and

\[(5) \text{ the centre } s \text{ of the segment } \overline{pq} \text{ belongs to } Q_0.\]

It follows by (4) that for every point \((p, q) \in Z\) there exists exactly one straight line \(L_{pq}\) passing through \(p\) and \(q\). Let

\[\Psi(p, q) = (s, 4[p(q) - 1], L_{pq}) \text{ for every } (p, q) \in Z.\]

Evidently \(\Psi\) is a 1-1 transformation of the set \(Z\) onto the set of all systems \((s, 4[p(q) - 1], L_{pq})\). Since for every \(s \in Q_0\) the straight lines \(L \subset R^n\) passing through \(s\) constitute a space homeomorphic with the \((n-1)\)-dimensional projective space \(P^{n-1}\) and \(s = 4[p(q) - 1]\) and \(L_{pq}\) depend continuously on \((p, q)\), we infer that \(\Psi\) maps \(Z\) topologically onto the Cartesian product \(Q_0 \times (0; 1) \times P^{n-1}\). Since \(Q_0 \times (0; 1)\) is an \((n-1)\)-dimensional element, we conclude that \(Z \subseteq Q^{(n)}\) contains topologically the Cartesian product \(S^n \times P^{n-1}\) of the \(n\)-dimensional sphere \(S^n\) and the \((n-1)\)-dimensional projective space \(P^{n-1}\). By the lemma \(S^n \times P^{n-1}\), and consequently also \(Q^{(n)}\), are not homeomorphic with any subset of \(R^{2n}\), which concludes the proof.

References


\[\text{Reçu par la Réduction le 15. 3. 1950}\]

On homotopically stable points and product spaces

by

Yukihiro Kodama (Tokyo)

§ 1. Introduction

Let \(X\) be a topological space. A point \(x_0\) of \(X\) is called homotopically labile in \(X\) whenever for every neighbourhood \(U\) of \(x_0\) there exists a continuous transformation \(f(x, t)\) which is defined in the Cartesian product \(X \times I\) of \(X\) and of the closed interval \(I = (0, 1)\) and which satisfies the following conditions:

\[(1) \ f(x, t) \in X \ \text{for every} \ (x, t) \in X \times I,\]
\[(2) \ f(x, 0) = x \ \text{for every} \ x \in X,\]
\[(3) \ f(x, t) = x \ \text{for every} \ (x, t) \in (X - U) \times I,\]
\[(4) \ f(x, t) \in U \ \text{for every} \ (x, t) \in U \times I,\]
\[(5) \ f(x, 1) \neq x_0 \ \text{for every} \ x \in X.\]

A point \(x_0\) of \(X\) is called homotopically stable if it is not homotopically labile. K. Borsuk and J. W. Jaworowski [5] introduced this notion and studied the various properties of labile and stable points.

In this paper, we shall study first a certain characteristic property of homotopically labile points in ANR's for metric spaces. This shows that "homotopical stability" is equivalent to "\(n\)-homotopical stability for some integer \(n\)". The main theorem, which states that the homotopical lability or stability of a point in a product space is determined by the local connectivity groups at that point, is proved in § 4. This theorem gives a generalization of H. Noguchi's theorem [21] to the case of ANR.

Let \(X\) and \(Y\) be two topological spaces. The equality \(\dim X \times Y = \dim X + \dim Y\) does not generally hold; for example, K. Borsuk [4] has proved that there exist 2-dimensional Cantor manifolds whose Cartesian product has dimension three. In § 5 we shall show that this equality holds in the following two cases:

\[1) \] For these definitions, see §§ 2 and 4.
1. \(X\) is a locally compact fully normal space and \(Y\) is a 2-dimensional locally compact ANR for metric spaces.

2. \(X\) and \(Y\) are locally compact ANR's for metric spaces satisfying certain conditions.

§ 2. Some characterizations of homotopically labile points

A topological space \(X\) is called an ANR for metric spaces if whenever \(X\) is a closed subset of a metric space \(Y\), there exists a continuous transformation from some neighbourhood of \(X\) in \(Y\) onto \(X\) which keeps \(X\) point-wise fixed (cf. [14], Definition 2.2).

We introduce the following definitions:

A point \(x_0\) of a topological space \(X\) is called homotopically \(n\)-labile in \(X\), \(n=0,1,2,\ldots\), when, for every neighbourhood \(U\) of \(x_0\), there exists a neighbourhood \(V\) of \(x_0\) which is contained in \(U\) and satisfies the following condition: Let \(E^{n+1}\) be an \((n+1)\)-cell whose boundary is an \(n\)-sphere \(S^n\). Then every continuous mapping \(f : S^n \to Y-x_0\) is extended to a continuous mapping \(f' : E^{n+1} \to Y-x_0\). A point \(x_0\) of \(X\) is called homotopically \(n\)-stable in \(X\) if it is not homotopically \(n\)-labile in \(X\).

For convenience, we shall use the following abbreviations:

\(\text{ANR} = \text{ANR for metric spaces,} \)
\(\text{HL} = \text{homotopically labile,} \)
\(n\text{-HL} = \text{homotopically } n\text{-labile,} \)
\(n\text{-HL}^* = \text{homotopically } n\text{-labile for each integer } i=0,1,2,\ldots,n, \)
\(n\text{-BS} = \text{homotopically stable,} \)
\(n\text{-BS}^* = \text{homotopically } n\text{-stable.} \)

Moreover, we shall understand by "mapping" a continuous transformation and denote by "dimension" the covering dimension of Lebesgue.

We shall establish the following theorem:

**Theorem 1.** Let \(X\) be an \(n\)-dimensional ANR. Then a point \(x_0\) of \(X\) is HL in \(X\) if and only if \(x_0\) is \(n\)-HL in \(X\).

To prove this theorem, it is convenient to state the following lemmas:

**Lemma 1.** Let \(X\) be an ANR and \(x_0\) a point of \(X\) and let \(U\) be a neighbourhood of \(x_0\). Then there exists a neighbourhood \(U_0\) of \(x_0\) contained in \(U\) with the following property: If \(f, g\) are two mappings of a metric space \(Y\) into \(X\) such that

- \(f(y) = g(y)\) for \(y \in Y-f^{-1}(U_0)\),
- \(g(y) \in U_0\) for \(y \in f^{-1}(U_0)\),

then there exists a mapping \(F : Y \times I \to X\) such that

- \(F(y,0) = f(y)\) and \(F(y,1) = g(y)\) for \(y \in Y\),
- \(F(y,t) = f(y)\) for \(y \in Y-f^{-1}(U_0)\),
- \(F(y,t) \in U\) for \(y \in f^{-1}(U_0)\).

**Proof:** (cf. [12], p. 40). According to a theorem of Wojdyslawski ([22], p. 186), \(X\) can be imbedded as a closed set of a convex subset \(D\) of a normed vector space \(B\). Since \(X\) is an ANR, there exist a neighbourhood \(W\) of \(X\) in \(D\) and a retraction \(h : W \to X\). Let \(U\) be a neighbourhood of \(x_0\) in \(X\). We can find a spherical neighbourhood \(V\) of \(x_0\) in \(D\) such that \(V \subseteq CV \cap h^{-1}(U)\). Put \(U = V \cap X\). Let \(Y\) be a metric space and let \(f, g\) be two mappings of \(Y\) into \(X\) satisfying conditions of Lemma 1. Since \(D\) is a convex set, \(V\) is a convex set. Hence, there exists a homotopy \(h_t : Y \to X \to \mathbb{C}^n\) such that \(h_t(y) = f(y)\) for \(y \in Y-CV \cap h^{-1}(U)\), \(h_t(y) = g(y)\) for \(y \in Y\). Then the homotopy \(F : Y \times I \to X\) defined by \(F(y,t) = h_t(y)\) is the required one.

**Lemma 2.** Let \(X\) be an ANR. If \(U\) is an open subset of \(X\), then \(U\) is an ANR ([10], Lemma 3.1).

**Proof of Theorem 1.** Sufficiency. Let \(x_0\) be a point of \(X\) which is \(HL^{n-1}\). There exist two neighbourhoods \(U_0, U_{n-1}\) of \(x_0\) which satisfy the condition of Lemma 1. Since \(x_0\) is \(HL^{n-1}\), we can construct a decreasing sequence of neighbourhoods \(U_i\) of \(x_0\) such that \(x_0 \in U_i \subseteq U_{i-1} \subseteq U_{i-2}\) and every mapping \(f : S^n \to U_i-x_0\) has an extension \(f': E^{n+1} \to U_i-x_0\) for \(i = 1, 2, \ldots, n-1\). Put \(M = U_{n-1}, N = U_{n-2} \subseteq U_{n-1}\). Since \(M\) is a metric space and \(N\) is a closed subset of \(M\), we can construct a space \(Y\) and a continuous mapping \(h : M \to Y\) (cf. [8], Theorem 3.1) such that

1. \(h|N\) is a homeomorphism and \(h(N)\) is closed in \(Y\),
2. \(F = Y - h(N)\) is an \(n\)-dimensional infinite complex with the weak topology and \(h(M-N) \subseteq F\).

Moreover, by ([8], p. 335), there exists a continuous extension \(g_1 : P^1 \cup h(N) \to N\) of a mapping \(g = h^1 : h(N) \to N\), where \(P^1\) is the \(i\)-skeleton of the complex \(P\). Consider \(g_1\) as a mapping \(P^1 \cup h(N) \to U_{n-1} - x_0\). Since \(U_{n-1} - x_0\) is an ANR by Lemma 2, we can find a mapping \(g_0\) and a neighbourhood \(V\) of \(h(N)\) in \(Y\) such that \(g_0 : P^1 \cup V \cup h(N) \to U_{n-1} - x_0\) and \(g_0 \cup h(N) = g_1\) (cf. for example, [12], p. 40). Let \(Q\) be a subcomplex consisting of closed simplices of \(P\) contained in \(V\). Then \(Q \cup h(N)\) forms a closed neighbourhood of \(h(N)\) in \(Y\). Consider the mapping \(g_0 : h(N) \to Q \cup h(N) \to U_{n-1} - x_0\). By the constructions of \(U_i\), we can find a continuous extension \(g_1 : P^1 \cup h(N) \to U_{n-1} - x_0\) of \(g_0\), since \(P^1\) has
On homologically stable points and product spaces

the weak topology. By a repeated application of this process, we can see that there exists a mapping $g_{0}$ of $Y$ into $U_{g_{0}} - x_{0}$ such that $g_{0}h(N) = h^{-1}$. Define a mapping $f$ of $X$ into $X - x_{0}$ as follows:

$f(x) = x$ for $x \in X - U_{m+1}$,

$f(x) = g_{0}h(x)$ for $x \in U_{m+1}$.

If $I$ is an identity mapping of $X$ into $X$, then we can find a homotopy $F$ between $f$ and $f'$ whose existence is proved by Lemma 1. This homotopy means the homotopical stability of $x_{0}$.

2) Necessity. It is sufficient to prove that, if a point $x_{0}$ of $X$ is $m$-HIS for some $n$ such that $0 \leq n < m - 1$, then $x_{0}$ is HIS. Let $x_{0}$ be a point in $X$. By the definition of $m$-HIS, there exists a neighbourhood $U$ of $x_{0}$ satisfying the condition that, if $V$ is a neighbourhood of $x_{0}$ contained in $U$, there exists at least one mapping $f: S^{n} \to X$ such that $f$ has an extension $f': E^{n+1} \to U - x_{0}$. Since $X$ is an ANR, we may suppose that $f'$ is contractible in $U$. Therefore we have an extension $g: E^{n+1} \to U$ such that $g = f'$. Since $f(S^{n})$ is compact, we can find a positive number $e$ such that $0 < e < d(f(S^{n}), x_{0})$, where $d$ is a metric on $X$. Assume that $x_{0}$ is in $X$. Then there exists a mapping $F: X \times I \to X$ such that $F(x, 0) = x$ for $x \in X$, $F(x, 1) = x$ for $x \in X - S(x_{0}, e)$, $F(x, e) = S(x_{0}, e)$ for $x \in S(x_{0}, e)$ and $F(x, 1) = x_{0}$ for $x \in X$, where $S(x_{0}, e)$ is the sphere of radius $e$ with center $x_{0}$ in $X$. Put $f(x) = F(x, e)$ for $x \in E^{n+1}$. We have $f(S^{n}) = f'(S^{n}) \subset U - x_{0}$. This contradicts our hypothesis that $f$ has no extension $f': E^{n+1} \to U - x_{0}$. This completes the proof.

It follows from Theorem 1 that, in an $m$-dimensional ANR, a point $x_{0}$ is HIS if and only if $x_{0}$ is HIS for $k \geq m - 1$. Moreover, in the same way as in the proof of the sufficiency of Theorem 1, the condition "$x_{0}$ is HIS for $k \geq m - 1$" is equivalent to the condition "for every neighbourhood $U$ of $x_{0}$ there is a neighbourhood $V$ of $x_{0}$ contained in $U$ such that $V - x_{0}$ is contractible in $U - x_{0}$". Therefore we have the following theorem:

**Theorem 2.** Let $X$ be a finitely dimensional ANR and $x_{0}$ a point of $X$. Then $x_{0}$ is HIS in $X$ if and only if, for every neighbourhood $U$ of $x_{0}$, there exists a neighbourhood $V$ of $x_{0}$ contained in $U$ such that $V - x_{0}$ is contractible in $U - x_{0}$.

Theorems (3,1)-(3,4) of [20] are consequences of Theorem 2.

**Remark.** We can replace the condition "$X$ is finitely dimensional" by the condition "$X$ is finitely dimensional at the point $x_{0}$ in

the sense of C. H. Dowker (cf. [7], p. 103). In Theorems 1 and 2, for the homotopical stability and stability are local properties in ANR's and if dim $V < n$, then dim $V < n$ by [19], Theorems 5.1 and 8.6.

**§ 3. Künneth's Theorem**

Let $(X, A)$ and $(Y, B)$ be pairs of topological spaces and let $J$ be a commutative field. The following homomorphism $h$ is naturally defined:

$$h: \bigoplus_{p+q=n} H_{p}(X, A) \otimes H_{q}(Y, B) \to H_{n}(X \times A \times Y \times B; J),$$

for $n = 0, 1, 2, \ldots$,

where $H_{p}(X, A) : J$ means the $p$-dimensional Čech homology group of $(X, A)$ with coefficients $J$, $\Sigma$ means the direct sum of the groups, $\otimes$ means the tensor product of the groups and $(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$. The Künneth's theorem ([1], p. 308) shows that $h$ is an isomorphism if $(X, A)$ and $(Y, B)$ are pairs of finite complexes. K. Borsuk ([3], p. 293) proved that $h$ is an isomorphism if $X$ and $Y$ are compact ANR's and $A = B = \emptyset$.

We shall state the following generalization of the theorems quoted above, but omit its proof, since it is proved by a straightforward computation:

**Theorem 3.** We have the following isomorphism:

$$\bigoplus_{p+q=n} H_{p}(X, A) \otimes H_{q}(Y, B) \cong H_{n}(X \times A \times Y \times B; J),$$

for $n = 0, 1, 2, \ldots$,

if

(i) $(X, A)$ and $(Y, B)$ are pairs of compact Hausdorff spaces,

(ii) $(X, A)$ is a pair of compact Hausdorff spaces and $(Y, B)$ is a pair of (finite or infinite) complexes,

(iii) $(X, A)$ is a pair of $S$-spaces and $(Y, B)$ is a pair of finite complexes.

**§ 4. Homotopical Stability in Product Spaces**

Let $X$ be a topological space and $x_{0}$ a point of $X$. Let $V$ and $U$ be two neighbourhoods of $x_{0}$ such that $V \subset U$. If we denote by $H_{p}(X, U)$ the inclusion mapping $H_{p}(X, U) \cong (X, X - V)$, we have the homomorphism $H_{p}(X, U) : H_{p}(X, U - V) \to H_{p}(X, X - V)$ induced by $H_{p}$, $n = 0, 1, 2, \ldots$.

1) Professor K. Morita proved the case (i) in his lecture at the Tokyo University of Education.

2) A topological space $X$ is called an $S$-space if every open covering has a star finite open refinement. Cf. [16] and [5].
The system \( \{ H_0 X, X - U : R \}; (H^2)_{x_0} V \) and \( U \) range over all neighbourhoods of \( x_0 \) such that \( V \cap U \) forms the direct system of groups. Put \( H_0 X, R) = \text{lim} H_0(X, X - U : R). \) We shall call this group an \( n \)-dimensional local connectivity Čech group at \( x_0 \) with coefficients \( R \). If \( R \) is a commutative field, the rank of the group \( H_0 x_0, R) \) is the \( n \)-dimensional local Betti number (over \( R) \) at \( x_0 \) (cf. [21], p. 191). If we replace the Čech group by the singular group, we have the \( n \)-dimensional local connectivity singular group at \( x_0 \) with coefficients. We denote this group by \( \tilde{H}_0 x_0, R). \)

**Theorem 4.** Let \( X \) be a locally compact Hausdorff space and \( x_0 \) be a point of \( X. \) If \( U \) is HL in \( X \), then we have \( H_0 x_0, R) = \tilde{H}_0 x_0, R) = 0 \) for each integer \( n \) and any abelian group \( R. \)

Proof. Let \( x_0 \) be HL in \( X. \) Take a neighbourhood \( U \) of \( x_0 \) with compact closure. We can find a homotopy \( f_j : X \to X \) such that \( f_j(x) = x \) for \( x \in X, \) \( f_j(x) = x \) for \( x \in X - U, \) \( f_j(x) = f(x) \) for \( x \in U. \) Since \( U \) has compact closure, \( f_j(x) \) is closed in \( X \) and does not contain \( x_0. \) Since \( X \) is regular, there exists a neighborhood \( V \) of \( x_0 \) such that \( \overline{V} \cap U \) and \( f_j(x) \) are closed in \( X \) and do not contain \( x_0. \) Since \( X \) is regular, there exists a neighborhood \( V \) of \( x_0 \) such that \( \overline{V} \cap U \) and \( f_j(x) \) are closed in \( X \) and do not contain \( x_0. \) We easily see that \( H^2 f_j(x) : (X, X - U) \to (X, X - V) \). Therefore they induce the same homomorphisms \( (H^2)_{x_0} \to (H^2)_{x_0} \) \( H_0 x_0, R) = H_0 x_0, R) \). Therefore \( (H^2)_{x_0} = 0. \) This shows \( H_0 x_0, R) = 0. \) In the same way we can prove \( \tilde{H}_0 x_0, R) = 0. \)

Since the Čech homology theory satisfies the excision axiom (cf. for example, Eilenberg and Steenrod [9], p. 243) by Theorem 3, we can easily prove the following theorem:

**Theorem 5.** Let \( X \) and \( Y \) be locally compact Hausdorff spaces. Let \( x_0, y_0 \) be points of \( X \) and \( Y \) respectively. If \( X \) has a commutative field \( J \) and integers \( m, n \) such that \( H_0 x_0, J) \neq 0 \) and \( H_0 y_0, J) \neq 0, \) then the point \( (x_0, y_0) \) is HS in \( X \times Y. \)

Corollaries 2 and 3 of [5], p. 175, are consequences of Theorem 5.

**Theorem 6.** Let \( X \) and \( Y \) be finite dimensional locally compact ANR's, and let \( x_0 \) and \( y_0 \) be points of \( X \) and \( Y \) respectively. Moreover, assume that \( X \) and \( Y \) are are-connected and non-degenerate. Then the point \( (x_0, y_0) \) is HS in \( X \times Y \) if and only if there exists a non-negative integer \( n \) such that \( H_n x_0, y_0; Z) \neq 0 \) where \( Z \) is an additive group of integers.

**On homologically stable points and product spaces**

Since the sufficiency is a consequence of Theorem 4, we have only to prove the necessity. Therefore, by Theorem 1, it is sufficient to prove that if \( H_0 x_0, y_0; Z) = 0 \) for \( n = 0, 1, 2, \ldots, \) then \( (x_0, y_0) \) is \( n \)-HL in \( X \times Y \) for \( i = 0, 1, 2, \ldots. \) We shall prove this statement in the following three stages:

I. \( (x_0, y_0) \) is 0-HL.
II. \( (x_0, y_0) \) is 1-HL.
III. \( (x_0, y_0) \) is \( k \)-HL for \( k > 1. \)

At first, we need the following lemmas:

**Lemma 2.** Let \( X \) be an ANR and \( (x_0, y_0) \), \( i = 1, 2, \ldots \), be two pairs of closed subsets of \( X \) such that \( X_1 \) and \( X_2 \) are closed neighborhoods of \( X_1 \) and \( X_2 \), respectively. Then there exist a pair of complexes \( (K, L) \) and mappings \( \varphi : (X_1, A_1) \to (K, L) \) and \( \psi : (K, L) \to (X_2, A_2) \) such that \( i - \varphi : (X_1, A_1) \to (X_2, A_2) \), \( (X_2, A_2) \), where \( i \) is the inclusion mapping \( (X_1, A_1) \to (X_2, A_2) \).

Proof (cf. [13], Theorem 2). Let us imbed \( X \) as a closed subset of a convex subset \( D \) of a normed vector space \( B \) as in Lemma 1. Let \( S \) be a retraction of some neighborhood \( W \) of \( D \) in \( X \) and let \( g \) be a metric function in \( B \). For each point \( x \) of \( A_1 \), let \( e(x) \) be a positive number such that \( e(x) < \min \{ \varphi(x, X_1 - A_1, \psi(x, D - W) \} \). For each point \( x \) of \( X_1 - A_1 \), let \( e(x) \) be a positive number such that \( e(x) < \min \{ \varphi(x, X_1 - X, \psi(x, D - W) \} \). Take positive numbers \( e(x) \) and \( e(x) \) such that \( S(x, e(x)) \) \( \cap \{ \{x, e(x) \} \} \) \( \cap \{ x, e(x) \} \) \( \cap \{ x, e(x) \} \) \( \cap \{ x, e(x) \} \) for each point \( x \) of \( X_1 \), where \( S(x, \varepsilon) \) means the spherical neighbourhood of \( x \) with the radius \( \varepsilon \) in \( D \). Consider a covering \( \mathfrak{U} = \{ S(x, e(x)); x \in X_1 \} \) of \( X_1 \). According to a theorem of A. H. Stone [16], we have a locally finite collection of open sets \( \mathfrak{U} = \{ U_i; \alpha \in \Omega \} \) which covers \( X \) and is a star refinement of \( \mathfrak{U} \) that is, \( \mathfrak{U} = \{ U_i; \alpha \in \Omega \} \) is a refinement of \( \mathfrak{U} \)

Let \( (K, L) \) be a pair of nerves of the covering \( \mathfrak{U} \cap (X_1, A_1) \) with the weak topology. Since \( \mathfrak{U} \) is a star refinement of \( \mathfrak{U} \), for each element \( V_0 \) of \( \mathfrak{U} \), we can select a point \( x_0 \) of \( X_1 \) such that \( \bigcup \mathfrak{U} \mathfrak{C} S(x_0, \varepsilon(x_0)) \). By the construction of \( e(x_0) \), \( \bigcup \mathfrak{U} \mathfrak{C} S(x_0, \varepsilon(x_0)) \) such that \( \mathfrak{U} \mathfrak{C} S(x_0, \varepsilon(x_0)) \) \( \cap \{ x, e(x) \} \) \( \cap \{ x, e(x) \} \) \( \cap \{ x, e(x) \} \) \( \cap \{ x, e(x) \} \) for each vertex \( x \) of \( K \), \( \mathfrak{U} \mathfrak{C} S(x_0, \varepsilon(x_0)) \). For each vertex \( v \) of \( K \), \( \mathfrak{U} \mathfrak{C} S(x_0, \varepsilon(x_0)) \) \( \cap \{ x, e(x) \} \) \( \cap \{ x, e(x) \} \) \( \cap \{ x, e(x) \} \) \( \cap \{ x, e(x) \} \) a vertex. Define \( \varphi : K \to X_1 \) by \( \varphi = \mathfrak{U}. \) Obviously \( \varphi(L) (X_2, A_2). \) Let \( \varphi \) be
Let \( X \) and \( Y \) be \( \text{ANR}'s \) and \( x_0 \) and \( y_0 \) be points of \( X \) and \( Y \).

**Lemma 5.** If either \( x_0 \) is \( 0\text{-HL} \) in \( X \) or \( y_0 \) is \( 0\text{-HL} \) in \( Y \), then \( (x_0, y_0) \) is \( 1\text{-HL} \) in \( X \times Y \).

**Lemma 6.** If \( x_0 \) and \( y_0 \) are \( 0\text{-HS} \) in \( X \) and \( Y \) respectively; then \( H_\ast (x_0, y_0) ; Z \neq 0 \).

**Proof of Lemma 5.** Assume that \( x_0 \) is \( 0\text{-HL} \) in \( X \). Let \( W \) be a neighbourhood of \( (x_0, y_0) \) and \( U_i, i=1, 2, \ldots \), be neigbourhoods of \( x_0 \) and \( y_0 \) in \( X \) and \( Y \) respectively such that \( U_1 \times U_2 \subset W \). Take a neighbourhood \( V_1 \) of \( x_0 \) and \( y_0 \) such that \( V_1 \) is contractible in \( U_i, i=1, 2, \ldots \). Let \( f \) be a mapping of \( 1\text{-sphere} \) into \( V_1 \times V_2 \). We shall prove that \( f \) has an extension \( f' : E \to U_1 \times U_2 \). There exists a point \( \epsilon \) such that 0 < \( \epsilon < \delta \) where \( \delta \) is a metric in \( X \times Y \). Put \( W_\epsilon = S(x_0, \epsilon) \). Define \( f' : E \to V_1 \times V_2 \) such that \( f' = f \) for \( i = 1, 2 \), where \( p_1 \) and \( p_2 \) are projections \( X \times Y \to X \) and \( X \times Y \to Y \) respectively. Take neighbourhoods \( W_i, W_\epsilon, W_2 \) of \( x_0 \) such that

1. \( W_i \) is contained in \( W_\epsilon \) and contractible in \( W_\epsilon \).
2. \( W_\epsilon \) is contained in \( W_2 \) and any mapping \( g : S^0 \to W_\epsilon \) has an extension \( g' : E \to W_\epsilon \).
3. \( W_2 \) is contained in \( W_2 \) and contractible in \( W_2 \).

Put \( N = f^{-1}(x_0) \). Let \( N_1, i=1, 2, \ldots, \) be components of \( N \). Put \( G = f^{-1}(W) \). Then \( G \) is an open set containing \( N \). Let \( (G_i) \) be all components of \( G \) intersecting with \( N \). Since \( S^0 \) is locally connected, each \( G_i \) is an open set (cf. [24], Chap. I, (14.1)) in \( S \). Hence, \( (G_i) \) is an open covering of the compact set \( N \). Therefore \( (G_i) \) consists of a finite number of sets.

Let us denote them by \( G_1, \ldots, G_n \). Put \( M_i = \bigcup \{ N_j \mid N_j \subset G_i, j=1, 2, \ldots, n \} \). Let \( I_1 \) be the minimal closed interval in \( S^0 \). Let \( f' : E \to \Phi \) for \( i \neq j \). Define a mapping \( g_i : S^0 \to V_1 \) such that

\[
g_i(s) = g_i(s) \quad \text{for} \quad s \in S^0 \setminus \bigcup_{j \neq i} I_j,
\]

\[
g_i(s) = x_0 \quad \text{for} \quad s \in \bigcup_{j \neq i} I_j.
\]

Obviously, \( g_i \) is continuous. Moreover, if we define a mapping \( G_i : S^0 \to V_1 \times V_2 \) such that \( G_i(s) = (g_i(s), f_i(s)) \) for \( s \in S^0 \), we have by the construction \( 3^0 \) of \( W_2 \) and \( W_2 \),

\[
G_i \to f : S^0 \to V_1 \times V_2 (x_0, y_0) \).
\]

Since \( \bigcup_{j=1}^{n} I_j \) is contained in the open set \( g^{-1}(W_2) \), there exists an open interval \( H_i = (a_i, b_i) \) in \( S^0 \) containing \( I_j \) such that \( H_i \cap \bigcup_{j \neq i} I_j = \emptyset \) and

\[
H_i \cap \bigcup_{j \neq i} I_j = \emptyset.
\]

Fundamenta Mathematicae, T. XLVII. 13
Define a mapping $g_j: S^0 \to \bigcup_{j=1}^n H_j \to V_1 - x_0$ such that $g_j = g_1|_{S^0 - \bigcup_{j=1}^n H_j}$. By the construction 2 of $W_1$, we find that a mapping $g_1|_{S^0 \times Y_1 \times \{x_0\}}$ is extended to $i: H_j \to W_1 - x_0$ for $j = 1, 2, \ldots, n$. Put $g_j: S^0 \to V_1 - x_0$ such that

$$
g_j(y) = g_j(x) \quad \text{for} \quad x \in S^0 - \bigcup_{j=1}^n H_j,
$$

$$
g_j(x) = i_j(x) \quad \text{for} \quad x \in H_j, \quad j = 1, 2, \ldots, n.
$$

If we define a mapping $G_2: S^0 \to V_1 \times V_2 - (x_2, y_2)$ such that $G_2(x) = (g_2(x), f_2(x))$ for $x \in S^0$, by the construction 1 of $W_2$ and $W_1$, we have

$$
G_2 \sim G_1: S^0 \to V_1 \times V_2 - (x_1, y_1).
$$

Since $V_1$ is contractible in $U_1$ and $Y$ is non-degenerate, there exists a homotopy $h_1: Y_1 \times Y_1 \to U_1$ such that $h_1(g_1(y)) = y$ and $h_1(y) = y_1 \neq y_2$ for $y \in V_2$. Define $G_3: S^0 \to V_1 \times Y_1$ by $G_3(x) = (g_2(x), g_1(x))$ for $x \in S^0$. Put $H_3: S^0 \to V_1 \times Y_1 - (x_1, y_1)$ such that $H_3(x) = (g_2(x), h_1(x))$ for $t \in (0, 1)$, and $s \in S^0$. Then $H_3|_{S^0 \times \{1\}} = g_1|_{S^0 \times \{1\}}$. Therefore we have

$$
G_3 \sim G_2: S^0 \to V_1 \times Y_1 - (x_1, y_1).
$$

Since $V_1$ is contractible in $U_1$, there exists a homotopy $i: V_1 \to U_1$ such that $i_1(x) = x$ and $i_1(x) = x_1$ for $x \in V_1$. If we denote by $G_1$ the constant mapping $S^0 \to (x_1, y_1)$, we have, in the same way as (c),

$$
G_3 \sim G_1: S^0 \to U_1 \times U_2 - (x_0, y_0).
$$

(a)-(d) completes the proof of Lemma 6.

Proof of Lemma 6. Since $x_0$ is 0-III, there exists a neighborhood $U$ of $x_0$ such that, whenever $V$ is a neighborhood of $x_0$ and contained in $U$, there exists a mapping $f: \overline{V} \to V - x_0$ which has no extension $f': E \to U - x_0$. We can assume that $V$ is contractible in $U$. Therefore, we have an extension $g: E \to \overline{V}$ of $f$. Take a neighborhood $W$ of $x_0$ such that $W \subset V$ and $W \subset g(S^0) \subset W - f(S^0) = \Phi$. Then $g$ determines an element $a$ of $\tilde{H}_2(U, U - W; Z)$, where $\tilde{H}_2(X, A; Z)$ means the $n$-dimensional singular homology group of $(X, A)$ with coefficients $Z$. Let $u$ be the boundary homomorphism $\tilde{H}_2(U, U - W; Z) \to \tilde{H}_1(U - W; Z)$. Since $\tilde{H}_2$ is an element represented by $f$ with the infinite order, the order of $a$ is infinite. Moreover, let $W'$ be a neighborhood of $x_0$ contained in $W$. The homomorphism $j_3: \tilde{H}_3(U, U - W; Z) \to \tilde{H}_3(U, U - W'; Z)$ induced by the inclusion mapping $j_3: (U, U - W) \to (U, U - W')$, maps $a$ into an element $j_3(a)$ of $\tilde{H}_3(U, U - W'; Z)$ with the infinite order. Let $(W_i)$ be a complete family of neighborhoods of $x_0$ such that $W_{i+1} \subset W_i$ and $W_1 = U$. We have a pair of complexes $(K_i, L_i)$ mappings $\rho_i: (U, U - W_i) \to (K_i, L_i)$, $\rho_i: (K_i, L_i) \to (U_1 - W_i)$ and a homotopy $h_i: (U_i, U - W_i) \to (U_i, U - W_{i+1})$ where $h_i$ is the inclusion mapping $(U_i, U - W_i) \subset (U, U - W_i)$. There exists an element $a_1$ of $\tilde{H}_1(K_i, L_i; Z)$ with the infinite order such that $(\rho_i - 1) \alpha_i = a_{i+1}$, $i = 1, 2, \ldots$. Since $y_0$ is 0-III, we can find a complete family $(Y_i)$ of neighborhoods of $y_0$, a sequence of pairs of complexes $(M_i, N_i)$, mappings $\mu_i: (U_i, U - W_i) \to (M_i, N_i)$, $\xi_i: (M_i, N_i) \to (U_i, U - W_i)$ and $\mu_i: (U_i, U - W_i) \to (U_i, U - W_{i+1})$, where $\mu_i$ is the inclusion mapping $(U_i, U - W_i) \subset (U, U - W_i)$. Moreover, there exists an element $b_i$ of $\tilde{H}_1(M_i, N_i; Z)$ with the infinite order such that $(\mu_i - 1) \beta_i = b_{i+1}$ for $i = 1, 2, \ldots$. Define $\alpha_{i+1} = (K_i \times M_i, L_i \times N_i) \to (K_{i+1} \times M_{i+1}, L_{i+1} \times N_{i+1})$ such that $\alpha_{i+1}(x, s') = (\rho_i(s) \mu_i(x, s'))$ for $s \in K_i$ and $s' \in M_i$. If $a_{i+1}$ is the homomorphism induced by $\alpha_{i+1}$, then the limit group of the direct system $(\alpha_{i+1}(x, s), s \in K_i, a_{i+1}(x, s), s' \in M_i)$ is equal to $\tilde{H}_2(K_i, L_i; Z)$, because the singular theory satisfies the excision axiom (cf. [9], Chap. 7). By Künneth's theorem ([13], p. 308), we have $a_{i+1} \circ b_{i+1} \circ \rho_i(x_0, y_0) = 0 \circ (\mu_i(x_1, y_1), 1) \circ (M_i, N_i) \subset \tilde{H}_2(K_i, L_i; Z)$, since the order of $a_{i+1}$ is finite and $\alpha_{i+1}, a_{i+1}$ are infinite. Moreover, $(\alpha_{i+1}(x_0, y_0), a_{i+1} \circ b_{i+1}) \neq 0$. This shows $\tilde{H}_2(x_0, y_0; Z) = \tilde{H}_2(x_0, y_0; \Phi) 
eq 0$ by Lemma 4.

Proof of III. It is sufficient to prove that if $X$ is an ANR, and $x_0$ is a point of $X$ such that $X$ is $(k+1)$-dimensional for $k > 1$ and $H_k(x_0; Z) = 0$, then $x_0$ is 0-III in $X$.

Let $U$ be a neighborhood of $x_0$. There exists a sequence $V_1, V_2, \ldots$ of neighborhoods of $x_0$ such that

1. $x_0 \in V_1 \subset \overline{V}_2 \subset \overline{V}_3 \subset \cdots \subset \overline{V}_k \subset \cdots \subset \overline{V}_k \subset \overline{V}_{k+1} = \overline{U},$

2. $V_i$ is contractible in $V_i$.

3. if $j: S^0 \to V_i - x_0$, there exists an extension $f: E^{j+1} \to V_i - x_0$ of $f$ for $i = 1, 2, \ldots, k$ and $j = 0, 1, 2, \ldots, k - 1$.

Let $f$ be any mapping of $S^k$ into $V_i - x_0$. Fix a point $a_0$ of $S^k$. There exists a mapping $g: (E^k, E^k) \to (S^k, a_0)$ such that $\varphi(E^k - E^k) = \tilde{H}_2$. Therefore, by the definition of singular homology group, there exists a $(k+1)$-dimensional finite complex $E_{k+1}$ containing $E^k$ and a mapping $h: E^{k+1} \to V_i - x_0$ such that $h_{i+1}(x) = x_0$.
Define a homotopy $H: P^{k+1} \times I \rightarrow U - x_0$ as follows: At first, put $H(p,0)=h(p)$ for $p \in P^{k+1}$ and $H(p,t)=f_t(p)$ for $p \in P^{k+1}$ and $t \in I$. Next, put $H(p,1)=x_0$ for $p \in P^{k+1}$, where $P^{k+1}_i$ is the $i$-skeleton of $P^{k+1}$. By the construction $5^o$ of $V_1$ and $V_2$, we can extend $H$ to a mapping of $(P^{k+1} \cup P^{k+1}_i) \times I \rightarrow V_1 - x_0$. By a repeated application of this process we can have a mapping $H: (P^{k+1} \times 0) \cup (P^{k+1} \cup P^{k+1}_i) \times I \rightarrow V_1 - x_0$ such that

$$H(p,0)=h(p) \quad \text{for} \quad p \in P^{k+1},$$
$$H(p,t)=(p) \quad \text{for} \quad p \in P^{k+1}_i,$$
$$H(p,1)=x_0 \quad \text{for} \quad p \in P^{k+1}_i.$$

Since $U - x_0$ is an ANR by Lemma 2, we can extend $H$ to a mapping of $P^{k+1} \times 1$ into $U - x_0$. Consider a mapping $H|P^{k+1} \times 1: P^{k+1} \times 1 \rightarrow U - x_0$. In the same way as in Theorem 12.6 of [11], we define the following homomorphism $a$ of the $k$-dimensional chain group $C_k(P^{k+1}, Z)$ of $P^{k+1}$ into the $k$-dimensional homotopy group $a_k(U - x_0, x_0)$ of $(U - x_0, x_0)$. Let $T_i^k, i=1,2,\ldots, q$, be the $n$-simplices of $P^{k+1}_i$, each in a definite orientation. Since the restricted mapping $H|P^{k+1} \times 1: P^{k+1} \times 1 \rightarrow U - x_0$ maps the $(k-1)$-skeleton of $P^{k+1} \times 1$ into the point $x_0$, the mapping $H|T_i^k: T_i^k \times 1 \rightarrow U - x_0$ determines the element $a(T_i^k) = a(T_i^k) = a(T_i^k) = a(T_i^k) = a(T_i^k) = a(T_i^k).$ Thus, the integral $k$-chain $C_k = \sum a(T_i^k) \epsilon C_k(P^{k+1}, Z)$, let us assign the element $a(C_k) = \frac{1}{k}a(T_i^k).$ Since $k > 1$, $a$ is a homomorphism. Moreover, if $T_i^k$ is a $(k+1)$-simplex of $P^{k+1}, a(T_i^k) = 0$. Hence, since $H|P^{k+1} \times 1$ maps $\omega 0^1$ into the point $x_0$, we have $a(T_i^k) = a(T_i^k) = a(T_i^k) = a(T_i^k) = a(T_i^k) = a(T_i^k).$ Therefore, by $3^o$, we have $a(T_i^k) = 0$. This completes the proof of III and consequently the proof of Theorem 6.

Theorem 5 of [21] is a consequence of Theorem 6.

In part III of Theorem 6 we have proved that if $x_0$ is $H_1^j$ in $X$ and $H_2(z) = 0$ for $z \in Z$, then $x_0$ is $H_1^j$. Therefore, we have the following theorem.

**Theorem 7.** Let $X$ be a finitely dimensional locally compact ANR and let $x_0$ be a point of $X$. Then $x_0$ is $H_1^j$ in $X$ if and only if $x_0$ is $H_1^j$ in $X$ and $H_2(z) = 0$ for $z \in Z$. Therefore, we have the following.

Remark 2 (cf. Remark 1). Since the Čech homology theory satisfies the excision axiom (cf. for example, Eilenberg and Steenrod [9], p. 245), we can replace the condition "$X$ is finitely dimensional" by the condition "$X$ and $Y$ are finitely dimensional" by the condition "$X$ and $Y$ are finitely dimensional" in Theorem 7. Similarly, we can replace the condition "$X$ and $Y$ are finitely dimensional" by the condition "$X$ and $Y$ are finitely dimensional" in Theorem 7.

**5. Dimension of products spaces**

**Theorem 8.** Let $X$ be a locally compact fully normal space and let $Y$ be a locally compact $2$-dimensional ANR. Then the following equality exists:

$$\dim X \times Y = \dim X \times \dim Y.$$
Theorem 9. Let $X$ be a locally compact $n$-dimensional ANR containing a point $x_0$ which is $H^m_{\infty}$ and $(m-1)$-HS, and let $Y$ be a locally compact $n$-dimensional ANR containing a point $y_0$ which is $H^m_{\infty}$ and $(m-1)$-HS. Then the following equality exists:

$$\dim X \times Y = \dim X + \dim Y.$$ 

Proof. By Theorem 8 and [15], Theorem 6, it is sufficient to prove the theorem in the case of $2 < n$ and $2 < n$. In the same way as in Lemma 6, we can show that there exist compact subsets $X_1, X_2$ of $Y$ and $Y_1, Y_2$ of $X$ such that $H^m(X_1, X_2; Z) \neq 0$ and $H^m(Y_1, Y_2; Z) \neq 0$. Since $\dim X = m$ and $\dim Y = n$, all non-zero elements of the groups $H^m(X_1, X_2; Z)$ and $H^m(Y_1, Y_2; Z)$ have infinite orders. Therefore, if $F$ is the field of rational numbers, we have $H^m(X_1, X_2; F) \neq 0$ and $H^m(Y_1, Y_2; F) \neq 0$. Hence, by Theorem 3, we have $H^m(X_1 \times Y_1, X_1 \times Y_2 \cup A \times Y_2; F) \neq 0$. This shows that $\dim X \times Y > \dim X + \dim Y$. It is obvious that $\dim X \times Y < \dim X + \dim Y$. This completes the proof.

References