On symmetric products

by

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1. Symmetric products. If M is a metric space, 2^M denotes the space of all closed, bounded and non-empty sets $E\subset X$ metrized by the formula

$$\varrho(E_1,E_2)=\max[\sup_{x\in E_2}\varrho(x,E_1),\sup_{x\in E_1}\varrho(x,E_2)].$$

Let $E_1,E_2,\ldots,E_n$ be bounded and non-empty subsets of M. By the *symmetric product* ([1] and [2]) of the sets $E_1,\ldots,E_n$ we understand the subset $E_1\circ\ldots\circ E_n$ of 2^M composed of all sets $\{x_1,x_2,\ldots,x_n\}$ [1]) with $x_i\in E_i$ for $i=1,2,\ldots,n$. In case $E_1=\ldots=E_n=E$ the product $E_1\circ\ldots\circ E_n$ is called the nth *symmetric power* of E and denoted by $E^{(n)}$.

If U is a neighbourhood of x_i in E_i $(i=1,\ldots,n)$, then the set of all points $\{x_1',x_2',\ldots,x_n'\}$ of $E_1\circ\ldots\circ E_n$ such that $U_i\cap\{x_1',\ldots,x_n'\}\neq 0$ for $i=1,2,\ldots,n$ is a *neighbourhood* of $\{x_1,\ldots,x_n\}$ in $E_1\circ\ldots\circ E_n$.

In the case, where E_i are disjoint sets, the symmetric product is identical with the Cartesian product.

Evidently, if h is a homeomorphism mapping M onto another space $h(M)$, then the symmetric product $E_1\circ\ldots\circ E_n$ is homeomorphic with the symmetric product $h(E_1)\circ\ldots\circ h(E_n)$.

Let Q_m denote the m-dimensional Euclidean cube. It is known (cf. [2]), that for $n=1,2,3$, Q_1^n (*i. e.*, the nth symmetric power of the segment) is homeomorphic with Q_n, but, for $n\geqslant 4$, $Q_1^{(n)}$ is not homeomorphic with any subset of the Euclidean space R^n. In this note it is shown that $Q_2^{(2)}$ is homeomorphic with Q_4, but, for $n\geqslant 3$, $Q_2^{(n)}$ and $Q_n^{(2)}$ are not homeomorphic with any subset of R^{2n}.

2. An elementary lemma. We need the following

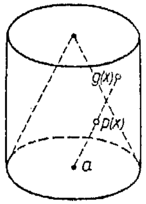
Lemma. *The set P of all points p lying in the Euclidean 4-space R^4 and having the form*

[1]) We denote by $\{x_1,\ldots,x_n\}$ the set composed of the elements $x_1,\ldots,x_n$, and we denote by $(x_1,\ldots,x_n)$ the ordered system $x_1,\ldots,x_n$.

$$p=(\varrho\cos\psi,\ \varrho\sin\psi,\ v\cos\omega,\ v\sin\omega),$$
(1)
$$0\leqslant\varrho\leqslant 1,\quad 0\leqslant\psi<2\pi,\quad 0\leqslant v\leqslant 1-\varrho,\quad 0\leqslant\omega<2\pi,$$

is a 4-dimensional element [2]).

Proof. Evidently the points of P with $\psi=0$, *i. e.* the points of the form $(\varrho,0,v\cos\omega,v\sin\omega)$ with ϱ, v and ω satysfying (1), constitute a rotation cone C (in the elementary sense). Let A denote the cylinder (in the elementary sense) circumscribed on C and let a be the centre of the common base of A and C. Evidently A is a set of points of the form $(\bar\varrho,0,\bar v\cos\bar\omega,v\sin\bar\omega)$, where



(2) $$0\leqslant\bar\varrho\leqslant 1,\quad 0\leqslant\bar\omega<2\pi,\quad 0\leqslant\bar v\leqslant 1.$$

For every point $x\in C-(a)$ let $p(x)$ and $g(x)$ denote the point of intersection of the ray $\vec{ax}$ with the surface of the cone C and with the surface of the cylinder A respectively. Evidently $g(x)$ and $p(x)$ depend continuously on x. It can easily be seen that setting

$$\mu(a)=a,\qquad \mu(x)=a+\frac{\varrho\big(a,g(x)\big)}{\varrho\big(a,p(x)\big)}(x-a)$$

for every point $x\in C-(a)$, we obtain a homeomorphism mapping C onto A. If $x=(\varrho,0,v\cos\omega,v\sin\omega)$, let us put $\mu(x)=(\bar\varrho,0,\bar v\cos\bar\omega,\bar v\sin\bar\omega)$, where $\bar\varrho$, $\bar v$ and $\bar\omega$ satisfy (2). Moreover

$$\mu(0,0,v\cos\omega,v\sin\omega)=(0,\ 0,\ v\cos\omega,\ v\sin\omega).$$

Now let us observe that the set Q composed of the points of the form $(\bar\varrho\cos\psi,\bar\varrho\sin\psi,\bar v\cos\bar\omega,\bar v\sin\bar\omega)$, where $0\leqslant\bar\varrho\leqslant 1$, $0\leqslant\psi<2\pi$, $0\leqslant\bar v\leqslant 1$, $0\leqslant\bar\omega<2\pi$, is homeomorphic with the Cartesian product of two 2-dimensional elements. Setting

$$h(\varrho\cos\psi,\ \varrho\sin\psi,\ v\cos\omega,\ v\sin\omega)=(\bar\varrho\cos\psi,\ \bar\varrho\sin\psi,\ \bar v\cos\bar\omega,\ \bar v\sin\bar\omega),$$

for every point $(\varrho\cos\psi,\ \varrho\sin\psi,\ v\cos\omega,\ v\sin\omega)$ we obtain a homeomorphism mapping P onto Q. Thus the proof of the lemma is finished.

Let us observe that the boundary of the set P, *i. e.*, a set homeomorphic with a 3-dimensional sphere, is composed of the elements of the form

(3) $$(\varrho\cos\psi,\ \varrho\sin\psi,\ (1-\varrho)\cos\omega,\ (1-\varrho)\sin\omega).$$

[2]) By an *n-dimensional element* we understand every set homeomorphic with an n-dimensional Euclidean cube.

3. Symmetric square of a 2-dimensional element. Using the last lemma we are able to prove the following

THEOREM 1. *The symmetric square of a 2-dimensional element is a 4-dimensional element.*

Proof. Consider a system of polar coordinates r, ψ in the Euclidean plane and let Q_2 denote the disk defined by the inequality $r\leqslant 1$. Let p,q be two points of Q_2 and let

$$s=(r\cos\psi,\ r\sin\psi)$$

be the centre of the segment $\overline{pq}$. Let φ, $0\leqslant\varphi\leqslant\pi$, denote, for $p=q$, the positive rotation angle from the axis $\psi=0$ to the straight line L_{pq} joining p and q, and let a_{pq},b_{pq} be the points of intersection of L_{pq} with the boundary of Q_2. Let

$$d_{pq}=\tfrac{1}{2}\varrho\big((s,a_{pq})+\varrho(s,b_{pq})-|\varrho(s,a_{pq})-\varrho(s,b_{pq})|\big),$$
$$u=\varrho(p,s)=\varrho(q,s),$$

d_{pq} being the smaller of the numbers $\varrho(s,a_{pq})$ and $\varrho(s,b_{pq})$. From the definition of d_{pq} we have $0\leqslant u\leqslant d_{pq}$, and observing that the distance from the point s to the boundary of Q_2 is $(1-r)$, we have $(1-r)\leqslant d_{pq}$. Setting, for every $p,q\in Q_2$,

$$\Phi\{p,q\}=\begin{cases}\left(r\cos\psi,\ r\sin\psi,\ \dfrac{u(1-r)}{d_{pq}}\cos 2\varphi,\ \dfrac{u(1-r)}{d_{pq}}\sin 2\varphi\right), & \text{if}\quad p\neq q,\\ (r\cos\psi,\ r\sin\psi,\ 0,\ 0), & \text{if}\quad p=q,\end{cases}$$

we can easily see that Φ is a 1-1 transformation of the symmetric square $Q_2^{(2)}$ onto a subset P of the Euclidean 4-space R^4, composed of all points of the form $(r\cos\psi,\ r\sin\psi,\ t\cos\omega,\ t\sin\omega)$ with $0\leqslant r\leqslant 1$, $0\leqslant\psi<2\pi$, $0\leqslant t=u(1-r)/d_{pq}\leqslant 1-r$, $0\leqslant\omega=2\varphi<2\pi$.

Moreover, let us observe that the transformation is continuous. If $p\neq q$, then the continuity of Φ at the point $\{p,q\}$ is a consequence of the continuity of every coordinate of $\Phi\{p,q\}$; if, however, $p=q$, $\delta>0$ and $\varrho(\{p',q'\})<\delta$, then denoting by $r',u',d_{p'q'}$ and φ' the numbers defined for p',q' in the same way as the numbers r,u,d_{pq} and φ are defined for p,q, we have

$$\frac{u'(1-r')}{d_{p'q'}}\cos 2\varphi'\leqslant u'<\delta,$$
$$\frac{u'(1-r')}{d_{p'q'}}\sin 2\varphi'\leqslant u'<\delta.$$

Since $Q_2^{(2)}$ is compact, we infer that Φ is a homeomorphism. But by the lemma the set P is a 4-dimensional element. Thus the proof of our theorem is finished.

Let us observe that the boundary of $Q_2^{(2)}$, *i. e.*, the set of points which is mapped by Φ on the set of points of the form (3), is composed of the points $\{p, q\}$ for which $u = d_{pq}$, *i. e.*, for which at least one of the points p, q lies on the boundary of Q_2. It follows that the symmetric square of an open 2-dimensional element is an open 4-dimensional element and we obtain the following

COROLLARY. *The symmetric square of a closed 2-dimensional manifold is a closed 4-dimensional manifold.*

THEOREM 2. *For $n \geqslant 3$, $Q_2^{(n)}$ is not homeomorphic with any subset of R^{2n}.*

Proof. Let Q_2 denote the disk defined in the Euclidean plane by the inequality $x^2 + y^2 \leqslant 1$. Let $U_1, U_2, \ldots, U_{n+1}$ be disjoint disks in Q_2 with centres at the points $x_i^0 = \big(0, (i-1)/(n-1)\big)$ and the radius $r_i = 1/3(n-1)$, $i = 1, 2, \ldots, n-1$.

Let us consider the subset W of $Q_2^{(n)}$ defined by the formula

$$W = \mathop{E}_{\{x_1, \ldots, x_n\}} [x_i \in U_i,\ i = 1, \ldots, n-2;\ x_{n-1}, x_n \in U_{n-1}].$$

Setting for every $\{x_1, \ldots, x_n\} \in W$

$$g\{x_1, \ldots, x_n\} = (x_1, \ldots, x_{n-2}, \Phi\{x_{n-1}, x_n\}),$$

where Φ is the homeomorphism mapping $U_{n-1}^{(2)}$ onto the 4-dimensional element defined in theorem 1, we can easily see that g is a homeomorphism mapping W onto a $2n$-dimensional element. The point $\Phi\{x_{n-1}^0, x_{n-1}^0\}$ is by the definition of Φ an inner point of $\Phi(U_{n-1}^{(2)})$. It follows that the point

$$g\{x_1^0, x_2^0, \ldots, x_0^{n-1}, x_0^{n-1}\}$$

is an inner point of $g(W)$. Now, let h be a homeomorphism mapping $Q_2^{(n)}$ onto a subset of R^{2n}. Applying Brouwer's theorem on the invariance of region in R^{2n} (cf. [3]), we conclude that the point $h\{x_1^0, x_2^0, \ldots, x_{n-1}^0, x_{n-1}^0\}$ is an inner point of $h(W)$. Now let us consider the sequence $[p_k]$ of points from the set $Q_2^{(n)}$:

$$p_k = \left\{x_1^0, \left(0, \frac{1}{k}\right), x_2^0, \ldots, x_{n-1}^0\right\}.$$

We have $p_k \in W$, and

$$\lim_{k \to \infty} p_k = \{x_1^0, x_1^0, x_2^0, \ldots, x_{n-1}^0\}.$$

Further $h(p_k) \in R^{2n} - h(W)$ and

$$\lim_{k \to \infty} h(p_k) = h\{x_1^0, x_1^0, x_2^0, \ldots, x_{n-1}^0\},$$

which is impossible since $h\{x_1^0, x_1^0, x_2^0, \ldots, x_{n-1}^0\} = h\{x_1^0, x_2^0, \ldots, x_{n-1}^0, x_{n-1}^0\}$ is an inner point of $h(W)$. This proves the theorem.

4. Symmetric square of an n-element. Let us prove the following

THEOREM 3. *For $n \geqslant 3$ the symmetric square of an n-element is not homeomorphic with any subset of the Euclidean space R^{2n}.*

We shall use the notions of homology theory and the theory of intersections of closed manifolds. The basic group used for homologies will be the cyclic group of order two.

With each closed manifold M_k we can associate a ring $R(M_k)$. The commutative group of this ring is the direct sum of Betti groups of different dimensions. The product of a p-cycle and a q-cycle is a $(p+q-n)$-cycle defined as their intersection (see for instance [5], p. 205-207). Our proof is based on the following theorem proved by H. Hopf ([4], p. 173):

A necessary condition that the k-dimensional closed manifold M_k might be imbedded in the $(k+1)$-dimensional Euclidean space R^{k+1}, is that the direct sum of their Betti groups mod 2 *of different dimensions except zero be a direct sum of two subrings, $R_1(M_k)$ and $R_2(M_k)$, of $R(M_k)$. (That means that the product in this rings is defined as an intersection).*

We use this theorem in the proof of the following lemma (this proof exactly follows the route taken by Hopf in the case of projective spaces):

LEMMA. *The Cartesian product $M_{2n-1} = S^n \times P^{n-1}$ of the n-dimensional sphere S^n and the $(n-1)$-dimensional projective space P^{n-1} cannot be imbedded in the $2n$-dimensional Euclidean space R^{2n}.*

Proof. M_{2n-1} is a $(2n-1)$-dimensional closed manifold. By the Küneth formula (cf. [5], p. 141) the homological structure of M_{2n-1} is as follows: in each dimension the Betti basis consist of exactly one element. The element of the $(n-1)$-dimension is the unity of the ring $R(M_{2n-1})$. Let z denote a $(2n-2)$-element. It is easy to show that for every r the power z^r is the $(2n-r-1)$-element. If $n > 1$, then z has a positive dimension. Suppose that M_{2n-1} can be imbedded in R^{2n}. From Hopf's theorem it follows that z belongs to one of the two rings, $R_1(M_{2n-1})$ and $R_2(M_{2n-1})$, for instance to $R_1(M_{2n-1})$. The ring $R_1(M_{2n-1})$ contains with z every power of z, in particular z^{2n-1}, which is a zero-dimensional element, in contradiction to the fact that $R_1(M_{2n-1})$ contains only elements of positive dimension. Thus the proof of the lemma is finished.

Proof of theorem 3. Let Q be an n-dimensional ball with radius 1 lying in the Euclidean n-space R^n and let Q_0 denote the concetric n-dimensional ball with radius $\frac{1}{2}$. Consider the compact subset Z of $Q^{(2)}$ composed of all points $\{p, q\}$ such that

(4) $$\tfrac{1}{4} \leqslant \varrho(p, q) \leqslant \tfrac{1}{2}$$

and

(5) the centre s of the segment $\overline{pq}$ belongs to Q_0.

It follows by (4) that for every point $\{p,q\} \in Z$ there exists exactly one straight line L_{pq} passing through p and q. Let

$$\Psi\{p,q\} = (s, 4[\varrho(p,q)-\tfrac{1}{4}], L_{pq}) \quad \text{for every} \quad \{p,q\} \in Z .$$

Evidently Ψ is a 1-1 transformation of the set Z onto the set of all systems $(s, 4[\varrho(p,q)-\frac{1}{4}], L_{pq})$. Since for every $s \in Q_0$ the straight lines $L \subset R^n$ passing through s constitute a space homeomorphic with the $(n-1)$-dimensional projective space P^{n-1} and s, $4[\varrho(p,q)-\frac{1}{4}]$ and L_{pq} depend continuously on $\{p,q\}$, we infer that Ψ maps Z topologically onto the Cartesian product $Q_0 \times \langle 0;1\rangle \times P^{n-1}$. Since $Q_0 \times \langle 0;1\rangle$ is an $(n+1)$-dimensional element, we conclude that $Z \subset Q^{(2)}$ contains topologically the Cartesian product $S^n \times P^{n-1}$ of the n-dimensional sphere S^n and the $(n-1)$-dimensional projective space P^{n-1}. By the lemma $S^n \times P^{n-1}$, and consequently also $Q^{(2)}$, are not homeomorphic with any subset of R^{2n}, which concludes the proof.

References

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On homotopically stable points and product spaces

by

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§ 1. Introduction

Let X be a topological space. A point x_0 of X is called *homotopically labile* in X whenever for every neighbourhood U of x_0 there exists a continuous transformation $f(x,t)$ which is defined in the Cartesian product $X \times I$ of X and of the closed interval $I = \langle 0,1\rangle$ and which satisfies the following conditions:

(1) $f(x,t) \in X$ for every $(x,t) \in X \times I$,

(2) $f(x,0) = x$ for every $x \in X$,

(3) $f(x,t) = x$ for every $(x,t) \in (X-U) \times I$,

(4) $f(x,t) \in U$ for every $(x,t) \in U \times I$,

(5) $f(x,1) \neq x_0$ for every $x \in X$.

A point x_0 of X is called *homotopically stable* if it is not homotopically labile. K. Borsuk and J. W. Jaworowski [5] introduced this notion and studied the various properties of labile and stable points.

In this paper, we shall study first a certain characteristic property of homotopically labile points in ANR's for metric spaces. This shows that "homotopical stability" is equivalent to "n-homotopical stability for some integer n [1])". The main theorem, which states that the homotopical lability or stability of a point in a product space is determined by the local connectivity groups at that point [1]), is proved in § 4. This theorem gives a generalization of H. Noguchi's theorem [21] to the case of ANR.

Let X and Y be two topological spaces. The equality $\dim X \times Y = \dim X + \dim Y$ does not generally hold; for example, K. Borsuk [4] has proved that there exist 2-dimensional Cantor manifolds whose Cartesian product has dimension three. In § 5 we shall show that this equality holds in the following two cases:

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[1]) For these definitions, see §§ 2 and 4.