1$ maps $\partial^k\times 1$ into the point x_0, we have $a(\dot{P}^{k+1})=a(\partial^k)=0$. Therefore, by 3°, we have $a(E^k)=0$. This completes the proof of III and consequently the proof of Theorem 6.

Theorem 5 of [21] is a consequence of Theorem 6.

In part III of Theorem 6 we have proved that if x_0 is HL^1 in X and $H_i(x_0:Z)=0$ for $i=0,1,2,\dots,j+1$, then x_0 is HL^j. Therefore, we have the following theorem.

THEOREM 7. *Let X be a finitely dimensional locally compact* ANR *and let x_0 be a point of X. Then x_0 is* HL *in X if and only if x_0 is* HL^1 *in X and $H_n(x_0:Z)=0$ for $n=0,1,2,\dots$*

Remark 2 (cf. Remark 1). Since the Čech homology theory satisfies the excision axiom (cf. for example, Eilenberg and Steenrod [9], p. 243), we can replace the condition "X is finitely dimensional" by the condition "X is finitely dimensional at x_0 in the sense of C. H. Dowker (cf. [7], p. 103)" in Theorem 7. Similarly, we can replace the condition "X and Y are finitely dimensional" by the condition "X and Y are finitely dimensional in the sense of C. H. Dowker (cf. [7], p. 103) at x_0 and y_0 respectively" in Theorem 6.

[6]) Cf. [11], p. 1023.

5. Dimension of products spaces

THEOREM 8. *Let X be a locally compact fully normal space and let Y be a locally compact 2-dimensional* ANR. *Then the following equality exists:*

$$\dim X\times Y=\dim X\times\dim Y.$$

Proof. By [17], Theorem 3.2, we may assume that X is compact. Since Y is 2-dimensional, there exists a point y_0 at which Y is 2-dimensional in the sense of C. H. Dowker (cf. [7], p. 103). Since Y is a locally compact ANR, we can find a neighbourhood U of y_0 such that $\overline{U}$ is compact and contractible in Y. Since Y is locally connected, there exists a 2-dimensional compactum M contained in U. M is not a dendrite (cf. [23], Chap. 5), because $\dim M=2$. Therefore M contains a topological image S of a 1-sphere.

If $\dim X=m$, there exist two closed subsets X_1 and A of X such that $A\subset X_1$ and $H_m(X_1,A:R_1)\neq 0$ (cf. for example, [15], Theorem 10), where R_1 is the group of real numbers modulo 1. Since X_1 is a compact space, S is a polyhedron and $H_1(S:Z)\approx Z$, we conclude that $H_{m+1}(X_1\times S,A\times S:R_1)\neq 0$. Since S is contractible in a compact subset N of Y, there is a homotopy f_t: $S\to N$ such that f_0 is the inclusion mapping and $f_1(S)$ is a point y_1 of N. Put g_t: $(X_1\times S,A\times S)\to(X_1\times N,A\times N)$ such that $g_t(x,s)=\big(x,f_t(s)\big)$ for $x\in X_1$ and $s\in S$. The homomorphism g_{t*}: $H_{m+1}(X_1\times S,A\times S:R_1)\to H_{m+1}(X_1\times N,A\times N:R_1)$ is the same for each $t\in\langle 0,1\rangle$. But $g_1(X_1\times S)\subset X_1\times y_1$. Since $\dim X_1=m$, g_{1*} is the trivial homomorphism. Therefore, g_{0*} is the trivial homomorphism. Let h and k be the inclusion mappings $(X_1\times S,A\times S)\subset(X_1\times N,A\times N)$ and $(X_1\times N,A\times S)\subset(X_1\times N,A\times N)$, respectively. Then $g_0=kh$. Therefore, $g_{0*}=k_*h_*$. Let a be a non-zero element of $H_{m+1}(X_1\times S,A\times S:R_1)$. At first, assume that $h_*(a)=0$. Since X_1,A,N,S and R_1 is compact, the sequence of Čech groups of the triple $(X_1\times N,X_1\times S,A\times S)$ is exact by [9], Chap. 1, Theorem 10.2 and Chap. 8, Theorem 5.6. Therefore we conclude that $H_{m+2}(X_1\times N,X_1\times S:R_1)\neq 0$. Next, suppose that $h_*(a)=b\neq 0$. Then $k_*(b)=0$. By the exactness of the Mayer-Vietoris sequence of the triad $(X_1\times N,X_1\times S,A\times N)$ (cf. [9], Chap. 1, Theorem 15.7), we have $H_{m+2}\big(X_1\times N,(X_1\times S)\cup(A\times N):R_1\big)\neq 0$. Therefore, $\dim X\times Y\geqslant m+2$. Since $\dim X\times Y\leqslant m+2$ (cf. for example, [18], Theorem 4), we have $\dim X\times Y=\dim X+\dim Y$.

THEOREM 9. *Let X be a locally compact m-dimensional* ANR *containing a point x_0 which is* HL^{m-2} *and* $(m-1)$-HS, *and let Y be a locally compact n-dimensional* ANR *containing a point y_0 which is* HL^{n-2} *and* $(n-1)$-HS. *Then the following equality exists:*

$$\dim X \times Y = \dim X + \dim Y .$$

Proof. By Theorem 8 and [18], Theorem 6, it is sufficient to prove the theorem in the case of $2 < m$ and $2 < n$. In the same way as in Lemma 6, we can show that there exist compact subsets X_1, A of X and Y_1, B of Y such that $H_m(X_1, A:Z) \neq 0$ and $H_n(Y_1, B:Z) \neq 0$. Since $\dim X = m$ and $\dim Y = n$, all non-zero elements of the groups $H_m(X_1, A:Z)$ and $H_n(Y_1, B:Z)$ have infinite orders. Therefore, if F is the field of rational numbers, we have $H_m(X_1, A:F) \neq 0$ and $H_n(Y_1, B:F) \neq 0$. Hence, by Theorem 3, we have $H_{m+n}(X_1 \times Y_1, X_1 \times B \cup A \times Y_1 : F) \neq 0$. This shows that $\dim X \times Y \geqslant \dim X + \dim Y$. It is obvious that $\dim X \times Y \leqslant \dim X + \dim Y$. This completes the proof.

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Reçu par la Rédaction le 7. 3. 1956