# Borel sets and countable series of operations

by

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## Introduction

In [1] Z. P. Dienes has raised the following question: Given two Borel sets $X, Y$ in a space $E$, there are the five possibilities

(i) $$X=Y, \quad X\subseteq Y, \quad X\supseteq Y, \quad X\cap Y=0, \quad X\cap Y\neq 0.$$

Is it possible to decide which of these hold by means of an "Operation $D$" (which he defines vaguely as a succession of $\aleph_0$ comparisons of integers for size). If it is possible, the problem is "decidable $D$". He goes on to shew that given a $G_\delta$-set on the real line $R^1$, then the problem of deciding whether or not the set is empty, is decidable $D$.

Now, his question has two sides, one logical, the other topological. His vague notion of an operation $D$ induces in us the intuitive picture of two magicians $M_1, M_2$, each with this property: if $\alpha$ is a countable ordinal, and if for each $\xi<\alpha$, $A(\xi)$ is an act which a human mathematician could in principle perform, then each magician can perform the whole series $A(0), A(1), \ldots, A(\xi), \ldots$, and live to tell the tale. $M_1$ then has to describe $X$ and $Y$ to $M_2$ by means of such a series $A$, and $M_2$ reads it and is able to recall any term at will; we want to know if there is a series $A'$ such that by performing it, $M_2$ can say which of the possibilities (i) holds. This suggests that Dienes's question might be best rephrased as "Are the possibilities (i) decidable in a logic with countably transfinite rules of inference and sentence formation?" It is then conceivable that a variety of other problems might be shewn to be decidable (or undecidable) relative to such a formal system. However, it seems to the author, that before constructing such a system — and in order to see what is involved and what features it is desirable to build into the system — we should have more experience of what ought, by any definition, to be "decidable $D$" problems.

We therefore give, in [1]) (1.3) below, a less formal definition of "de-

[1]) Results and statements are numbered as follows. There are eight Sections in the paper, and $(n\cdot m)$, $1\leqslant m\leqslant 17$, denotes the $m$th item of Section $n$; if $2\leqslant m\leqslant 17$, then for example $(n\cdot m2)$ denotes the second sub-item of item $(n\cdot m)$.

cidable $D$" which suffices for the purposes of the present paper, and helps us to give a similarly provisional meaning to "given a Borel set".

We therefore skirt the basic logical problem, as in fact it is the topological side of Dienes's question which interests us most here. Prof. M. H. A. Newman in 1948 generalised Dienes's result to a $G_\delta$ in a complete separable metric space $E$, and kindly gave his unpublished proof to the author, with some advice on its extension to sets of higher order; this was used to obtain a proof for such sets in the author's M. Sc. Thesis [2]. Recently, however, a new and shorter proof has occurred to the author and is given here, largely for its possible topological interest. Since we want to know how to "give" such a space $E$, we solve the following problem: "Given an abstract, countable, partially-ordered set, what axioms must the ordering satisfy in order that the set be order-isomorphic to a basis of open sets of some complete separable metric space?

It is desirable that it be an operation $D$ to decide whether or not the ordering satisfies any particular axiom". This solution enables us to give what we call a "countable specification" of the space $E$, and next we consider the analogous notion of a countable specification for a $G_\delta$-subset of $E$, and then for a general Borel set in $E$. Finally we deduce a process for deciding, by means of an operation $D$, whether or not a Borel set, "given" by means of such a specification, is empty; and this enables us to answer the above question of Dienes affirmatively. We conclude by indicating some unsolved problems.

## 1. Notation and definition of "decidable $D$"

Lower case Greek letters will usually denote ordinal numbers; and then they will be *always* less than $\Omega_1$, the first uncountable ordinal. The set of all integers $\geqslant 0$ will be denoted by $\mathfrak{I}$, and in Section 5, $I^0$ will mean the set consisting only of the integer zero. For each $\alpha$, let $\Sigma_\alpha$ denote the (well-ordered) series of all ordinals $<\alpha$, and let $I^\alpha$ denote the set of all maps $z\colon \Sigma_\alpha\to\mathfrak{I}$ for which $z(\xi)=0$ for all but a finite number of $\xi<\alpha$. Thus if $\alpha$ is finite, $I^\alpha$ is the ordinary Cartesian product of $\alpha$ copies of $\mathfrak{I}$; and in particular $I^1=\mathfrak{I}$. Let $R_{\alpha\xi}\colon I^\alpha\to\mathfrak{I}$ $(\xi<\alpha)$ be defined by $R_{\alpha\xi}(z)=z(\xi)$, and for each $\alpha$ let $C_\alpha\colon I^\alpha\to\mathfrak{I}$ be some fixed function which enumerates $I^\alpha$; so that $C_\alpha(I^\alpha)=\mathfrak{I}$ and $C_\alpha$ is (1-1). For each $q\in\mathfrak{I}$, let $F_q\colon \mathfrak{S}_q\to\mathfrak{I}$ enumerate the class $\mathfrak{S}_q$ of maps $\varrho\colon \Sigma_q\to\mathfrak{I}$.

We shall often find it convenient to write expressions of the form

$$u^{\alpha+1}=\{z^\beta,0,i\}\qquad(\beta\leqslant\alpha) \tag{1.1}$$



(sometimes suppressing the indices) to mean that $u^{\alpha+1}\in I^{\alpha+1}$, $z^\beta\in I^\beta$ and

$$u(\xi)=\begin{cases} z(\xi) & \text{if}\quad 0\leqslant\xi<\beta,\\ 0 & \text{if}\quad \beta\leqslant\xi<\alpha,\\ i & \text{if}\quad \xi=\alpha.\end{cases}$$

(1.2) Correspondingly, $\{X^\beta,0,i\}$ will denote the set of all $u^{\alpha+1}$ of the form (1.1), as $z^\beta$ runs through $X^\beta$. Also we use (1.1) to define the map $L_{\alpha\beta i}\colon I^\beta\to I^{\alpha+1}$ by $L_{\alpha\beta i}(z^\beta)=u^{\alpha+1}$.

The *class of primitive functions* is defined to be the set of all algebraic operations in the ring of integers, all the functions $R_{\alpha\xi}$, $C_\alpha$, $C_\alpha^{-1}$, $L_{\alpha\beta i}$, $F_q$, the operator $\bigcup$ applied to a sequence of sets, and the operators Max, Min, applied to a well-ordered countable series of integers.

Let $\mathfrak{P}$ be the set consisting of 0 and 1. Given maps $E_i\colon I^{\alpha(i)}\to\mathfrak{P}$ $(1\leqslant i\leqslant n)$, we shall say that a *function* $f^\alpha\colon\mathfrak{I}\to\mathfrak{I}$ *is countably recursively defined* rel $E_1,E_2,\ldots,E_n$ if and only if $f^\alpha$ has been defined by transfinite induction in terms of the primitive functions, $E_1,E_2,\ldots,E_n$, and the preceding functions $f^\beta$, $\beta<\alpha$; where $\alpha$ is countable.

Let $\mathcal{M}$ be the set of all maps of the form $f\colon\Delta\to\mathfrak{P}$, where $\Delta\subseteq I^\alpha$ for some countable $\alpha$. Let $\mathcal{C}$ be a class of objects. We shall say that the objects $X$ of $\mathcal{C}$ can be *countably specified* if and only if there exist maps

$$h\colon\mathcal{C}\to\mathcal{M},\qquad k\colon h(\mathcal{C})\to\mathcal{C} \tag{1.3}$$

such that $kh$ is the identity on $\mathcal{C}$. Let $R(X_1,X_2,\ldots,X_n)$ be an $n$-ary relation $(n\in I)$ on $\mathcal{C}$, which becomes a proposition when values are given to the $X_i$. Then we shall say that $R(X_1,\ldots,X_n)$, and its negation, are *decidable D*, if and only if the objects of $\mathcal{C}$ can be countably specified as in (1.3), and there is a map $g\colon\mathfrak{I}\to\mathfrak{P}$ which is countably recursive rel $h(X_1),h(X_2),\ldots,h(X_n)$, and such that $R(X_1,\ldots,X_n)$ is equivalent to "$g$ is identically zero on $\mathfrak{I}$".

In considering Dienes's problem, we shall take $\mathcal{C}$ to be the class of all Borel sets over the space $E$, and $R(X,Y)$ will be one of the relations (i) of the Introduction. We shall shew in the present paper that each such $R$ is decidable $D$; the final discussion is postponed until Section 8, in order not to interrupt the topological narrative of Sections 2-7. It will turn out that to construct the map $h\colon\mathcal{C}\to\mathcal{M}$ we have to use the axiom of choice; but, to return to the magicians of the Introduction, then $h(X)$ is essentially the series $A$ by which $M_1$ "gives" $X$ to $M_2$ (for $A$ is the list of arguments and values of $h$). Thus, $M_1$ has to choose $h$. However, it is *after* his choice that $M_2$ receives $A$, and the problem at issue is the decision process then used by $M_2$. His fundamental tool will be the construction given in Theorem 2.2 of the next section.

## 2. $\mathfrak{P}$-functions

(2.1) A $\mathfrak{P}$-*function* is any mapping $A: I^4 \to \mathfrak{P}$. For each $(n,i,j) \in I^3$, we write for brevity

$$A_n(i,j) = A(n,i,n+1,j).$$

A triple $(n,i,j)$ for which $A_n(i,j)=1$ is called a "*segment*" of $A$. By a "*thread*" in $A$ we mean a mapping $\mathfrak{t}: \mathfrak{I} \to \mathfrak{I}$, such that for each $n \in \mathfrak{I}$,

$$A_n(t_n, t_{n+1}) = 1,$$

where (as often) we put

$$t_n = \mathfrak{t}(n).$$

We shall then write

$$\mathfrak{t} \subseteq A.$$

$A$ is *reduced* if and only if every segment of $A$ of the form $(0,i,j)$ is part of a thread, *i. e.* if and only if there is a thread $\mathfrak{t}$ in $A$ such that $t_0 = i$ and $t_1 = j$. Given [2]) $A$, $B$ we write

$$A \supseteq B \quad \text{or} \quad B \subseteq A$$

whenever each segment of $B$ is a segment of $A$. Our language is motivated by the fact that we can make a geometrical model of $A$ as follows. If we take the set of all points in the plane, with coordinates $(p,-q)$, $(p,q \in \mathfrak{I})$, then we join $(n,-i)$ to $(m,-j)$ by the straight segment between them, if and only if they are distinct and $A(n,i,m,j)=1$. If the equation holds with $n=m$, $i=j$, we attach a loop at $(n,-i)$ having no other contact with the plane. If $\mathfrak{A}$ is the class of all such segments and loops, then $A \supseteq B$ if and only if $A \supseteq B$ (as classes). Moreover, a thread in $A$, corresponds to an infinite continuous path of segments "down" the diagram $\mathfrak{A}$. This geometrical picture will be the source for motivation of several of our proofs, and corresponds vaguely with Dienes's „pyramids" ([1], p. 230).

Next, for each $q \in \mathfrak{I}$, define a $\mathfrak{P}$-function ${}^qA$ by: ${}^qA(l,m,n,p)=1$ if and only if $A(l+q,m,n+q,p)=1$. Clearly ${}^0A = A$ and $(0,i,j)$ is a segment in ${}^qA$ if and only if $(q,i,j)$ is a segment in $A$. We shall say that $A$ is *fully reduced* if and only if for each $q \in \mathfrak{I}$, ${}^qA$ is reduced.

The main result of this Section is now

(2.2) THEOREM. *Given $A$ there exists $A^*$ such that*

(i) $A^* \subseteq A$,

(ii) *$A^*$ is fully reduced,*

(iii) *every thread in $A$ is also a thread in $A^*$.*

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[2]) Italic capital letters will usually denote $\mathfrak{P}$-functions.



Proof. We define a series $A^\alpha$ of $\mathfrak{P}$-functions, using transfinite induction, as follows:

(a) $A^0 = A$,

(b) If $\alpha$ is a limit ordinal, then for each $(i,j)$,

$$A_n^\alpha(i,j) = \operatorname*{Min}_{\xi<\alpha} A_n^\xi(i,j). \tag{2.3}$$

(c) Let $\alpha = \eta+1$. We call the segment $(n,i,j)$ *ignorable* in $A^\eta$ if and only if, for all $p \in \mathfrak{I}$,

$$A_{n+1}^\eta(j,p) = 0.$$

If $(n,i,j)$ is any ignorable segment in $A^\eta$, define $A_n^\alpha(i,j)$ to be zero, and for all other quadruples $(p,q,r,s) \in I^4$, define $A^\alpha(p,q,r,s)$ to be $A^\eta(p,q,r,s)$.

This completes the definition, by transfinite induction. Since the number of triples $(p,q,r)$ (and therefore of ignorable segments) is at most countable, then there exists a least $\beta < \Omega_1$ such that

$$A^\beta = A^\gamma \quad \text{whenever} \quad \beta \leqslant \gamma. \tag{2.4}$$

*We assert that $A^\beta$ is the required $A^*$.*

Clearly, $A^\beta \subseteq A$. To prove that for each $q$, ${}^q(A^\beta)$ is reduced, consider any segment $\tau$ of the form $(0,i,j)$ in $A^\beta$. Then,

$${}^q(A^\beta)_0(i,j) = 1 = A_q^\beta(i,j).$$

Since $A^\beta = A^{\beta+1}$, the segment $(q,i,j)$ cannot be ignorable in $A^\beta$; and so there exists $p \in \mathfrak{I}$ for which

$$A_{q+1}^\beta(j,p) = 1 = {}^q(A^\beta)_1(i,j).$$

Hence, by finite induction, there exists for each $n \in \mathfrak{I}$, a $p_n \in \mathfrak{I}$, such that $p_0 = i$, $p_1 = j$, $p_2 = p$, and

$${}^q(A^\beta)_n(p_n, p_{n+1}) = 1.$$

Thus the thread $\mathfrak{p}: \mathfrak{I} \to \mathfrak{I}$, defined by $\mathfrak{p}(n) = p_n$, is a thread in ${}^q(A^\beta)$ with initial segment $\tau$. This proves that ${}^q(A^\beta)$ is reduced, and hence that $A^\beta$ is fully reduced.

Next, let us prove that every thread $\mathfrak{t}$ in $A$ is also a thread in $A^\beta$. By definition, we have for all $n$,

$$A_n(t_n, t_{n+1}) = 1;$$

hence no segment $(n, t_n, t_{n+1})$ is ignorable in $A^0$. If $\mathfrak{t}$ is not a thread in $A^\beta$, there is a least $\gamma \leqslant \beta$ such that $\mathfrak{t}$ is not a thread in $A^\gamma$, *i. e.* there is a least

$h \in \mathfrak{J}$, such that $A_h^\gamma(t_h, t_{h+1}) = 0$. Hence, $\gamma$ cannot be a limit ordinal, by (2.3 b); and so $\gamma = \delta + 1$, say. Hence $\mathfrak{t}$ is a thread in $A^\delta$. Therefore for all $m \in \mathfrak{J}$, $A_m^\delta(t_m, t_{m+1}) = 1$, so that in particular $A_\delta^{h+1}(t_{h+1}, t_{h+1}) = 1$, and therefore $(h, t_h, t_{h+1})$ is *not* ignorable in $A^\delta$, whence

$$A_h^\gamma(t_h, t_{h+1}) = 1 .$$

This contradiction proves that $\mathfrak{t} \subseteq A^\beta$.

Thus, $A^\beta$ satisfies (i), (ii) and (iii) of (2.2), and so $A^\beta$ is the required $A^*$, as asserted.

(2.5) COROLLARY (of proof). *$A$ is fully reduced, if and only if $A = A^*$.*

(2.6) *$A$ contains no threads, if and only if for each $(i,j) \in I^2$, $A_0^*(i,j) = 0$.*

### 3. Bichains and threads in $A$

Our magician $M_1$ may wish to "give" to $M_2$ a complete separable metric space which happens to be a closed subset $F$ of a Euclidean space. He cannot name each point of $F$ individually because $F$ will not in general be countable. But since $F$ is closed, then

$$F = \bigcap_{n=1}^{\infty} U(F, 1/n) ,$$

and $G_n = U(F, 1/n)$ is an open set which can be covered by a sequence of neighbourhoods $V_{mn}$ of the form $U(x, a/b)$, where $x$ has rational coordinates and $a, b \in \mathfrak{J}$. Now $M_2$ can, in this case, *deduce* (from the formulae) relations of the form $V_{nm} \subseteq V_{pq}$, without any knowledge of the individual points concerned. Define the function $A(n,i,m,j)$ to be 1 or 0 according as $\overline{V}_{mj} \subseteq V_{ni}$ or not. Then if $M_1$ gives to $M_2$ the function [3]) $A\colon I^4 \to \mathfrak{P}$, we shall see below (Theorem 4.16) that $M_2$ can deduce all topological properties of $F$ as a space. The problem therefore arises: what properties must a map $B\colon I^4 \to \mathfrak{P}$ possess in order that it specify a complete metric separable space in the way that $A$ does $F$? Of course, $B$ can be regarded as an order relation on the set of pairs $(n,i)$: $(n,i) > (m,j)$ if and only if $B(n,i,m,j) = 1$. Hence we have the problem, mentioned in the Introduction, concerning ordered sets. In this Section and the next, we solve this problem, as follows.

For convenience, we put

(3.1) $$A(n,i,n,j) = A^n(i,j) .$$

[3]) Or rather, a slightly modified form of this.



Given $n \in \mathfrak{J}$ we shall mean by an *$n$-bichain* $\Gamma^n$ in $A$ a diagram of the form

$$\begin{array}{ccccccc} i_0 & \leftarrow & i_1 & \leftarrow & \ldots & \leftarrow & i_p \\ \uparrow & & \uparrow & & & & \uparrow \\ j_0 & \leftarrow & j_1 & \leftarrow & \ldots & \leftarrow & j_p \end{array}$$

where the $i$'s and $j$'s are in $\mathfrak{J}$, and for each appropriate $r$,

$$1 = A^{n+r}(i_r, j_r) = A_{n+r}(i_r, i_{r+1}) = A_{n+r}(j_r, j_{r+1}) .$$

Thus the arrows correspond to segments in the diagram of $A$. We say that $\Gamma^n$ is *twisted* if and only if

$$A(n, j_0, n+p, i_p) = 1 ,$$

and otherwise $\Gamma^n$ is *untwisted*. We call $p$ the *length* of $\Gamma^n$, and if $q \leqslant p$ we call the diagram

$$\begin{array}{ccccccc} i_0 & \leftarrow & i_1 & \leftarrow & \ldots & \leftarrow & i_q \\ \uparrow & & \uparrow & & & & \uparrow \\ j_0 & \leftarrow & j_1 & \leftarrow & \ldots & \leftarrow & j_q \end{array}$$

the *$q$-segment* of $\Gamma^n$; clearly it is also an $n$-bichain, and of length $q$. Given two $n$-bichains $\Gamma_1^n, \Gamma_2^n$, we write

$$\Gamma_1^n > \Gamma_2^n \quad \text{or} \quad \Gamma_2^n < \Gamma_1^n$$

if and only if $\Gamma_2^n$ is the $l$-segment of $\Gamma_1^n$, where $l$ = length $\Gamma_2^n$. A sequence $\{\Gamma_m^n\}$ of $n$-bichains is *ascending* if and only if for each $m \in \mathfrak{J}$

$$\Gamma_{m+1}^n > \Gamma_m^n .$$

We now assume that $A$ satisfies the following axiom:

$\mathfrak{C}_1$. *For each $n \in \mathfrak{J}$, every ascending sequence of $n$-bichains in $A$ contains a member which is twisted.*

We shall shew in Section 8 that it is an operation $D$ to decide whether or not a given $A$ satisfies $\mathfrak{C}_1$. For use in the proof, we establish the assertion (3.2) below. Thus if we fix $n$, then all the bichains in $A$ of length $n$ can be effectively enumerated [4]) in the form $\gamma_n(0), \gamma_n(1), \ldots, \gamma_n(p), \ldots$ Define the $\mathfrak{P}$-function $B\colon I^4 \to \mathfrak{P}$ by:

$B(r, u, r+1, v) = 1$ if and only if there exists $q$ such that $\gamma_r(u)$, $\gamma_{r+1}(v)$ are $q$-bichains and $\gamma_r(u) < \gamma_{r+1}(v)$. For all other quadruples $(r,s,t,u)$, put $B(r,s,t,u) = 0$.

We assert:

(3.2) *$A$ satisfies $\mathfrak{C}_1$ if and only if $B^*$* (see (2.2)) *is identically zero.*

[4]) *I. e.* the function $\gamma_n$ is definable in terms of $A$ and the class of primitive functions (of Section 1).

For if there is a segment $(n,i,j)$ in $B^*$, then since $B^*$ is fully reduced, there is a thread $\mathfrak{t}$ in ${}^n(B^*)$, such that for each $m$ ${}^n(B^*)_m\ (t_m,t_{m+1})=1$ (and $t_0=i, t_1=j$). Therefore by definition of $B$, there exists $q\in\mathfrak{J}$ such that $\gamma_m(t_m)$ is a $q$-bichain, $\gamma_m(t_m)<\gamma_{m+1}(t_{m+1})$, and no $\gamma_m(t_m)$ is twisted. Thus $\mathfrak{C}_1$ fails to hold. The argument reverses in the obvious way, and our assertion is established.

Let $\mathfrak{t}$ be a thread in $A$. By a *sub-thread* $\mathfrak{t}'$ of $\mathfrak{t}$ we mean a mapping $t'\colon \mathfrak{J}\to\mathfrak{J}$ such that for each $n\in\mathfrak{J}$,

(3.3) $$\mathfrak{t}'(n)=\mathfrak{t}\big(p(n)\big),$$

where $p\colon\mathfrak{J}\to\mathfrak{J}$ is monotonic increasing and $p(n)>n$. In order that $\mathfrak{t}'$ be itself a thread in $A$, we now impose the following extra conditions on $A$.

$\mathfrak{C}_2$. *If* $A(i,j,n,k)=1$, *then* $A(i-1,j,n-1,k)=1$ $(i,n>1)$.

$\mathfrak{C}_3$. $A$ *is transitive, i. e. if*

$$A(i,j,i+p,q)=1=A(i+p,q,i+p+n,m)\qquad (p,n\geqslant 0)$$

*then*

$$A(i,j,i+p+n,m)=1.$$

$\mathfrak{C}_4$. *If* $A(i,j,i+p,k)=1$, *then* $A(i,j,i+r,k)=1$, *where* $p>0$ *and* $r=0$ *and* 1.

(3.4) LEMMA. *If* $\mathfrak{t}'$ *is a sub-thread of a thread* $\mathfrak{t}$ *in* $A$, *then* $\mathfrak{t}'\subseteq A$.

Proof. If for each $n$, $t'_n=t_{p(n)}$ as in (3.2), we have to prove that

$$A_n(t_{p(n)},t_{p(n+1)})=1.$$

Now, since $\mathfrak{t}$ is a thread, and $p\colon\mathfrak{J}\to\mathfrak{J}$ is monotonic and strictly increasing, then

$$\begin{aligned}1&=A_{p(n)}(t_{p(n)},t_{p(n)+1})=A_{p(n)+1}(t_{p(n)+1},t_{p(n)+2})=\ldots\\&=A_{p(n+1)-1}(t_{p(n+1)-1},t_{p(n+1)})\\&=A\big(p(n),t_{p(n)},p(n+1),t_{p(n+1)}\big)\quad\text{by transitivity }(\mathfrak{C}_3)\\&=A\Big(n,t_{p(n)},p(n+1)-\big(p(n)-n\big),t_{p(n+1)}\Big)\quad\text{by repeating }\mathfrak{C}_2\\&=A_n(t_{p(n)},t_{p(n+1)}),\quad\text{by }\mathfrak{C}_4,\end{aligned}$$

as required.

If $\mathfrak{s}$, $\mathfrak{t}$ are threads in $A$, we write

(3.5) $$\mathfrak{s}<\mathfrak{t}\quad\text{or}\quad\mathfrak{t}>\mathfrak{s}$$

whenever, for each $n\in\mathfrak{J}$, then (see (3.1))

$$A^n(t_n,s_n)=1.$$

(3.51) Note that, by $\mathfrak{C}_3$, if $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{u}$ are threads in $A$, then

$$\mathfrak{s}<\mathfrak{t}\ \&\ \mathfrak{t}<\mathfrak{u}\quad\text{imply}\quad\mathfrak{s}<\mathfrak{u}.$$



(3.6) LEMMA. *If* $\mathfrak{t}'$ *is a sub-thread of* $\mathfrak{t}$, *then* $\mathfrak{t}'<\mathfrak{t}$.

Proof. Suppose $\mathfrak{t}$, $\mathfrak{t}'$ satisfy (3.3), so that $p(n)>n$. Then

$$1=A_n(t_n,t_{n+1})=\ldots=A_{p(n)-1}(t_{p(n)-1},t_{p(n)}),$$

so that by repetition of $\mathfrak{C}_2$

$$1=A_n(t_n,t_{n+1})=A_n(t_{n+2},t_{n+3})=\ldots=A_n(t_{p(n)-1},t_{p(n)}).$$

Therefore by $\mathfrak{C}_3$

$$1=A_n(t_n,t_{p(n)}),\qquad i.\ e.\qquad 1=A_n(t_n,t'_n),$$

whence, by $\mathfrak{C}_4$, $1=A^n(t_n,t'_n)$, so that $\mathfrak{t}>\mathfrak{t}'$, as required.

(3.7) LEMMA. *If* $\mathfrak{t}>\mathfrak{s}$ *there is a sub-thread* $\mathfrak{t}'$ *of* $\mathfrak{t}$ *such that* $\mathfrak{t}'<\mathfrak{s}$.

Proof. For each $m,n\in\mathfrak{J}$, we have the $n$-bichain

$$\Gamma_{nm}=\left\{\begin{array}{ccccc}t_n&\leftarrow t_{n+1}&\leftarrow&\ldots&\leftarrow t_{n+m}\\ \uparrow&\uparrow&&&\uparrow\\ s_n&\leftarrow s_{n+1}&\leftarrow&\ldots&\leftarrow s_{n+m}\end{array}\right.$$

since $\mathfrak{t}>\mathfrak{s}$. Hence, if $n$ is fixed, the sequence

$$\Gamma_{n1},\Gamma_{n2},\ldots,\Gamma_{nm},\ldots$$

is ascending, and so contains a twisted member, by $\mathfrak{C}_1$. Therefore for each $n$ there is a least $r(n)=r\in\mathfrak{J}$ such that

(3.71) $$A(n,s_n,n+r,t_{n+r})=1.$$

Hence, by $\mathfrak{C}_4$

(3.72) $$A^n(s_n,t_{n+r})=1.$$

Now $A_{n+r}(t_{n+r},t_{n+r+1})=1$, so that $A_n(t_{n+r},t_{n+r+1})=1$ by $\mathfrak{C}_2$. Therefore $A^n(t_{n+r},t_{n+r+1})=1$, by $\mathfrak{C}_4$, so that by $\mathfrak{C}_3$ and (3.72), $A^n(s_n,t_{n+r+1})=1$.

Thus we may suppose that the sequence $\{r(n)\}$ in (3.71) is monotonic increasing. Hence so is the sequence $\{p(n)\}=\{n+r(n)\}$; and $p(n)>n$ since $r(n)>0$. Therefore, if we put $t'_n=t_{p(n)}$, then $\mathfrak{t}'\colon\mathfrak{J}\to\mathfrak{J}$ is a sub-thread of $\mathfrak{t}$, and by (3.72) $\mathfrak{s}>\mathfrak{t}'$. This completes the proof.

The last result enables us to classify the threads in $A$ into disjoint equivalence classes. Thus, we write

(3.8) $$\mathfrak{s}\equiv\mathfrak{t}$$

if and only if there is a thread $\mathfrak{u}$ in $A$, such that $\mathfrak{u}<\mathfrak{s}$ and $\mathfrak{u}<\mathfrak{t}$. In (3.7), $\mathfrak{t}'<\mathfrak{t}$, so that

(3.81) If $\mathfrak{s}<\mathfrak{t}$, then $\mathfrak{s}\equiv\mathfrak{t}$.

(3.9) Theorem. *The relation* (3.8) *is a genuine equivalence relation.*

Proof. We have to show

(i) $\mathfrak{t}\equiv\mathfrak{t}$ (for by $\mathfrak{C}_4$, $\mathfrak{t}<\mathfrak{t}$);

(ii) $\mathfrak{s}\equiv\mathfrak{t}$ implies $\mathfrak{t}\equiv\mathfrak{s}$ (immediate from the definition);

(iii) if $\mathfrak{r}\equiv\mathfrak{s}$ and $\mathfrak{s}\equiv\mathfrak{t}$ then $\mathfrak{r}\equiv\mathfrak{t}$.

To prove (iii), there exist by definition threads $\mathfrak{a},\mathfrak{b}$, such that $\mathfrak{a}<\mathfrak{r}$ and $\mathfrak{a}<\mathfrak{s}$, while $\mathfrak{b}<\mathfrak{s}$ and $\mathfrak{b}<\mathfrak{t}$. By (3.6) there exists a sub-thread $\mathfrak{r}'$ of $\mathfrak{r}$, where

$$\mathfrak{r}'(n)=\mathfrak{r}(p(n)), \qquad p(n+1)\geqslant p(n)>n,$$

and $\mathfrak{r}'<\mathfrak{a}$. Since $\mathfrak{a}<\mathfrak{s}$, then for all $n\in\mathfrak{J}$, $A^n(s_n,a_n)=1$; while $A^n(a_n,r'_n)=1$ because $\mathfrak{r}'<\mathfrak{a}$. Therefore by $\mathfrak{C}_3$

(iv) $$A^n(s_n,r'_n)=1.$$

Similarly there is a sub-thread $\mathfrak{s}'$ of $\mathfrak{s}$, where

(v) $$\mathfrak{s}'(n)=\mathfrak{s}(q(n)), \qquad q(n+1)\geqslant q(n)>n$$

such that

(vi) $$A^n(t_n,s'_n)=1$$

for all $n\in\mathfrak{J}$. In particular, by (iv)

$$A^{q(n)}(s_{q(n)},r'_{q(n)})=1$$

so that by $\mathfrak{C}_2$

(vii) $$A^n(s_{q(n)},r'_{q(n)})=1.$$

But by (v) and (vi)

$$1=A^n(t_n,s'_n)=A^n(t_n,s_{q(n)})$$

so that by $\mathfrak{C}_3$ and (vii)

(viii) $$A^n(t_n,r'_{q(n)})=1.$$

Define $\mathfrak{r}''\colon\mathfrak{J}\to\mathfrak{J}$ by $\mathfrak{r}''(n)=\mathfrak{r}'(q(n))$, so that since by (v) $q\colon\mathfrak{J}\to\mathfrak{J}$ is monotonic increasing, then by (3.4), $\mathfrak{r}''$ is a thread in $A$; and by (3.7) $\mathfrak{r}''<\mathfrak{r}$. Moreover, by (viii) $\mathfrak{r}''<\mathfrak{t}$. Therefore, by definition, $\mathfrak{r}\equiv\mathfrak{t}$ as required.

This completes the proof that we have defined a genuine equivalence relation on the set of all the threads in $A$. Hence the equivalence classes $[\mathfrak{t}]$ of these threads $\mathfrak{t}$ are disjoint. In the next section we shall turn this system of classes into a topological space, with the classes $[\mathfrak{t}]$ as its points. First, however, we shall give the following result, which



is needed in Section 5. For convenience, if $\mathfrak{t}$ is a thread in $A$, then given $n\in\mathfrak{J}$ we define ${}^n\mathfrak{t}\colon\mathfrak{J}\to\mathfrak{J}$ by

(3.10) $${}^n\mathfrak{t}(j)=\mathfrak{t}(n+j).$$

Clearly, ${}^n\mathfrak{t}$ is a sub-thread of $\mathfrak{t}$.

(3.11) Lemma. *For each $n\in\mathfrak{J}$, let $\mathfrak{t}^n$ be a thread in $A$ for which $[\mathfrak{t}^n]=[\mathfrak{t}^{n+1}]=x$. Then there exists $\mathfrak{s}\in x$ such that for each $n\in\mathfrak{J}$*

$${}^n\mathfrak{s}<\mathfrak{t}^n.$$

Proof. Since $\mathfrak{t}^0,\mathfrak{t}^1\in x$, there exists by definition a thread $\mathfrak{r}$ in $A$ such that $\mathfrak{r}<\mathfrak{t}^0$ and $\mathfrak{r}<\mathfrak{t}^1$. Define $\mathfrak{r}^0,\mathfrak{r}^1$ to be $\mathfrak{r},{}^1\mathfrak{r}$ respectively, so that $\mathfrak{r}^0<\mathfrak{t}^0$; while $\mathfrak{r}^1={}^1\mathfrak{r}<\mathfrak{r}<\mathfrak{t}^1$ so that by (3.51), $\mathfrak{r}^1<\mathfrak{t}^1$. Now suppose that we have defined threads

$$\mathfrak{r}^0>\mathfrak{r}^1>\ldots>\mathfrak{r}^p \qquad (\text{in } A)$$

such that $\mathfrak{r}^j<\mathfrak{t}^j$ $(0\leqslant j\leqslant p)$. Then $\mathfrak{r}^p\equiv\mathfrak{t}^p$, by (3.81), while $\mathfrak{t}^p\equiv\mathfrak{t}^{p+1}$ (given); hence $\mathfrak{r}^p\equiv\mathfrak{t}^{p+1}$. Therefore, by definition, there is a thread $\mathfrak{u}$ in $A$, such that $\mathfrak{u}<\mathfrak{r}^p$ and $\mathfrak{u}<\mathfrak{t}^{p+1}$. Define $\mathfrak{r}^{p+1}$ to be $\mathfrak{u}$. By induction on $p$, this defines a sequence of threads $\mathfrak{r}^j$ in $A$, such that for each $j$,

$$\mathfrak{r}^{j+1}<\mathfrak{r}^j<\mathfrak{t}^j.$$

Now define $\mathfrak{s}\colon\mathfrak{J}\to\mathfrak{J}$ by

(i) $$\mathfrak{s}(n)=\mathfrak{r}^n(n);$$

we assert that $\mathfrak{s}$ is the thread we require.

For

$$1=A_n(t_n^n,t_{n+1}^n) \quad (\text{because } \mathfrak{t}^n \text{ is a thread})$$
$$=A^{n+1}(t_{n+1}^n,t_{n+1}^{n+1}) \quad (\text{because } \mathfrak{t}^n>\mathfrak{t}^{n+1})$$

so that

$$1=A_n(t_n^n,t_{n+1}^{n+1}) \quad (\text{by } \mathfrak{C}_3)$$
$$=A_n(s_n,s_{n+1}) \quad (\text{by (i) above}),$$

and so $\mathfrak{s}$ is a thread in $A$. Moreover ${}^n\mathfrak{s}<\mathfrak{t}^n$ if and only if, for each $m\in\mathfrak{J}$, $A^n(t_m^n,{}^ns_m)=1$; but

$$1=A_m(t_m^n,r_m^n) \quad (\text{since } \mathfrak{t}^n>\mathfrak{r}^n)$$
$$=A(m,r_m^n,n+m,r_{n+m}^n) \quad (\text{by } \mathfrak{C}_3, \text{ since } \mathfrak{r}^n \text{ is a thread})$$
$$=A(n+m,r_{n+m}^n,n+m,r_{n+m}^{n+m}) \quad (\text{since } \mathfrak{r}^n>\mathfrak{r}^{n+m}, \text{ by (3.51)}).$$

Hence, by $\mathfrak{C}_3$ again,

$$1=A(m,t_m^n,n+m,r_{n+m}^{n+m})$$
$$=A(m,t_m^n,m,r_{n+m}^{n+m}) \quad (\text{by } \mathfrak{C}_4)$$
$$=A^m(t_m^n,{}^ns_m) \quad (\text{by (i) above and (3.10)}),$$

*i. e.* $\mathfrak{t}^n>{}^n\mathfrak{s}$, as required.

## 4. Complete separable metric spaces

For each thread $\mathfrak{t}$ in $A$, let $[\mathfrak{t}]$ denote the equivalence class of $\mathfrak{t}$ as defined above. Given $i,j \in \mathfrak{J}$, define $U_{ij}$ to be the set of all classes $[\mathfrak{t}]$ (if any) for which there is an $\mathfrak{s} \in [\mathfrak{t}]$ with $\mathfrak{s}(i)=j$. Otherwise, put $U_{ij}=0$ (the empty set). We now form a topological space $\boldsymbol{A}=\boldsymbol{A}(A)$, whose points are defined to be the classes $[\mathfrak{t}]$ and for which the sets $U_{ij}, i,j \in \mathfrak{J}$, are by definition to form a basis for the open sets. Note that, if $x=[\mathfrak{t}]$, then [5] $x \in U_{i,t(i)}$; so that the system

$$\mathcal{B}=\{U_{i1}, U_{i2}, \ldots, U_{im}, \ldots\} \tag{4.1}$$

is a covering of $\boldsymbol{A}$. We need the following lemmas.

(4.2) Lemma. *If* $A(n,p,m,q)=1$, *and* $n \leqslant m$ *then* $U_{np} \supseteq U_{mq}$ *provided that* $U_{np} \neq 0$.

Proof. By hypothesis, $U_{np} \neq 0$, so that there is a thread $\mathfrak{g}$ such that $\mathfrak{g} \in U_{np}$ and

$$\mathfrak{g}(n)=p\,.$$

Let $x \in U_{mq}$; then there exists a thread $\mathfrak{h} \in x$, such that

$$\mathfrak{h}(m)=q\,.$$

Define $\mathfrak{r}\colon \mathfrak{J}\to\mathfrak{J}$ by

$$\mathfrak{r}(j)=\begin{cases}\mathfrak{g}(j), & j\leqslant n\,,\\ \mathfrak{h}(m-n+j), & j>n\,.\end{cases}$$

Then $A_j(r_j, r_{j+1})=A^j(g_j, g_{j+1})=1$, if $j<n$ (since $\mathfrak{g}$ is a thread) while if $j>n$,

$$\begin{aligned}A_j(r_j, r_{j+1}) &= A_j(h_{m-n+j}, h_{m-n+j+1})\\ &= A_{m-n+j}(h_{m-n+j}, h_{m-n+j+1})=1\end{aligned}$$

by $\mathfrak{C}_2$, since $\mathfrak{h}$ is a thread.

Finally, $A_n(r_n, r_{n+1})=A_n(p, h_{m+1})$; now $A(n,p,m,q)=1$ (given), and $A(m,q,m+1,h_{m+1})=1$, since $\mathfrak{h}$ is a thread and $h_m=q$. Therefore by $\mathfrak{C}_3$ (since $m\geqslant n$),

$$\begin{aligned}A(n,p,m+1,h_{m+1})=1&=A(n,p,n+1,h_{m+1}) \quad \text{by } \mathfrak{C}_4 \text{ (since } m\geqslant n)\\ &=A_n(p, h_{m+1}) \quad \text{by definition of } A_m\,,\end{aligned}$$

so that

$$A_n(r_n, r_{n+1})=1.$$

Hence we have proved that $\mathfrak{r}$ is a thread over $A$.

Define the sub-thread $\mathfrak{r}'$ of $\mathfrak{r}$ by

$$r'_j=r_{j+n}\,,$$

so that

$$r'_j=h_{m+j}\,,$$

[5]) For typographical reasons, we shall denote $\mathfrak{t}(i)$ by $t(i)$ (on subscripts).



*i. e.* $\mathfrak{r}'$ is a sub-thread of $\mathfrak{h}$. Therefore by (3.5), $\mathfrak{r}'<\mathfrak{r}$ and $\mathfrak{r}'<\mathfrak{h}$, *i. e.* $\mathfrak{r}\equiv\mathfrak{h}$. Hence $x=[\mathfrak{h}]=[\mathfrak{r}]$. But $r_n=g_n=p$, whence $x=[\mathfrak{r}]\in U_{np}$. Thus

$$U_{mq}\subseteq U_{np}\,,$$

as required.

(4.3) Lemma. *If* $x \in U_{np}\cap U_{rs}$ $(r\geqslant n)$, *then there exists* $m \in \mathfrak{J}$, *such that* $x \in U_{rm}\subseteq U_{np}\cap U_{rs}$.

Proof. Since $x\in U_{np}$ and $x \in U_{rs}$, there exist threads $\mathfrak{a},\mathfrak{b}\in x$ with

$$\mathfrak{a}(n)=p;\qquad \mathfrak{b}(r)=s,\qquad \text{and}\qquad \mathfrak{a}\equiv\mathfrak{b}\,.$$

Hence, there exists a thread $\mathfrak{c}\in x$ with $\mathfrak{c}<\mathfrak{a}$, $\mathfrak{c}<\mathfrak{b}$. In particular therefore

$$A^r(a_r, c_r)=1=A^r(b_r, c_r) \tag{4.31}$$

so that if we apply the last lemma to the second equality, we get (since $b_r=s$)

$$U_{r,c(r)}\subseteq U_{rs}\,,\qquad c(r)=c_r\,. \tag{4.32}$$

But since $\mathfrak{a}$ is a thread, then by $\mathfrak{C}_3$ (since $r\geqslant n$)

$$A(n, a_n, r, a_r)=1\,,$$

so that by $\mathfrak{C}_3$ again, and (4.31),

$$A(n, a_n, r, c_r)=1; \tag{4.33}$$

hence by the last lemma (since $a_n=p$)

$$U_{r,c(r)}\subseteq U_{np}\,. \tag{4.34}$$

Combination of (4.32) and (4.34) now gives the required result.

(4.4) Lemma. *The space* $\boldsymbol{A}$, *defined with* (4.1), *is a* $T_1$*-space*.

Proof. Let $x,y$ be points of $\boldsymbol{A}$ such that every neighbourhood of $x$ meets $y$. It suffices to prove that $x=y$. Let $\mathfrak{a}\in x$, $\mathfrak{b}\in y$ be threads over $A$. Then by definition, for each $n\in\mathfrak{J}$,

$$x\in U_{n,a(n)}\,,\qquad y\in U_{n,b(n)}\,,$$

while $y\in U_{n,a(n)}$ by hypothesis since $U_{n,a(n)}$ is a neighbourhood of $x$. Hence $y\in U_{n,a(n)}\cap U_{n,b(n)}$, so that by (4.3) and (4.31), there is a (first) $u(0)\in\mathfrak{J}$ such that

$$y\in U_{0,u(0)}\subseteq U_{0,a(0)}\cap U_{0,b(0)}\quad \text{and}\quad A^0(a_0, u_0)=1=A^0(b_0, u_0)\,.$$

Suppose now that we have defined $u(0), u(1), \ldots, u(m)\in\mathfrak{J}$, such that

$$\begin{cases} A_i(u_i, u_{i+1})=1, & 0\leqslant i<m\,;\\ A^i(a_i, u_i)=1=A^i(b_i, u_i), & 0\leqslant i\leqslant m\,,\\ y\in U_{0,u(0)}\cap\ldots\cap U_{m,u(m)}\,. \end{cases} \tag{4.41}$$

Then

$$y \in (U_{m+1,a(m+1)} \cap U_{m,u(m)}) \cap (U_{m,u(m)} \cap U_{m+1,b(m+1)}) = V_\alpha \cap V_\beta = V \quad \text{(say)}.$$

Hence, by (4.3), (4.31) and (4.34), there exist $p, q \in \mathfrak{J}$ such that

$$(4.42) \qquad \begin{aligned} y \in U_{m+1,p} \subseteq V_\alpha, \quad & A^{m+1}(a_{m+1}, p) = 1 = A_m(u(m), p), \\ y \in U_{m+1,q} \subseteq V_\beta, \quad & A^{m+1}(b_{m+1}, q) = 1 = A_m(u(m), q). \end{aligned}$$

Applying (4.3) and (4.31) again gives the existence of a (least) $g \in \mathfrak{J}$ for which

$$y \in U_{m+1,g} \subseteq U_{m+1,p} \cap U_{m+1,q} \subseteq V_\alpha \cap V_\beta = V,$$

and $A^{m+1}(p, g) = 1 = A^{m+1}(q, g)$.

Using the equations of (4.42) and applying $\mathfrak{C}_3$ gives

$$A^{m+1}(a_{m+1}, g) = 1 = A^{m+1}(b_{m+1}, g) = A_m(u(m), g).$$

Put $g = u(m+1)$. Then, by finite induction on $m$, we have proved the existence of a map $u: \mathfrak{J} \to \mathfrak{J}$ satisfying (4.41) for all $m$. Thus $u = \mathfrak{u}$ is a thread over $A$, and $\mathfrak{u} < \mathfrak{a}$, $\mathfrak{u} < \mathfrak{b}$. Hence $\mathfrak{a} \equiv \mathfrak{b}$, i. e. $x = [\mathfrak{a}] = [\mathfrak{b}] = y$, as required.

(4.43) Corollary. Note that the proof shews that *the intersection of all the $U_{n,\mathfrak{a}(n)}$ (for any thread $\mathfrak{a}$) is at most a single point.*

We shall now ensure that $A$ be regular, by imposing upon $A$ the additional axioms:

$\mathfrak{C}_5$. *Given $p, n \in \mathfrak{J}$, then it is impossible that for all $m \in \mathfrak{J}$, $A_{n+m}(p, p) = 1$.*

$\mathfrak{C}_6$. *If $A(n, p, m, q) = 1$, and $p \neq q$, there exists $s \geqslant n$ such that for all $t, a, b$, if*

$$A(m, q, a, b) = 1 = A(s, t, a, b)$$

*then*

$$A(n, p, s, t) = 1.$$

(4.5) Lemma. *$A$ is regular.*

Proof. Let $x \in A$ and let $V$ be a neighbourhood of $x$. Then there exists $U_{np}$ such that $x \in U_{np} \subseteq V$, and there is a thread $\mathfrak{a} \in x$ such that $\mathfrak{a}(n) = p$.

By $\mathfrak{C}_5$, there exists $m > n$ such that $\mathfrak{a}(m) \neq \mathfrak{a}(n)$ (since $\mathfrak{a}$ is a thread). Let $W = U_{m,\mathfrak{a}(m)}$; we assert that[6] $x \in W \Subset U_{np}$. That $x \in W$ follows by definition of $U_{m,\mathfrak{a}(m)}$. To prove that $W \Subset U_{np}$, we apply $\mathfrak{C}_6$; for $A(n, p, m, \mathfrak{a}(m)) = 1$ (since $\mathfrak{a}$ is a thread and $\mathfrak{C}_3$ holds) so that there

[6]) $X \Subset Y$ means that the closure of $X$ is contained in $Y$.



exists $s \geqslant n$ satisfying the conditions of $\mathfrak{C}_6$. Given $y \in \overline{W} - W$, then every neighbourhood of $y$ meets $W$. In particular then, $y \in U_{sd}$ for some $d$ (by 4.1) and $U_{sd} \cap W \neq 0$. Hence, by (4.3), there exists $b \in \mathfrak{J}$ such that $U_{sb} \subseteq U_{sd} \cap W$ and, by (4.31) and (4.33)

$$A(m, a_m, s, d) = 1 = A(s, d, s, b).$$

Therefore, by $\mathfrak{C}_6$

$$A(n, p, s, d) = 1;$$

and so, since $U_{np} \neq 0$, then $U_{sa} \subseteq U_{np}$ by 4.2. Hence $y \in (\overline{W} - W) \cap U_{np}$, and, since $W = U_{m,a(m)} \subseteq U_{np}$ (by (4.2) again), then $\overline{W} \subseteq U_{np}$, as asserted. Therefore

$$x \in W \Subset U_{np} \subseteq V,$$

i. e. $A$ is regular at $x$, and therefore everywhere.

This completes the proof.

It is desirable that each $U_{nq}$ should in most cases be non-empty. To ensure this we shall suppose that $A$ satisfies the following two axioms.

$\mathfrak{C}_7$. *$A$ is fully reduced (in the sense of (2.1)).*

$\mathfrak{C}_8$. *For all $p, q, n \in \mathfrak{J}$, if $A_n(p, q) = 1$, then*

$$A_m(p, p) = 1, \quad 0 \leqslant m \leqslant n.$$

(4.6) Lemma. *Given $n, p \in \mathfrak{J}$, suppose that there exists $q \in \mathfrak{J}$ such that $A_n(p, q) = 1$; then $U_{np} \neq 0$.*

Proof. Since $1 = A_n(p, q) = {}^nA_0(p, q)$, and ${}^nA$ is by $\mathfrak{C}_7$ reduced, then there is a thread $\mathfrak{a}$ over ${}^nA$ with $a_0 = p$, $a_1 = q$, and such for all $m \in \mathfrak{J}$,

$${}^nA(m, a_m, m+1, a_{m+1}) = 1, \quad \text{i. e.} \quad A(m+n, a_m, m+n+1, a_{m+1}) = 1.$$

In particular, $A(n, a_0, n+1, a_1) = 1$ $(= A_n(a_0, a_1))$ and so by $\mathfrak{C}_8$

$$(4.61) \qquad A_m(a_0, a_0) = 1, \quad 0 \leqslant m \leqslant n.$$

Define a new mapping $b: \mathfrak{J} \to \mathfrak{J}$ by

$$b(j) = a_0 \quad (0 \leqslant j \leqslant n), \qquad b(n+j) = a(j), \quad j \in \mathfrak{J}.$$

In view of the above equations,

$$A_j(b_j, b_{j+1}) = 1 \quad \text{for all} \quad j \in \mathfrak{J},$$

so that $b = \mathfrak{b}$ is a thread over $A$. Moreover, if $j \leqslant n$, then $\mathfrak{b}(j) = \mathfrak{b}(n) = a_0 = p$, and therefore $[\mathfrak{b}] \in U_{jp}$. Hence, in particular, $[\mathfrak{b}] \in U_{np}$, and so $U_{np} \neq 0$, as required.

(4.7) Corollary. *For all $n,p \in \mathfrak{I}$,*

$$U_{np} \supseteq U_{n+1,p}.$$

This holds because either

(i) $U_{m+1,p}=0$, in which case the inclusion is trivial;

or

(ii) $U_{m+1,p}\neq 0$, so that there is a thread $\mathfrak{a}$ with $\mathfrak{a}(m+1)=p$, and $A_{m+1}(p,a_{m+2})=1$. Thus the hypotheses of (4.6) are satisfied, and so (4.61) holds with $n+1$ for $n$. Therefore by (4.2) $U_{m+1,p}\subseteq U_{mp}$ for all $m\leqslant n$. Thus, $U_{np}\supseteq U_{n+1,p}$ for all $n,p\in\mathfrak{I}$, as required.

(4.8) Lemma. *If $U_{np}\supseteq U_{mq}\neq 0$ then either*

(a) *$n\leqslant m$ and $A(n,p,m,q)=1$*

*or*

(b) $U_{np}=U_{mq}$.

Proof. Since there exists $x\in U_{mq}\subseteq U_{np}$, there exist threads $\mathfrak{a},\mathfrak{b}\in x$ such that

(i) $$\mathfrak{a}(m)=q, \quad \mathfrak{b}(n)=p;$$

and since $\mathfrak{a},\mathfrak{b}\in x$, there exists a thread $\mathfrak{c}$ with

(ii) $$\mathfrak{c}<\mathfrak{a}, \quad \mathfrak{c}<\mathfrak{b}.$$

First suppose $n\leqslant m$. Then by (ii), we have

$$A^n(b_n,c_n)=1=A^m(a_m,c_m),$$

and since $c$ is a thread, then by $\mathfrak{C}_3$

$$A(n,c_n,m,c_m)=1.$$

Hence by (i) and $\mathfrak{C}_3$,

$$A(n,p,m,q)=1$$

as required.

If $n>m$, a similar argument gives

$$A(m,q,n,p)=1,$$

whence by 4.2, $U_{mq}\subseteq U_{np}$; with $U_{np}\subseteq U_{mq}$, we get

$$U_{mq}=U_{np}.$$

This completes the proof.

(4.9) Lemma. *Let $v\colon \mathfrak{I}\to\mathfrak{I}$ be such that for each $j\in\mathfrak{I}$*

$$U_{j,v(j)}\ni U_{j+1,v(j+1)}\neq 0.$$

*Then $\bigcap_{j=0}^{\infty} U_{j,v(j)}$ is precisely one point.*



Proof. By 4.8, since $j<j+1$, we have

$$A_j(v_j,v_{j+1})=1,$$

and therefore $\mathfrak{v}=v$ is a thread over $A$; moreover, by definition $[v]\in U_{j,v(j)}$. Hence

(4.91) $$X=\bigcap_{j=0}^{\infty} U_{j,v(j)}\neq 0.$$

By (4.43), $X$ is at most a single point, and so, since $X$ is not empty, the result follows.

Note that to obtain 4.91, we did not use the fact that each $U_{j,v(j)}$, contains the *closure* of $U_{j+1,v(j+1)}$ (and not just $U_{j+1,v(j+1)}$ itself).

By (4.1), each system $\mathcal{B}_n$ is a covering of $A$. It will be convenient if $\mathcal{B}_n$ is actually a *basis* for $A$, and to ensure [7]) this we shall assume that $A$ satisfies the further axiom

$\mathfrak{C}_9$. *Given $n,p,q\in\mathfrak{I}$ such that*

$$A_n(p,q)=A_n(p,p)=1$$

*then*

$$A^{n+1}(p,q)=1.$$

If $A$ satisfies $\mathfrak{C}_9$, then we have

(4.10) Lemma. *If $U_{p+1,j}\neq 0$, then $U_{p,j}=U_{p+1,j}$.*

Proof. By (4.7) $U_{pj}\supseteq U_{p+1,j}$, so that we need to prove the reverse inclusion. Let then $x\in U_{pj}$, so that there exists by definition a thread $\mathfrak{a}\in x$, with $\mathfrak{a}(p)=j$, and, in particular,

(i) $$A_p(a_p,a_{p+1})=1.$$

Since $U_{p+1,j}\neq 0$, there is a thread $\mathfrak{b}\subseteq A$ such that $[\mathfrak{b}]\in U_{p+1,j}$ and $\mathfrak{b}(p+1)=j$, while

$$A_{p+1}(b_{a+1},b_{p+2})=1.$$

Therefore, by $\mathfrak{C}_8$,

(ii) $$A_p(b_{p+1},b_{p+1})=1.$$

Hence applying $\mathfrak{C}_9$ to (i) and (ii) (since $j=\mathfrak{a}(p)=\mathfrak{b}(p+1)$)

(iii) $$A^{p+1}(j,a_{p+1})=1.$$

Now, since $\mathfrak{a}$ is a thread,

(iv) $$A_{p+1}(a_{p+1},a_{p+2})=1,$$

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[7]) We have made no attempt to investigate the independence or otherwise of the axioms $\mathfrak{C}_1$-$\mathfrak{C}_9$.

so that applying $\mathfrak{C}_3$ to (iii) and (iv) gives

(v) $$A_{p+1}(j, a_{p+2}).$$

Define a thread $\mathfrak{c}: \mathfrak{J} \to \mathfrak{J}$ by

$$\mathfrak{c}(n) = \begin{cases} \mathfrak{a}(n), & n \neq p+1, \\ j, & n = p+1. \end{cases}$$

That $\mathfrak{c}$ is a thread follows from the fact that $\mathfrak{a}$ is a thread and by (ii) and (v). Moreover $\mathfrak{a} < \mathfrak{c}$; for applying $\mathfrak{C}_4$ to the equations $A_n(a_n, a_{n+1}) = 1$ gives $A^n(a_n, a_n) = 1$, *i. e.*

$$A^n(c_n, a_n) = 1 \quad \text{if} \quad n \neq p+1$$

while by (iii) $A^{p+1}(c_{p+1}, a_{p+1}) = 1$.

Thus $\mathfrak{a} < \mathfrak{c}$ follows by definition. Hence $x = [\mathfrak{a}] = [\mathfrak{c}]$. But $\mathfrak{c}(p+1) = j$, *i. e.* $[\mathfrak{c}] \in U_{p+1,j}$, and so $U_{pj} \subseteq U_{p+1,j}$; from which the lemma follows.

(4.11) Lemma. *For each $n$, $\mathcal{B}_n$ is a basis for $\boldsymbol{A}$.*

Proof. We have to shew that given $x \in \boldsymbol{A}$ and an open set $G$ containing $x$, then there exists $U_{nm} \in \mathcal{B}_n$, such that

(i) $$x \in U_{nm} \subseteq G.$$

Now since by definition $\bigcup_{j=0}^{\infty} \mathcal{B}_j$ constitutes a basis for $\boldsymbol{A}$, then there exist $p, q \in \mathfrak{J}$ such that

$$x \in U_{pq} \subseteq G.$$

If $p \geqslant n$, then by (4.10), $U_{pq} = U_{nq}$ since $U_{pq} \neq 0$, and so $U_{nq}$ is the element of $\mathcal{B}_n$ required in (i). Suppose therefore that $p < n$. Since $x \in U_{pq}$, there exists a thread $\mathfrak{a} \in x$ for which $\mathfrak{a}(p) = q$, and, in particular,

$$1 = A_p(a_p, a_{p+1}) = A_{p+1}(a_{p+1}, a_{p+2}) = \ldots = A_{n-1}(a_{n-1}, a_n);$$

therefore by $\mathfrak{C}_3$, $A(p, a_p, n, a_n) = 1$, and so by (4.2) (since $U_{pq} \neq 0$), $U_{pq} = U_{p,\mathfrak{a}(p)} \supseteq U_{n,\mathfrak{a}(n)}$. But, by definition, $x \in U_{n,\mathfrak{a}(n)}$, because $\mathfrak{a} \in x$, and so $x \in U_{n,\mathfrak{a}(n)} \subseteq U_{pq} \subseteq G$. Therefore $U_{n,\mathfrak{a}(n)}$ is the element of $\mathcal{B}_n$ required in (i). This completes the proof.

(4.12) Corollary (of proof). $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$.

(4.13) We have now shewn that $\mathcal{B}_n$ is a "fundamental sequence of neighbourhoods" in the sense of Whyburn [4], p. 2. Also, by (4.3), (4.4) and (4.5), conditions (1)-(6) of *loc. cit.* are satisfied; therefore by *op. cit.* Chap. I, 5.3, p. 7, $A$ is metrisable, say, with metric $\varrho$. A property of a *complete* metric space is that (4.9) holds (cf. Hausdorff [3], p. 130, known as the



Second Intersection Theorem, since clearly diam $U_{i,v(j)} \to 0$ as $j \to \infty$). We now prove the converse in

(4.14) Lemma [8]). *With the metric $\varrho$, $\boldsymbol{A}$ is complete.*

Proof. Since $\boldsymbol{A}$ has a metric $\varrho$, then by Hausdorff [3], p. 106, we can complete $\boldsymbol{A}$ to $\hat{\boldsymbol{A}}$ with metric $\hat{\varrho}$, and can regard $\boldsymbol{A}$ as being included in $\hat{\boldsymbol{A}}$. Given $y \in \hat{\boldsymbol{A}}$ and a real number $\delta > 0$, let $U(y,\delta)$ denote the set of all $y' \in \hat{\boldsymbol{A}}$ with $\hat{\varrho}(y,y') < \delta$. For each, $n, m \in \mathfrak{J}$, define $V_{nm} \subseteq \hat{\boldsymbol{A}}$ to be the union of all $U(y,\delta)$ for which $y \in U_{nm}$ and

$$U(y,\delta) \cap \boldsymbol{A} \subseteq U_{nm}.$$

Then it can be verified that

(a) $$U_{nm} = V_{nm} \cap \boldsymbol{A},$$

and (by (4.11)) that the system

$$\mathcal{B}_n' = \{V_{n0}, V_{n1}, \ldots, V_{nm}, \ldots\}$$

is a basis for $\hat{\boldsymbol{A}}$.

Let $\{x_n\}$ be a Cauchy sequence in $\boldsymbol{A}$. Since $\boldsymbol{A} \subseteq \hat{\boldsymbol{A}}$, there exists $x \in \hat{\boldsymbol{A}}$ such that, in $\hat{\boldsymbol{A}}$, $x_n \to x$. Since each $\mathcal{B}_n'$ is a basis for $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{A}}$ is regular then there exists $v: \mathfrak{J} \to \mathfrak{J}$ such that $x \in V_{n,v(n)} \in \mathcal{B}_n'$ and

(b) $$\mathcal{K} V_{n+1,v(n+1)} \subseteq V_{n,v(n)} \qquad (\mathcal{K} = \text{closure in } \hat{\boldsymbol{A}}).$$

Now $V_{n,v(n)}$ is a neighbourhood of $x$ in $\hat{\boldsymbol{A}}$, so that there exists $r = r(n) \in \mathfrak{J}$ for which $x_{r(n)} \in V_{n,v(n)}$, and so

(c) $$x_{r(n)} \in V_{n,v(n)} \cap \boldsymbol{A} = U_{n,v(n)} \qquad \text{(by (a) above)}.$$

By (b), $U_{n+1,v(n+1)} \Subset U_{n,v(n)} \in \mathcal{B}_n$, so that by (4.9) there exists $z \in \boldsymbol{A}$ for which

$$\bigcap_{n=0}^{\infty} U_{n,v(n)} = z.$$

From (c), $x_{r(n)} \to z$ in $\boldsymbol{A}$, whence

$$\lim_{n\to\infty} x_n = z \quad \text{in} \quad \boldsymbol{A}.$$

Therefore $z = x$; *i. e.* every Cauchy sequence in $\boldsymbol{A}$ has a limit in $\boldsymbol{A}$, whence $\boldsymbol{A}$ is complete, as required.

To sum up, we have proved

(4.15) Theorem. *If the $\mathfrak{P}$-function $A$ satisfies $\mathfrak{C}_1$-$\mathfrak{C}_9$, then the space $\boldsymbol{A} = \boldsymbol{A}(A)$ is separable and has a complete metric.*

---

[8]) This lemma appears in essence in [2], p. 14.

Conversely, we have

(4.16) THEOREM. *Given a complete metric separable space $S$, there is a $\mathfrak{P}$-function $A$ such that $S$ is homeomorphic to $A(A)$.*

Proof. Since $S$ is separable, there is a countable set $(b_0, b_1, \ldots, b_n, \ldots)$ of points of $S$, which is dense in $S$. Hence the system of all neighbourhoods

$$U\left(b_j, (1+n)^{-1}\right), \qquad j, n \in \mathfrak{J},$$

is a basis for $S$. Let it be enumerated in the form $W_0, W_1, \ldots, W_p, \ldots$, where

$$p = C_2(j, n)$$

in the enumeration $C_2\colon I^2 \to \mathfrak{J}$ of Section 1.

Define $A\colon I^4 \to \mathfrak{P}$ by

(4.161) $A(n,p,m,q) = 1$ if and only if $m \geqslant n$, $\operatorname{diam} W_p \leqslant 2/(1+n)$, $\operatorname{diam} W_q \leqslant 2/(1+m)$, and either

$$p = q . \& . m = n+1 \text{ or } n$$

or

$$p \neq q . \& . W_p \Supset W_q .$$

Put $A(n,p,m,q) = 0$ otherwise.

If now we draw the diagram $\mathfrak{A}$ of $A$ as in (2.1) we observe that, owing to the transitivity of equality and $\Supset$, $A$ immediately satisfies $\mathfrak{C}_3$; and through each point $(p, 0)$ there is a vertical "chain" of segments which terminates after at most $(\frac{1}{2}\operatorname{diam} W_p)^{-1}$ steps. Hence $A$ satisfies $\mathfrak{C}_5$. $\mathfrak{C}_2$ and $\mathfrak{C}_4$ are equally simply verified. $\mathfrak{C}_8$ is obvious, and $\mathfrak{C}_7$ holds for the following reason.

We have to shew that each segment $(n,p,q)$ of $A$ is part of a thread in ${}^nA$. Now if $A_n(p,q) = 1$, then $\operatorname{diam} W_q \leqslant 2/n+2$, and there exist $u, v$, such that $W_q = U(b_u, v^{-1})$. Since $S$ is regular and metric, we can construct (by induction on $j$) a sequence

(4.162) $$W_q = W_{a(1)} \Supset W_{a(2)} \Supset \ldots \Supset W_{a(j)} \Supset W_{a(j+1)} \Supset \ldots$$

of neighbourhoods of $b_u$, such that $\operatorname{diam} W_{a(j)} \leqslant 2(n+j+1)$ and $\bigcap_{j=0}^{\infty} W_{a(j)} = b_u$. Put $a(0) = p$, so that $A_n(a_0, a_1) = 1$ (given) while $A_{n+j}(a_j, a_{j+1}) = 1$ (by 4.162). Hence $\mathfrak{a}\colon \mathfrak{J} \to \mathfrak{J}$, defined by $\mathfrak{a}(j) = a_j$, is a thread in ${}^nA$ with initial segment $(n,p,q)$ as required. This proves that $A$ satisfies $\mathfrak{C}_7$. It remains to verify that $A$ satisfies $\mathfrak{C}_1$ and $\mathfrak{C}_6$.

Concerning $\mathfrak{C}_6$, it suffices to prove that if $W_q \Subset W_p$, then there exists $s$ such that for all $j \in \mathfrak{J}$ for which $U(b_j, s^{-1}) \frown W_q \neq 0$, then $U(b_j, s^{-1}) \Subset W_p$. To this end, let $W_q = U(b_u, k^{-1})$, $W_p = U(b_v, r^{-1})$ and let $s$ be the first integer satisfying

$$s > 2 \Big/ \left\{\frac{1}{r} - \left(\frac{1}{k} + \varrho(b_v, b_u)\right)\right\} \qquad (\varrho = \text{metric of } S).$$



We assert that this is the required $s$. For if

$$y \in U(b_j, s^{-1}) \frown W_q \quad \text{and} \quad z \in \overline{U}(b_j, s^{-1}),$$

$$\begin{aligned} \varrho(b_v, z) &\leqslant \varrho(b_v, b_u) + \varrho(b_u, y) + \varrho(y, z) \\ &< \varrho(b_v, b_u) + k^{-1} + 2s^{-1} \\ &< \varrho(b_v, b_u) + k^{-1} + \left(r^{-1} - k^{-1} - \varrho(b_v, b_u)\right) = r^{-1}. \end{aligned}$$

Hence $U(b_j, s^{-1}) \Subset W_p$, as asserted.

Finally, to prove that $A$ satisfies $\mathfrak{C}_1$, suppose the contrary. Then there exist threads $\mathfrak{a}, \mathfrak{b}$ in $A$, with $\mathfrak{b} < \mathfrak{a}$, and such that, in particular,

(4.163) $$A(0, b, n, a_n) = 0$$

for all $n$. But since $\mathfrak{a} < \mathfrak{b}$, then by definition and (4.161), we have the diagram

$$\begin{array}{ccccccc} W_{a(0)} \leftarrow & W_{a(1)} & \leftarrow \ldots \leftarrow & W_{a(n)} & \leftarrow \ldots \\ \uparrow & \uparrow & & \uparrow & \\ W_{b(0)} \leftarrow & W_{b(1)} & \leftarrow \ldots \leftarrow & W_{b(n)} & \leftarrow \ldots, \end{array}$$

where $X \to Y$ means "$X \Subset Y$ or $X = Y$", and

(4.164) $$\operatorname{diam} W_{a(n)} < 2/(n+1) \to 0 \quad \text{as} \quad n \to \infty;$$

and so by applying the Second Intersection Theorem see ((4.13)) in $S$, $\bigcap_{n=0}^{\infty} W_{a(n)}$ is a single point, say $x_a$. Since for each $n$, $W_{b(n)} \subseteq W_{a(n)}$ then by the same argument, $\bigcap_{n=0}^{\infty} W_{b(n)}$ is a single point $x_b$; and $x_a = x_b$, since

$$\bigcap W_{a(n)} \supseteq \bigcap W_{b(n)}.$$

Therefore $W_{b(0)}$ is a neighbourhood of $x_a$ (as is each $W_{a(n)}$); and so since $S$ is regular, there exists by (4.164) an integer $p$ such that

$$W_{a(p)} \Subset W_{b(0)}.$$

Hence

$$A(0, b_0, p, a_p) = 1,$$

which contradicts (4.163). Thus $A$ satisfies $\mathfrak{C}_1$, and this completes the proof that $A$ satisfies $\mathfrak{C}_1$-$\mathfrak{C}_9$.

We now have to prove that $S$ is homeomorphic to the space $\boldsymbol{A}=\boldsymbol{A}(A)$. First we shall collect some results. We recall from (4.12) that $\mathcal{B}_0=\bigcup_{n=0}^{\infty}\mathcal{B}_n$.

(4.17) A sequence $\{W_{\mathfrak{a}(n)}\}$ in $\mathcal{B}_0$ will be called a *proper* sequence if and only if for each $n\in\mathfrak{J}$, $W_{\mathfrak{a}(n+1)}\Subset W_{\mathfrak{a}(n)}$ and $\operatorname{diam} W_{\mathfrak{a}(n)}<2/(n+1)$. As remarked after (4.164),

(4.171) $\bigcup_{n=0}^{\infty}W_{\mathfrak{a}(n)}$ *is a single point*, $x_{\mathfrak{a}}$.

We shall say that $\{W_{\mathfrak{a}(n)}\}$ is *over* $x_{\mathfrak{a}}$. Also from (4.162),

(4.172) *The map* $\mathfrak{a}\colon\mathfrak{J}\to\mathfrak{J}$ *is a thread in* $A$, and

(4.173) *Given* $x\in W_j$, *there exists a proper sequence* $\{W_{\mathfrak{a}(n)}\}$ *over* $x$ *such that* $W_{\mathfrak{a}(0)}=W_j$.

From (4.163) and (4.164) we have

(4.174) $\mathfrak{a}<\mathfrak{b}$ *implies* $x_{\mathfrak{a}}=x_{\mathfrak{b}}$.

Now $\mathfrak{c}\equiv\mathfrak{d}$ if and only if there exists $\mathfrak{a}$ with $\mathfrak{a}<\mathfrak{c}$ and $\mathfrak{a}<\mathfrak{d}$, whence $x_{\mathfrak{a}}=x_{\mathfrak{c}}$ and $x_{\mathfrak{a}}=x_{\mathfrak{d}}$. Therefore

(4.175) $\mathfrak{a}\equiv\mathfrak{b}$ *implies* $x_{\mathfrak{a}}=x_{\mathfrak{b}}$.

Next, let $\{W_{\mathfrak{u}(n)}\}$, $\{W_{\mathfrak{v}(n)}\}$ be two proper sequences over the same point $x(=x_{\mathfrak{u}}=x_{\mathfrak{v}})$. Then, using (4.172),

(4.176) $$\mathfrak{u}\equiv\mathfrak{v}.$$

For, it follows easily by induction on $n$ that there is a proper sequence $\{W_{\mathfrak{w}(n)}\}$ over $x$, such that for each $n\in\mathfrak{J}$,

$$W_{\mathfrak{w}(n)}\Subset W_{\mathfrak{u}(n)}\cap W_{\mathfrak{v}(n)};$$

whence, using (4.172) and (4.161),

$$A^n(u_n,w_n)=A^n(v_n,w_n),$$

*i. e.* $\mathfrak{w}<\mathfrak{u}$ and $w<\mathfrak{v}$. Therefore $\mathfrak{u}\equiv\mathfrak{v}$ as required.

As a sort of converse to (4.172), we have

(4.177) *Given a thread* $\mathfrak{a}$ *over* $A$, *there is a sub-thread* $\mathfrak{b}$ *of* $\mathfrak{a}$ *such that* $\{W_{\mathfrak{b}(n)}\}$ *is a proper sequence over* $x_n$.

For by definition of a thread and (4.161), we have

$$\begin{aligned}W_{\mathfrak{a}(0)}&=W_{\mathfrak{a}(1)}=\ldots\\ &=W_{\mathfrak{a}(p(1)-1)}\supsetneq W_{\mathfrak{a}(p(1))}=\ldots=W_{\mathfrak{a}(p(2)-1)}\supsetneq W_{\mathfrak{a}(p(2))}=\ldots\\ &=W_{\mathfrak{a}(p(j)-1)}\supsetneq W_{\mathfrak{a}(p(j))}=\ldots=W_{\mathfrak{a}(p(j+1)-1)}\supsetneq W_{\mathfrak{a}(p(j+1))}=\ldots\end{aligned}$$



so that the sequence $\{p(j)\}$ is monotonic increasing; therefore the map $\mathfrak{b}\colon\mathfrak{J}\to\mathfrak{J}$ defined by $\mathfrak{b}(j)=\mathfrak{a}(p(j))$ is a sub-thread of $\mathfrak{a}$, and

$$\bigcap_{j=0}^{\infty}W_{\mathfrak{b}(j)}=\bigcap_{j=0}^{\infty}W_{\mathfrak{a}(p(j))}=x_{\mathfrak{a}}.$$

Therefore $\{W_{\mathfrak{b}(n)}\}$ is a proper sequence over $x_{\mathfrak{a}}$, as required.

Now let us establish the homeomorphism of $\boldsymbol{A}$ on $S$. By (4.175), $x_{\mathfrak{a}}$ remains constant when $\mathfrak{a}$ runs through $[\mathfrak{a}]$. Define this constant value to be

(4.178) $$f[\mathfrak{a}].$$

We assert that $f\colon\boldsymbol{A}\to S$ is the required homeomorphism.

First, $f(\boldsymbol{A})=S$. For if $x\in S$, then by (4.173) there is a proper sequence $\{W_{\mathfrak{a}(n)}\}$ over $x$, and so by (4.171), (4.172) and (4.175), $x=x_{\mathfrak{a}}=f[\mathfrak{a}]$. This proves that $f$ is onto, as required.

Secondly, $f$ is (1-1). For if $f[\mathfrak{a}]=f[\mathfrak{b}]$, then $f[\mathfrak{a}]=x_{\mathfrak{a}}=f[\mathfrak{b}]=x_{\mathfrak{b}}$; therefore the proper sequences $\{W_{\mathfrak{a}(n)}\}$, $\{W_{\mathfrak{b}(n)}\}$ are over the same point. Hence, by (4.176), $[\mathfrak{a}]=[\mathfrak{b}]$; which proves $f$ to be (1-1).

Thirdly, for each $j\in\mathfrak{J}$

(4.179) $$f(U_{0j})=W_j.$$

To prove this let $[\mathfrak{a}]\in U_{0j}$, so that by definition there exists $\mathfrak{b}\in[\mathfrak{a}]$ with $\mathfrak{b}(0)=j$. Now $f[\mathfrak{a}]=x_{\mathfrak{b}}=\bigcap_{n=0}^{\infty}W_{\mathfrak{b}(n)}\in W_{\mathfrak{b}(0)}=W_j$, so that $f(U_{0j})\subseteq W_j$. Conversely, given $y\in W_j$, there is by (4.173) a proper sequence $\{W_{\mathfrak{c}(n)}\}$ over $y$ with $c(0)=j$; hence (using (4.172)) the thread $\mathfrak{c}$ is such that $[\mathfrak{c}]\in U_{0j}$. But $y=x_{\mathfrak{c}}=f[\mathfrak{c}]$. This proves that $W_j\subseteq f(U_{0j})$; and so $f(U_{0j})=W_j$ as asserted.

Hence $f^{-1}(W_j)=U_{0j}$, since $f^{-1}$ is single-valued. Thus $f,f^{-1}$ induce order-preserving maps $f_*\colon\mathcal{B}_0\to\mathcal{W}=(W_0,W_1,\ldots)$, $f_*^{-1}\colon\mathcal{W}\to\mathcal{B}_0$; and since $\mathcal{B}_0[\mathcal{W}]$ are bases of $\boldsymbol{A}[S]$, then $f$ and $f^{-1}$ are both continuous. This completes the proof that $f\colon\boldsymbol{A}\to S$ is a homeomorphism, and thus Theorem 4.16 is established.

## 5. The Borel sets of $\boldsymbol{A}(A)$

Let $S$ be the complete metric separable space of 4.16, with associated $\mathfrak{P}$-function $A=A(S)$. Then by (4.16), the topological properties of $S$ are those of $\boldsymbol{A}$, and we shall from now on assume that $S$ is $\boldsymbol{A}$. Thus in the notation of (4.178) we have

(5.1) $$f[\mathfrak{a}]=[\mathfrak{a}]$$

and so by (4.179) we can write the basis $\mathcal{B}_0=(U_{00},U_{01},\ldots)$ as $(W_0,W_1,\ldots)$.

A $G_\delta$-set of $\boldsymbol{A}$ is any set of the form

$$G_\delta = \bigcap_{n=0}^{\infty} G_n\,, \qquad G_n \text{ open in } \boldsymbol{A}\,.$$

We shall make a model of $G_\delta$, in the form of a $\mathfrak{P}$-function $B \subseteq A$ as follows. Define a $\mathfrak{P}$-function $B(G_\delta) = B \subseteq A$ by

(5.2) $\quad B(n,p,m,q) = 1 . \leftrightarrow . \big(A(n,p,m,q) = 1 . \& . U_{np} \subseteq G_n . \& . U_{mq} \subseteq G_m\big)\,.$

Define $\boldsymbol{A}(B)$ to be the set of all $x \in \boldsymbol{A}$ for which there is a thread $\mathfrak{a}$ in $B$ such that $\mathfrak{a} \in x$.

(5.3) LEMMA. $\boldsymbol{A}(B) = G_\delta$.

Proof. Let $x \in \boldsymbol{A}(B) = \boldsymbol{B}$. Then there exists a thread $\mathfrak{a} \in x$ in $B$; so that for each $n \in \mathfrak{J}$,

$$B_n(a_n, a_{n+1}) = 1\,.$$

Therefore, by (5.2), $U_{n,a(n)} \subseteq G_n$. But, by definition, $[\mathfrak{a}] \in U_{n,a(n)}$, and so $x = [\mathfrak{a}]$ is in every $G_n$. Thus $x \in G_\delta$, *i. e.*

(5.31) $$\boldsymbol{B} \subseteq G_\delta\,.$$

To prove the reverse inclusion, let $y \in G_\delta$. Then for each $n \in \mathfrak{J}$, $y \in G_n$. Since each $\mathcal{B}_n$ is a basis of the (regular) space $\boldsymbol{A}$, there exists (by induction on $n$) a proper sequence $U_{n,a(n)} = U_{0,a(n)} = W_{a(n)}$ over $y$ such that, for each $n \in \mathfrak{J}$,

(i) $$G_n \supseteq W_{a(n)} \in \mathcal{B}_n\,.$$

By (4.172) $\mathfrak{a}$ is a thread in $A$, and so by (5.1) and (i), (since $W_{a(n)} = U_{n,a(n)}$) it follows that $\mathfrak{a}$ is a thread in $B$. Therefore, by definition, $[\mathfrak{a}] \in \boldsymbol{B}$. Now in the notation of (4.171),

$$y = \bigcap_{n=0}^{\infty} W_{a(n)} = x_a = f[\mathfrak{a}] \qquad (\text{by } (4.178))$$
$$= [\mathfrak{a}] \qquad (\text{by } (5.1))\,.$$

Hence $y \in \boldsymbol{B}$, *i. e.* $G_\delta \subseteq \boldsymbol{B}$. With (5.31), the proof is complete.

We shall later require

(5.4) LEMMA. *If* $\mathfrak{s}, \mathfrak{t}$ *are threads in* $A$ *such that* $\mathfrak{s} < \mathfrak{t}$ *and* $\mathfrak{t} \subseteq B$, *then* $\mathfrak{s} \subseteq B$.

Proof. Since $\mathfrak{t} \subseteq B$, then by (5.2)

$$A_n(t_n, t_{n+1}) = 1 \;\;\&\;\; U_{n,t(n)} \subseteq G_n\,,$$

for each $n \in \mathfrak{J}$. Also, since $\mathfrak{s} < \mathfrak{t}$, then

$$A_n(t_n, s_n) = 1\,,$$



so that by (4.2) $U_{n,t(n)} \supseteq U_{n,s(n)}$ (since $[\mathfrak{t}] \in U_{n,t(n)} \neq 0$). Therefore $U_{n,s(n)} \subseteq U_{n,t(n)} \subseteq G_n$, while, since $\mathfrak{s} \subseteq A$ (given),

$$A_n(s_n, s_{n+1}) = 1\,.$$

Therefore, by (5.2), $B_n(s_n, s_{n+1}) = 1$, *i. e.* $\mathfrak{s} \subseteq B$.

Now let us pass to the Borel sets of higher order in $\boldsymbol{A}$. We recall from Hausdorff [3], p. 178, that the Borel sets of order 1 are the $G_\delta$ sets of $\boldsymbol{A}$, and that a Borel set of order $a > 1$ is any set of the form

(5.5) $$\bigcup_{n=0}^{\infty} X_n \quad (a \text{ even}), \qquad \bigcap_{n=0}^{\infty} X_n \quad (a \text{ odd}),$$

where each $X_n$ is a Borel set of order $< a$.

Moreover there is an ordinal $\nu < \Omega_1$ (depending on $\boldsymbol{A}$) such that every Borel set in $\boldsymbol{A}$ of order $> \nu$ is also one of order $\nu$.

A set of the form $\bigcup_{n=0}^{\infty} X_n$ is empty if and only if each $X_n$ is empty. Hence we shall confine our attention to Borel sets of odd order, of the form

(5.6) $$X = \bigcap_{i=0}^{\infty} \bigcup_{j=0}^{\infty} X_{ij}\,,$$

where each $X_{ij}$ is of odd order $<$ order $(X)$. How can we use the $\mathfrak{P}$-function $A$ to make a model of $X$? One method is given below. It will be necessary to introduce some new concepts, guided by the following considerations. We note that in particular, a set of order 3 is of the form (5.6) where each $X_{ij}$ is a $G_\delta$, *i. e.* a set of order 1. This suggests that we write a set of order $2p+1$, where $p$ is *finite*, in the form

$$X = \bigcap_{i(p)} \bigcup_{j(p)} \bigcap_{i(p-1)} \bigcup_{j(p-1)} \cdots \bigcap_{i(1)} \bigcup_{j(1)} X_{i(1),i(2),\ldots,i(p);\,j(1),j(2),\ldots,j(p)}$$

where the $i$'s and $j$'s run from 0 to $\infty$. By induction on $p$, a necessary and sufficient condition that $x \in X$ is easily seen to be that there exist $p$ integer-valued functions (corresponding to existential quantifiers)

$$a_r\big(i(r), i(r+1), \ldots, i(p)\big) \qquad (1 \leqslant r \leqslant p)\,,$$

such that for all $i(1), i(2), \ldots, i(p) \in \mathfrak{J}$, $x \in X(\boldsymbol{u}, \boldsymbol{v})$; where

$$\boldsymbol{u} = \big(i(1), i(2), \ldots, i(p)\big), \qquad \boldsymbol{v} = (v_1, v_2, \ldots, v_p) \in I\,,$$
$$v_r = a_r\big(i(r), i(r+1), \ldots, i(p)\big)$$

and

$$X(\boldsymbol{u}, \boldsymbol{v}) = X_{i(1),i(2),\ldots,i(p)\;\, j(1),j(2),\ldots,j(p)}\,.$$

If we call the correspondence $u \to v$ a *compound* function, denoted by $\alpha: I^p \to I^p$, then the above takes the form

(5.7) *A necessary and sufficient condition, that $x \in X$, is that there exists a compound function $\alpha: I^p \to I^p$ such that, for all $u \in I^p$,*

$$x \in X(u, \alpha(u)).$$

Such a relatively neat statement has no immediate simple analogue when $p$ becomes infinite (as will be apparent to the reader if he reflects on the case $p = \omega$). Hence we have to introduce the complicated definitions of $\mathfrak{B}$-domains, etc. below. We shall require the notation of Section 1.

Given $z \in I^a$, then since only a finite number of values $z(\xi)$ are non-zero, we can form the sum of all $z(\xi)$, as $\xi$ runs through $\Sigma_a$, to form an integer

(5.8) $$\sigma(z) = z(0) + z(1) + \ldots + z(\xi) + \ldots$$

Next let us write for brevity

(5.81) $$E_a = I^a \times I^a,$$

so that every $z \in E_a$ is of the form

$$z = u \times v, \quad u, v \in I^a.$$

Using transfinite induction on $a$, we now define the term a $\mathfrak{B}$-*domain of* $E_a$. Such a domain is a subset $\Delta \subseteq E_a$ given by the scheme:

(5.82) (a) $\Delta = E_a$ if $a = 0$ or $1$;

(b) if $a > 1$, then every $z \in \Delta$ is of the form [9])

$$z = \{u, 0, i\} \times \{v, 0, j\}$$

and it is required that for each fixed $i, j \in I$, the points $u \times v$ run through the *whole* of a $\mathfrak{B}$-domain $\Delta_{ij} \subseteq E_{a(i,j)}$, where $a(i,j) < a$.

This completes the definition. The collection of all $\Delta_{ij}$ will be called the *associated domains* of $\Delta$.

(5.83) Let $\mathcal{G}$ be the set of all $\mathfrak{P}$-functions of the form $B(G_\delta)$ defined in (5.2), as $G_\delta$ runs through the class of all $G_\delta$-subsets of $A$. An element $f^0 \in \mathcal{G}$ will be called a *Borel map of order* 0, and we shall often write

$$f^0: E_0 \to \mathcal{G}.$$

By a *Borel map of order* $a > 0$, we shall mean a map $f^a: \Delta \to \mathcal{G}$ where $\Delta \subseteq E_a$ and is a $\mathfrak{B}$-domain.

[9]) The notation is defined in (1.1).



We can now associate with $f^a$ a Borel set

$$M(f^a) \subseteq A$$

of order $2a+1$ using transfinite induction, as follows. First

$$M(f^0) = A(f_0) \quad (\text{a } G_\delta)$$

in the sense of (5.3); if $a > 0$, and $f^a: \Delta \to \mathcal{G}$ is a Borel map of order $a$, let the associated domains of $\Delta$ be $\Delta_{ij}$, as above. Define $f_{ij}: \Delta_{ij} \to \mathcal{G}$ by

(5.84) $$f_{ij}(u \times v) = f^a(\{u, 0, i\} \times \{v, 0, j\})$$

where $\{u, 0, i\}, \{v, 0, j\} \in I^a$ and $u, v \in I^{a(i,j)}$ with $a(i,j) < a$. Then $f_{ij}: \Delta_{ij} \to \mathcal{G}$ is a Borel map of order $a(i,j)$; so that on putting

(5.9) $$M(f^a) = \bigcap_{i=0}^{\infty} \bigcup_{j=0}^{\infty} M(f_{ij}),$$

$M(f^a)$ is defined inductively, for all $a$, and is a Borel set of $A$, of order $a$ (cf. (5.6)).

The analogue of (5.7) for infinite $a$ leads us to define the system $\mathcal{E}(\Delta)$, where $\Delta$ is a fixed $\mathfrak{B}$-domain $\subseteq E_a$, by the following scheme. $\mathcal{E}(\Delta)$ is to consist of all subsets $\Phi \subseteq I^a$ such that

(5.10) (a) $\Phi = I^a$, $a = 0$ or $1$;

(b) $a > 1$, $\Phi$ is of the form

$$\Phi = \bigcup_{i=0}^{\infty} \{\Phi_i, 0, i\}$$

where there exists $\varphi: \mathfrak{I} \to \mathfrak{I}$ such that for each $i \in \mathfrak{I}$,

$$\Phi_i \in \mathcal{E}(\Delta_{i,\varphi(i)}).$$

The $\Phi$'s in $\mathcal{E}(\Delta)$ are the analogues of the set of all $(i(1), i(2), \ldots, i(p))$ in (5.7). As analogues of the compound function $\alpha$ in (5.7) we define the elements of the system $\mathcal{M}(\Delta)$ to consist of all maps $\mu: \Phi \to I^a$ where $\Phi \in (\Delta)$ and such that

(5.11) (a) if $a = 0, 1$, $\mu$ is arbitrary;

(b) if $a > 1$, and if $\Phi$ is as in (5.10b), then for each $i \in \mathfrak{I}$,

$$\mu(u, 0, i) = (\mu^{(i)}(u), 0, \varphi(i))$$

where $u \in \Phi_i$ and $\mu^{(i)}: \Phi_i \to I^{a(i)}$ is an element of $\mathcal{M}(\Delta_{i,\varphi(i)})$, $a(i) < a$.

The statement corresponding to (5.7) is now as follows. Let $f^a: \Delta \to E_a$ be a Borel map of order $a$.

(5.12) LEMMA. *A necessary and sufficient condition that* $x \in M(f^a)$ *is that there exists* $\Phi \in \mathcal{E}(\Delta)$ *and in* $\mathcal{M}(\Delta)$ *a map* $\mu\colon \Phi \to I^a$, *such that for each* $\boldsymbol{u} \in \Phi$,

$$x \in M\big(f^a(\boldsymbol{u} \times \mu(\boldsymbol{u}))\big).$$

Proof. The result is trivial if $a=0,1$; now suppose it holds for all Borel maps of order $<a$. By (5.6) and the result for $a=1$, $x \in M(f^a)$ if and only if there is a map $\varrho\colon \mathfrak{I} \to \mathfrak{I}$, such that for all $i \in \mathfrak{I}$,

$$x \in M(f_{i,\varrho(i)}),$$

where the $f_{ij}$ are as defined in (5.84). By the inductive hypothesis, there exists $\Phi_i \in \mathcal{E}(\Delta_{i,\varrho(i)})$ and a map $\mu^{(i)}\colon \Phi_i \to I^{a(i)}$ $(a(i)<a)$ such that for all $\boldsymbol{z} \in \Phi_i$

(i) $$x \in M\big(f_{i,\varrho(i)}(\boldsymbol{z} \times \mu^{(i)}(\boldsymbol{z}))\big).$$

Define $\Phi \in \mathcal{E}(\Delta)$ to be $\bigcup_{i=0}^{\infty}\{\Phi_i, \boldsymbol{0}, i\}$ and define $\mu\colon \Phi \to I^a$ as follows. Each $u \in \Phi$ is of the form $\{\boldsymbol{z}, \boldsymbol{0}, i\}$, $\boldsymbol{z} \in \Phi_i$; put

$$\mu(\boldsymbol{u}) = \{\mu^{(i)}(\boldsymbol{z}), \boldsymbol{0}, \varrho^{(i)}(i)\} \in I^a,$$

so that by definition, $\mu \in \mathcal{M}(\Delta)$. Moreover

$$\begin{aligned} x \in M\big(f_{i,\varrho(i)}(\boldsymbol{z} \times \mu^{(i)}(\boldsymbol{z}))\big) &= M\big(f^a(\{\boldsymbol{z}, \boldsymbol{0}, i\} \times \{\mu^{(i)}(\boldsymbol{z}), \boldsymbol{0}, \varrho(i)\})\big) \\ &= M\big(f^a(\boldsymbol{u} \times \mu(\boldsymbol{u}))\big), \end{aligned}$$

as required. The argument reverses in the obvious way to prove the converse, and this completes the proof, by transfinite induction.

We recall that $A$ is the $\mathfrak{P}$-function associated with the space $\boldsymbol{A}(A)$.

(5.13) LEMMA. *A necessary and sufficient condition that* $x \in M(f^a)$ *is that there exists a thread* $\mathfrak{a} \subseteq A$, *together with* $\mu\colon \Phi \to I^a$ *in* $\mathcal{M}(\Delta)$, *such that* $x = [\mathfrak{a}]$ *and for each* $\boldsymbol{z} \in \Phi$,

(i) $${}^{z}\mathfrak{a} \subseteq f^a(\boldsymbol{z} \times \mu(\boldsymbol{z})),$$

*where* [10]) $z = \sigma(\boldsymbol{z})$.

Proof. The result is (5.2) if $a=0$; now suppose it to be true for all $f^\beta$, $\beta < a$. Then $x \in M(f^a)$ if and only if (i) of the last lemma holds. But by the inductive hypothesis, a necessary and sufficient condition for (5.12 i) to hold is that there exist a thread $\mathfrak{t}^{(i)} \subseteq A$ such that for each $\boldsymbol{u} \in \Phi_i$

(ii) $${}^{u}\mathfrak{t}^{(i)} \subseteq f_{i,\varrho(i)}(\boldsymbol{u} \times \mu^{(i)}(\boldsymbol{u})), \quad u = \sigma(\boldsymbol{u}), \quad \text{and} \quad [\mathfrak{t}^{(i)}] = x.$$

[10]) $\sigma(\boldsymbol{z})$ was defined in (5.8), and ${}^{z}a$ in (3.10).



By (3.11), there exists in $x$ a thread $\mathfrak{s} \subseteq A$ such that for each $i \in \mathfrak{I}$, ${}^{i}\mathfrak{s} < \mathfrak{t}^{(i)}$. Therefore ${}^{i+u}\mathfrak{s} < {}^{u}\mathfrak{t}^{(i)}$, and so by (ii) and (5.3),

$${}^{i+u}\mathfrak{s} \subseteq f_{i,\varrho(i)}(\boldsymbol{u} \times \mu^{(i)}(\boldsymbol{u})) = f^a(\boldsymbol{z} \times \mu(\boldsymbol{z}))$$

for all $\boldsymbol{z} \in \Phi$, where $\Phi$ and $\mu$ are defined as in the proof of the last lemma. But $\boldsymbol{z} = \{\boldsymbol{u}, \boldsymbol{0}, i\}$, so that $\sigma(\boldsymbol{z}) = \sigma(\boldsymbol{u}) + i$, and therefore $z = i+u$. Hence

$${}^{z}\mathfrak{s} \subseteq f^a(\boldsymbol{z} \times \mu(\boldsymbol{z}))$$

for each $\boldsymbol{z} \in \Phi$, as required. The argument reverses in the obvious way; and the proof is complete, by transfinite induction.

If to be "given" a Borel set $X$ of order $a$ is to be given a Borel map $f^a\colon \Delta \to \mathcal{G}$ with $X = M(f^a)$; and if we wish to shew in an operation $D$ that there exists an $x$ in $X$; then clearly (5.13) is a better tool for the job than (5.12), in view of (5.5). However, there are still too many elements in $\mathcal{E}(\Delta)$ and $\mathcal{M}(\Delta)$ to examine collectively in an operation $D$, and so we must refine (5.13) further.

(5.14) To this end, then, let $q \in \mathfrak{I}$, let $\Phi$ be as in (5.13) and let $Z_q(\Phi)$ denote the set of all $\boldsymbol{z} \in \Phi$ such that $r = q - \sigma(\boldsymbol{z}) \in \mathfrak{I}$ [11]). Then $r + \sigma(\boldsymbol{z}) = q$, so that

(5.141)
$$\begin{cases} \text{(i)} & Z_q(\Phi) \subseteq Z_{q+1}(\Phi), \\ \text{(ii)} & \boldsymbol{z} \in \bigcup_{i=0}^{\infty} Z_{z+i}(\Phi), \\ \text{(iii)} & \Phi = \bigcup_{q=0}^{\infty} Z_q(\Phi). \end{cases}$$

It is now easily verified that (5.13) is equivalent to

(5.142) $x \in M(f^a)$ *if and only if for all* $q \in \mathfrak{I}$,

$$\big(f^a(\boldsymbol{z} \times \mu(\boldsymbol{z}))\big)_{q-z} \quad (a_q, a_{q+1}) = 1,$$

*or all* $\boldsymbol{z} \in Z_q(\Phi)$.

Our next object is therefore to seek to characterise those subsets of $I^a$ which are of the form $Z_q(\Phi)$, $\Phi \in \mathcal{E}(\Delta)$, *without having to find* $\Phi$; for we shall see that the number of such subsets is countable, while the number of $\Phi$ in $\mathcal{E}(\Delta)$ is not. This characterisation is the concern of the next section, and culminates in the crucial Lemma (6.15).

## 6. The sets $Z_q(\Phi)$

Given $\Phi \in \mathcal{E}(\Delta)$, and as in (5.10b), we assert that

(6.1) $$Z_q(\Phi) = \bigcup_{i=0}^{q}\{Z_{q-i}(\Phi_i), \boldsymbol{0}, i\}.$$

[11]) *I. e.* $q \geqslant \sigma(\boldsymbol{z})$.

For if $z \in Z_q(\Phi)$, then $z \in \Phi$ and is of the form $z=(u,0,p)$, $0 \leq p \leq q$, where $u \in \Phi_p$. Since $z \in Z_q(\Phi)$, then by definition

$$q-\sigma(z)=q-(\sigma(u)+p)=(q-p)-\sigma(u),$$

and so $u \in Z_{q-p}(\Phi_p)$; whence $z \in \{Z_{q-p}(\Phi_p),0,i\}$. The argument is reversible and so (6.1) follows.

Now, if $a=0$ or 1, $\Phi=I^a$ and so $Z_q(\Phi)$ is finite. Therefore, by (6.1) and induction on $a$ we have

(6.2) *$Z_q(\Phi)$ is a finite subset of $I^a$.*

Next we define a certain system $\mathcal{Z}_q(\Delta)$ of subsets of $I^a$ by the following scheme (the associated domains of $\Delta$ are $\Delta_{ij}$ as in (5.84)):

(6.3) (a) $a=0$ or 1; $\mathcal{Z}_q(\Delta)$ possesses the single member $Z_q(I^a)$;

(b) $a>1$; $X \in \mathcal{Z}_q(\Delta)$ if and only if there is a map

$$\varrho(X)=\varrho\colon Z_q(\mathfrak{I}) \to \mathfrak{I}$$

such that

$$X=\bigcup_{i=0}^{q}\{X_{q-i},0,i\}$$

where $X_{q-i} \in \mathcal{Z}_{q-i}(\Delta_{i,\varrho(i)})$, $0 \leq i \leq q$.

This completes the definition, by induction; we see immediately that

(6.31) *If $X \in \mathcal{Z}_q(\Delta)$, then $X \subseteq I^a$ and is finite.*

The significance of $\mathcal{Z}_q(\Delta)$ appears in

(6.4) Lemma. *If $\Phi \in \mathcal{E}(\Delta)$, then $Z_q(\Phi) \in \mathcal{Z}_q(\Delta)$.*

Proof. The result is obvious if $a=0,1$; suppose its truth for all $\mathfrak{B}$-domains in $I^\beta$, $\beta<a$. As in (5.10b) let

$$\Phi=\bigcup_{i=0}^{\infty}\{\Phi_i,0,i\}, \qquad \Phi_i \in \mathcal{E}(\Delta_{i,\varphi(i)}).$$

Then by (6.1),

$$Z_q(\Phi)=\bigcup_{i=0}^{q}\{Z_{q-i}(\Phi_i),0,i\};$$

now define $\varrho\colon Z_q(\mathfrak{I}) \to \mathfrak{I}$ by [12] $\varrho=\varphi|Z_q(\mathfrak{I})$. Then by the inductive hypothesis, $Z_{q-i}(\Phi_i) \in \mathcal{Z}_{q-i}(\Delta_{i,\varrho(i)})$, so that (6.3b) is satisfied by $Z_q(\Phi)$. This completes the proof.

Given $X \in \mathcal{Z}_q(\Delta)$, $Y \in \mathcal{Z}_{q+1}(\Delta)$, we define the relation $X<Y$ inductively by the scheme:

[12]) *I. e.* the restriction of $\varphi$ to $Z_q(\mathfrak{I})$.



(6.5) (a) $a=0$ or 1; $X<Y$ if and only if $X=Z_q(I^a)$, $Y=Z_{q+1}(I^a)$;

(b) if $a>1$, and $X=\bigcup_{i=0}^{q}\{X_{q-1},0,i\}$, $X_{q-i} \in \mathcal{Z}_{q-i}(\Delta_{i,\sigma(i)})$,

$$Y=\bigcup_{i=0}^{q+1}\{Y_{q+1-i},0,i\}, \qquad Y_{q+1-i} \in \mathcal{Z}_{q+1-i}(\Delta_{i,\tau(i)})$$

where $\sigma=\varrho(X)$, $\tau=\varrho(Y)$; then $X<Y$ if and only if

$$\sigma(i)=\tau(i) \quad \text{and} \quad X_{q-i}<Y_{q+1-i} \quad (0 \leq i \leq q).$$

This completes the definition, by induction on $a$ and $q$. Clearly

(6.51) *$X<Y$ implies $X \subseteq Y$.*

An immediate consequence of (6.1) and (6.5) (in view of (6.4)) is

(6.6) *Given $\Phi \in \mathcal{E}(\Delta)$ and $q \in \mathfrak{I}$, then $Z_q(\Phi)<Z_{q+1}(\Phi)$.*

Corresponding to (5.141) ((i) and (iii)), we have

(6.7) Lemma. *If $a>0$ and for each $q$, $X_q \in \mathcal{Z}_q(\Delta)$ and $X_q<X_{q+1}$, then*

(a) $X=\bigcup_{q=0}^{\infty}X \in \mathcal{E}(\Delta)$,

*and*

(b) $X_q=Z_q(X)$.

Proof. The result is obvious if $a=1$; hence suppose the result proved for $\beta<a$. Then by (6.4), there exists $\varrho\colon \mathfrak{I}\to\mathfrak{I}$ defined by $\varrho|Z_q(\mathfrak{I})=\varrho(X_q)$, such that, for each $q \in \mathfrak{I}$,

(6.71) (i) $X_q=\bigcup_{i=1}^{q}\{X_{q,q-1},0,i\}$, $X_{q,q-1} \in \mathcal{Z}_{q-i}(\Delta_{i,\varrho(i)})$,

(ii) $X_{q,q-i}<X_{q+1,q+1-i}$, $0 \leq i \leq q$.

Hence

$$\bigcup_{q=0}^{\infty}X_q=\bigcup_{q=0}^{\infty}\bigcup_{i=0}^{q}\{X_{q,q-i},0,i\}$$

$$=\bigcup_{i=0}^{\infty}\Big(\bigcup_{q\geq i}\{X_{q,q-i},0,i\}\Big)$$

(6.72) $$=\bigcup_{i=0}^{\infty}\{\Phi_i,0,i\} \quad \text{say}$$

where $\Phi_i=\bigcup_{q\geq i}X_{q,q-i}$.

Therefore $\Phi_i \in \mathcal{E}(\Delta_{i,\varrho(i)})$ by the inductive hypothesis on (a); for then $X_{q,q-i} \in \mathcal{Z}_{q-i}(\Delta_{i,\varrho(i)})$, and by (ii) $X_{q,q-i}<X_{q,q+1-i}$. Hence, by (5.10b), and (iii), the inductive hypothesis on (a) is justified for $a$, and so (a) of the Lemma follows by induction.

The condition (b) follows immediately, by induction, from (6.72). This completes the proof of the entire Lemma.

It is now necessary to define the "finite" analogues of the elements of $\mathcal{M}(\Delta)$. Thus we define $\mathcal{M}_q(\Delta)$ to be the set of all maps $\nu: X \to I^\alpha$, with the properties that $X \in \mathcal{Z}_q(\Delta)$ and

(6.8) (a) $\nu$ is arbitrary if $\alpha = 0,1$;

(b) if $\alpha > 1$, and $X$ is as in (6.3b), then given $z \in X$, $z = \{u, 0, i\}$ where $u \in X_{q-i} \in \mathcal{Z}_{q-i}(\Delta_{i,\varrho(i)})$.

It is required that $\nu(z)$ be of the form $\{\nu_i(u), 0, \varrho(i)\}$, where $\nu_i: X_{q-i} \to I^{\alpha(i,\varrho(i))}$ is in $\mathcal{M}_{q-u}(\Delta_{i,\varrho(i)})$ and $\alpha(i, \varrho(i)) < \alpha$. This completes the definition, by induction on $\alpha$.

Corresponding to (6.4) we have

(6.9) LEMMA. *Let $\mu: \Phi \to I^\alpha$ be in $\mathcal{M}(\Delta)$. Then for each $q \in \mathfrak{I}$, $\mu|Z_q(\Phi)$ is in $\mathcal{M}_q(\Delta)$.*

Proof. The result is trivial if $\alpha = 0$ or $1$; therefore suppose that it holds for all $\beta < \alpha$. By (6.1)

$$\mu|Z_q(\Phi) = \mu|\bigcup_{i=0}^{q}\{Z_{q-i}(\Phi_i), 0, i\} \quad (= \mu_q \text{ (say)})$$

therefore, if $z \in Z_q(\Phi)$, then $z$ is of the form $z = (u, 0, i)$ where $u \in Z_{q-i}(\Phi_i)$; and $\mu_q(z)$ is of the form

$$\mu_q(z) = \mu(z) = \{\mu^{(i)}(u), 0, \varphi(i)\} \quad \text{(by (5.11b))}$$

where $\mu^{(i)}: \Phi_i \to I^{\alpha(i)}$ is in $\mathcal{M}(\Delta_{i,\varphi(i)})$. Hence, by the inductive hypothesis, if

$$\nu_i = \mu^{(i)}|Z_{q-i}(\Delta_{i,\varphi(i)})$$

then $\nu_i \in \mathcal{Z}_{q-i}(\Delta_{i,\varphi(i)})$; therefore

$$\mu_q(z) = \{\nu_i(u), 0, \varrho(i)\},$$

where $\varrho: Z_q(\mathfrak{I}) \to \mathfrak{I}$ is defined to be $\Phi|Z_q(\mathfrak{I})$. Thus the conditions of (6.8b) are satisfied by $\mu_q: Z_q(\Phi) \to I^\alpha$, so that $\mu_q \in \mathcal{Z}_q(\Delta)$.

The Lemma now follows, by induction.

In (6.7) suppose that, for each $q \in \mathfrak{I}$, there is given a map $\nu_q: X_q \to I^\alpha$, such that $\nu_q \in \mathcal{M}_q(\Delta)$ and [13] $\nu_q = \nu_{q+1}|X_q$, — a state of affairs which we express for brevity by writing

(6.10) $$\nu_q < \nu_{q+1} \quad \text{if} \quad q \in \mathfrak{I}.$$

We saw in (6.7) that $\bigcup_{q=0}^{\infty} X_q$ $(= \Phi$ say) belongs to $\mathcal{E}(\Delta)$; and we shall now prove

(6.11) *There exists in $\mathcal{M}(\Delta)$ a map $\mu: \Phi \to I^\alpha$ such that, for each $q \in \mathfrak{I}$,*

$$\mu|X_q = \nu_q.$$

[13]) This statement is legitimate, by (6.51).



Proof. Given $z \in \Phi$, there exists a least $q \in \mathfrak{I}$, such that $z \in X_q$, since $\Phi = \bigcup_{q=0}^{\infty} X_q$. Define $\nu_*(z)$ to be $\nu_q(z)$. Thus

$$\nu_*|X_q = \nu_q.$$

It remains to prove that $\nu_*: \Phi \to I^\alpha$ is in $\mathcal{M}(\Delta)$. This result is obvious if $\alpha = 0$ or 1, and we now assume it for all ordinals $\beta < \alpha$. If $z \in X_q$ as above, then by (6.71), $z \in \{X_{q,q-i}, 0, i\}$ for some $i$, and therefore by (6.8b), $\nu_q(z)$ is of the form

(6.111) $$\{\nu_q^{(i)}(u), 0, \varrho(i)\}$$

where $\nu_q^{(i)}: X_{q,q-i} \to I^{\alpha(i,\varrho(i))}$ is in $\mathcal{M}_{q-i}(\Delta_{i,\varrho(i)})$. Hence

$$\nu_q^{(i)} = \nu_{q+1}^{(i)}|X_{q,q-i}.$$

By (6.71ii) and the fact that $\Phi_i = \bigcup_{q \geqslant i} X_{q,q-i}$ (in (6.72)), we can apply the inductive hypothesis, to conclude that

$$(\nu^{(i)})_*: \Phi_i \to I^{\alpha(i,\varrho(i))}$$

is in $\mathcal{M}(\Delta_{i,\varrho(i)})$. Since $\nu_*^{(i)}|X_{q,q-i} = \nu_q^{(i)}$, then by (6.111),

$$\nu_*(z) = \nu_q(z) = \{\nu_*^{(i)}(u), 0, \varrho(i)\}, \quad u \in X_{q,q-i} \subseteq \Phi_i;$$

and therefore, by (5.11b), $\nu_*$ is in $\mathcal{M}(\Delta)$. The required result now follows by induction on $\alpha$.

A sort of converse to (6.11) is

(6.12) LEMMA. *Given $\mu: \Phi \to I^\alpha$ in $\mathcal{M}(\Delta)$, then for each $q \in \mathfrak{I}$,*

$$\mu|Z_q(\Phi) < \mu|Z_{q+1}(\Phi).$$

The proof is an immediate consequence of (6.6), (6.9), and the definition (6.10).

We are now in a position to state (5.142) in the following form:

(6.13) LEMMA. *Given the Borel map $f^\alpha: \Delta \to \mathcal{G}$, a necessary and sufficient condition that $x \in M(f^\alpha)$ is that there exists*

(i) *a thread $\mathfrak{a} \in x$,*

(ii) *for each $q$ an element $\nu_q: X_q \to I^\alpha$ of $\mathcal{Z}_q(\Delta)$ satisfying $\nu_q < \nu_{q+1}$, such that for each $q \in I$ and [14]) $z \in X_q$,*

(iii) $\left(f^\alpha(z \times \nu_q(z))\right)_{q-z} (\mathfrak{a}_q, \mathfrak{a}_{q+1}) = 1.$

[14]) We recall from (5.8) and (5.14) that $z = \sigma(z)$.

Proof. The necessity follows from (5.142), with $X_q = Z_q(\Phi)$, by (6.12); for "$r$" in (5.142) is just $q-z$ by definition. The sufficiency follows because by (ii) and (6.7), if $X = \bigcup_{q=0}^{\infty} X_q$, then $X \in \mathcal{E}(\Delta)$, and $X_q = Z_q(X)$; and since $\nu_q < \nu_{q+1}$, then by (6.11), $\nu_*\colon X \to I^a$ is in $\mathcal{M}(\Delta)$, and $\nu_*|X_q = \nu_q$, so that (iii) is exactly (5.142) with $r = q-z$. Therefore, by (5.142), $[\mathfrak{a}] \in M(f^a)$, as required.

Using the last lemma, we now construct a $\mathfrak{P}$-function $Q\colon I^4 \to \mathfrak{P}$, to be used as a "test" function (described in (6.15) below). Given $f^a\colon \Delta \to \mathcal{G}$, then $\Delta$ is a countable set, and therefore by (6.2), so is $\mathcal{Z}_q(\Delta)$ for each $q \in \mathfrak{J}$. If $X \in \mathcal{Z}(\Delta)$, then the number of maps $\nu\colon X \to I^a$ is countable, because $X$ is finite; hence $\mathcal{M}_q(\Delta)$ is countable, with an enumeration whose $j$th element is $\eta_{qj}\colon X_{qj} \to I^a$. We recall from Section 1 the enumeration functions $C_2\colon I^2 \to \mathfrak{J}$, and $F_q\colon \mathfrak{S}_q \to \mathfrak{J}$; and note that $\sum_{\nu+1} = Z_q(\mathfrak{J})$.

Now define $Q\colon I^4 \to \mathfrak{P}$ by:

$Q_q\big(C_2(j,r), C_2(k,s)\big) = 1$ *if and only if*

(6.14) (a) $\eta_{qj} < \eta_{q+1,k}$ *and* $F_{q+1}(r) < F_{q+2}(s)$,

(b) *for all* $z \in X_{qj}$,

$$f^a\big(z \times \eta_{qj}(z)\big)_{q-z}\big(F_{q+1}(r)(q), F_{q+1}(r)(q+1)\big) = 1,$$

(c) *for all* $z \in X_{q+1,k}$,

$$f^a\big(z \times \eta_{q+1,k}(z)\big)_{q+1-z}\big(F_{q+2}(s)(q+1), F_{q+2}(s)(q+2)\big) = 1.$$

*For all other quadruples* $(n, i, m, j)$,

$$Q(n, i, m, j) = 0.$$

Then we have

(6.15) Lemma. *A necessary and sufficient condition that* $M(f^a) \neq 0$ *is that* $Q$ *contains a thread.*

Proof. By (6.13), if $M(f^a) \neq 0$, then (i), (ii) and (iii) are satisfied. For, let $\nu_q$ be $\eta_{q,j(q)}\colon X_{q,j(q)} \to I^a$ in the above enumeration of $\mathcal{M}_q(\Delta)$, and let $\mathfrak{a}|Z_q(\mathfrak{J})$ be $F_{q+1}(r_q)$. Then by definition of $Q$, and by (iii),

$$Q_q\big(C^2(j(q), r_q), C^2(j(q+1), r_{q+1})\big) = 1$$

for each $q \in \mathfrak{J}$; so that if we define $\mathfrak{b}\colon \mathfrak{J} \to \mathfrak{J}$ by

$$\mathfrak{b}(q) = C^2\big(j(q), r_q\big)$$

then $\mathfrak{b}$ is a thread in $Q$, as required.

Conversely, given a thread $\mathfrak{b}$ in $Q$, the argument reverses in the obvious way. The proof is then complete.



## 7. Operations with Borel maps

In Section 5 we shewed how the Borel maps gave a model of the Borel sets of the space $A$. Our next task is to find operations which model those of intersection, and complementation. With the notation of (5.84) we shall write

(7.1) $$f^a = \bigwedge_i \bigvee_j f_{ij}.$$

If $f, g \in \mathcal{G}$, define a new element of $\mathcal{G}$, $f \circ g$, by

(7.1a) $$(f \circ g)(n, i, m, j) = \min[f(n, i, m, j), g(n, i, m, j)].$$

Clearly, $f \circ g \in \mathcal{G}$; moreover

(7.2) $$M(f \circ g) = M(f) \cap M(g).$$

For if $\mathfrak{t}$ is a thread in $f \circ g$, we have for each $n \in \mathfrak{J}$,

$$1 = (f \circ g)_n(t_n, t_{n+1}) = \min[f_n(t_n, t_{n+1}), g_n(t_n, t_{n+1})]$$

whence $1 = f_n(t_n, t_{n+1}) = g_n(t_n, t_{n+1})$, so that $\mathfrak{t}$ is a thread in both $f$ and $g$. Hence

(7.21) $$M(f \circ g) \subseteq M(f) \cap M(g).$$

Conversely, if $x \in M(f) \cap M(g)$, there exist, in $A$, threads $\mathfrak{s} \subseteq f$, $\mathfrak{t} \subseteq g$, such that $x = [\mathfrak{s}] = [\mathfrak{t}]$. Hence there is by definition a thread $\mathfrak{u} \subseteq A$, with $\mathfrak{u} < \mathfrak{s}$, $\mathfrak{u} < \mathfrak{t}$; and so by (5.4), $\mathfrak{u} \subseteq f$ and $\mathfrak{u} \subseteq g$. Therefore by definition and by (7.1a), $\mathfrak{u} \subseteq f \circ g$. This establishes the inclusion reverse to (7.21), and (7.2) follows.

Now suppose that we have defined a product $f^a \circ g^\beta$ of the maps $f^a$, $g^\beta$ for all $a, \beta < \gamma$ and such that

(7.3) $$M(f^a \circ g^\beta) = M(f^a) \cap M(g^\beta).$$

Then we can define $f^\gamma \circ g^\beta$ $(\gamma \geqslant \beta)$ by

$$f^\gamma \circ g^\beta = \bigwedge_i \bigvee_j (f^\gamma \circ g^\beta)_{ij}, \qquad (f^\gamma \circ g^\beta)_{ij} = f^\gamma_{ir} \circ g^\beta_{is},$$

where [15] $C_2(r, s) = j$ and $g^\beta_{is} = g^\beta$ if $\beta = 0$; for $f^\gamma_{ir}$, $g^\beta_{is}$ are Borel maps of orders $< \gamma$. Therefore by (5.9)

$$M(f^\gamma \circ g^\beta) = \bigcap_i \bigcup_j M\big((f^\gamma \circ g^\beta)_{ij}\big) = \bigcap_i \bigcup_j M(f^\gamma_{ir} \circ g^\beta_{is})$$

$$= \bigcap_i \bigcup_j \big(M(f^\gamma_{ir}) \cap M(g^\beta_{is})\big) \quad \text{(by the inductive hypothesis)}$$

$$= \bigcap_i \bigcup_r M(f^\gamma_{ir}) \cap \bigcap_i \bigcup_s M(g^\beta_{is}) = M(f^\gamma) \cap M(g^\beta),$$

[15] $C_2\colon I^2 \to \mathfrak{J}$ is the enumeration function of Section 1.

so that the inductive hypothesis (7.3) is justified. The "product" $f \circ g$ is therefore defined for Borel maps $f$, $g$, of all orders, and satisfies (7.3).

Next, given $f \in \mathcal{G}$ we shall construct a sequence of maps $K_n(f) \in \mathcal{G}$ such that

$$\mathrm{C}M(f) = \bigcup_{n=0}^{\infty} MK_n(f), \tag{7.4}$$

where $\mathrm{C}X$ = complement of $X$ in the space $A$. By definition of $\mathcal{G}$ and by (5.2), $f = B(\bigcap_{n=0}^{\infty} G_n)$ where $G_n$ is open in $A$, and so $M(f) = \bigcap_{n=0}^{\infty} G_n$. It follows quickly from the definitions that also $M(f) = \bigcap_{n=0}^{\infty} G'_n$, where $G'_n$ is the union of all $U_{np}$ for which there exist $m, q \in \mathfrak{J}$ such that $B(n,p,m,q) = 1$. Hence $\mathrm{C}M(f) = \bigcup_{n=0}^{\infty} \mathrm{C}G'_n$; but then each $\mathrm{C}G'_n$ is closed in $A$, and so

$$\mathrm{C}G'_n = \bigcap_{j=0}^{\infty} U\big(G'_n,\ 1/(j+1)\big) = \bigcap_{j=0}^{\infty} G_{nj} \quad \text{(say)}.$$

We define $K_n(f)$ to be $B(\bigcap_{j=0}^{\infty} G_{nj})$; clearly $K_n(f) \in \mathcal{G}$ and satisfies (7.4).

If also $g \in \mathcal{G}$ write

$$K_n(f,g) = f \circ K_n(g) \qquad (\in \mathcal{G}),$$

so that, as one easily verifies,

$$MK_n(f,g) = M(f) \cap \mathrm{C}M(g).$$

Now suppose that $f$, $g$ are Borel maps of orders $\beta, 0$ respectively. Define a Borel map $K_{ij}(f,g)$ of order $\beta$ by

$$K_{ij}(f,g) = f_{ir} \circ K_s(g), \qquad C_2(r,s) = j;$$

so that

$$\begin{aligned}\bigcap_i \bigcup_j MK_{ij}(f,g) &= \bigcap_i \bigcup_j M\big(f_{ir} \circ K_s(g)\big) \\ &= \bigcap_i \bigcup_j \big(M(f_{ir}) \cap MK_s(g)\big) \qquad \text{by ((7.3))} \\ &= \bigcap_i \big[\bigcup_r M(f_{ir}) \cap \bigcup_s MK_s(g)\big] \\ &= \bigcap_i \big[\bigcup_r M(f_{ir}) \cap \mathrm{C}M(g)\big] \qquad \text{(by (7.4))} \\ &= M(f) \cap \mathrm{C}M(g).\end{aligned}$$

Therefore, by (7.1) and (5.9),

$$M(f) \cap \mathrm{C}M(g) = M\big(\bigwedge_i \bigvee_j K_{ij}(f,g)\big). \tag{7.5}$$



If $g^1$ is a Borel map of order 1, the result is different; we have

$$\begin{aligned}M(f) \cap \mathrm{C}M(g^1) &= M(f) \cap \bigcup_i \bigcap_j \mathrm{C}M(g^1_{ij}) \\ &= \bigcup_i \bigcup_j [M(f) \cap \mathrm{C}M(g^1_{ij})] \\ &= \bigcup_i \bigcap_j \big(\bigcap_p \bigcup_q MK_{pq}(f, g^1_{pq})\big) \qquad \text{(by (7.5))} \\ &= \bigcup_i \big(\bigcap_r \bigcup_q M(h_{irq})\big) \qquad \text{say},\end{aligned}$$

where $h_{irq} = K_{pq}(f, g^1_{ij})$, and $C_2(p,j) = r$. Therefore

$$\begin{aligned}M(f) \cap \mathrm{C}M(g^1) &= \bigcup_i M\big(\bigwedge_r \bigvee_q K_{pq}(f, g^1_{ij})\big) \\ &= \bigcup_i K_i(f, g^1) \qquad \text{say},\end{aligned}$$

where $K_i(f,g^1) = \bigwedge_r \bigvee_q K_{pq}(f, g^1_{ij})$, $C_2(p,j) = r$, and $K_i(f,g^1)$ is a Borel map of order $\alpha$.

We can now make the inductive hypothesis that [16]) for each $\alpha$, given Borel maps $f^\alpha$, $g^\beta$, where $0 < \beta < \delta < \alpha$, we can construct a sequence $\{K_i(f^\alpha, g^\beta)\}$ of Borel maps of order $\alpha$, such that

$$M(f^\alpha) \cap \mathrm{C}M(g^\beta) = \bigcup_i MK_i(f^\alpha, g^\beta). \tag{7.6}$$

Then

$$\begin{aligned}M(f^\alpha) \cap \mathrm{C}M(g^\delta) &= M(f^\alpha) \cap \bigcup_i \bigcap_j \mathrm{C}M(g^\delta_{ij}) \\ &= \bigcup_i \bigcap_j M(f^\alpha) \cap \mathrm{C}M(g^\delta_{ij}) \\ &= \bigcup_i \bigcap_j \big[\bigcup_p MK_p(f^\alpha, g^\delta_{ij})\big]\end{aligned}$$

by the inductive hypothesis, since $g^\delta_{ij}$ is now of order $< \delta$. Define

$$K_i(f^\alpha, g^\delta) = \bigwedge_j \bigvee_p K_p(f^\alpha, g^\delta_{ij}),$$

so that by (5.9) and the above

$$M(f^\alpha) \cap \mathrm{C}M(g^\delta) = \bigcup_i MK_i(f^\alpha, g^\delta).$$

Therefore (7.6) is justified for *all* $\beta, \alpha$, by transfinite induction.

[16]) Note that the induction is on $\delta$.

### 8. On propositions which are decidable $D$

We now return to the considerations of Section 1. With the notation used there, we take $\mathcal{C}$ to be the class of all $\mathfrak{P}$-functions, *i. e.* maps of $I^4$ into $\mathfrak{P}$. Then $\mathcal{C} \subseteq \mathcal{M}$ and in what follows we take $h, k$ in (1.1) to be the identity maps. On $\mathcal{C}$ let $R_0(A)$ be the relation "There is no thread in $A$". We recall the functions $C_a\colon I^a \to \mathfrak{J}$ defined in Section 1, with inverse $C_a^{-1} = G_a$. Now in (2.3c), a quadruple $(n, i, m, j)$ gives rise to an ignorable segment in $A^\eta$ if and only if

$$\left[1 - A^\eta(n,i,m,j) + \big(1-(n-m)\big)^2 + \max_{p \in I} A^\eta(m,j,m+1,p)\right] = 0\,.$$

Denote the expression on the left by $H^\eta\big(C_4\,(n,i,m,j)\big)$; we have avoided use of the functions $R_\xi$ of Section 1 for clarity, writing $n-m$ for $R_{40}(n,i,m,j) - R_{43}(n,i,m,j)$. Then, by (2.3c), for each $k \in \mathfrak{J}$,

$$A^{\eta+1}\big(G_4(k)\big) = \min\left[A^\eta\big(G_4(k)\big), H^\eta(k)\right], \tag{8.1}$$

and by (2.4), if $\alpha$ is a limit ordinal,

$$A^\alpha\big(G_4(k)\big) = \min_{\xi<\alpha}\left[A^\xi\big(G_4(k)\big)\right].$$

Let $\beta$ be as in (2.4), and let $g\colon \mathfrak{J} \to \mathfrak{P}$ be $A^\beta G_4$. Then $g$ is clearly countably recursively definable and by (2.6) $R_0(A)$ is equivalent to $g \equiv 0$. Therefore, by definition, $R_0(A)$ is decidable $D$.

(8.2) On $\mathcal{C}$, let $R_i(A)$ be the relation: "$A$ satisfies axiom $\mathfrak{C}_i$", $i = 1, \ldots, 9$ (of Sections 3 and 4). To shew that $R_1(A)$ is decidable $D$ is a simple extension of the above, and the above itself shews that $R_7(A)$ is decidable $D$. For the rest, the corresponding function $g_i$ required can be defined directly according to the following scheme:

$$\mathfrak{C}_2\colon\ g_2(u) = g_2\big(G_4(i,j,n,k)\big) = \min(i,1)\cdot\min(n,1)\cdot A(i,j,n,k)\cdot[1 - A(i-1,j,n-1,k)];$$

$$\mathfrak{C}_3\colon\ g_3(u) = g_3\big(G_6(i,j,k,q,l,m)\big) = \max(i-k+1,0)\cdot\max(1+k-l,0)\cdot A(i,j,k,q)\cdot A(k,q,l,m)\cdot [1 - A(i,j,l,m)];$$

$$\mathfrak{C}_4\colon\ g_4(u) = g_4\big(G_4(i,j,m,k)\big) = A(i,j,m,k)\cdot\max(1+i-m,0)\,[2 - A(i,j,i,k) - A(i,j,i+1,k)];$$

$$\mathfrak{C}_5\colon\ g_5(u) = g_5\big(G_2(p,n)\big) = \operatorname*{Min}_{m\in I} A_{n+m}(p,p);$$



$$\mathfrak{C}_6\colon\ g_6(u) = g_6\big(G_4(n,p,m,q)\big) = A(n,p,m,q)(p-q)\cdot \operatorname*{Min}_{s\in I}\Big\{\operatorname*{Max}_{t,a,b}\big[A(m,q,a,b)A(s,t,a,b)\cdot\big(1 - A(n,p,s,t)\big)\big]\Big\};$$

$$\mathfrak{C}_8\colon\ g_8(u) = g_8\big(G_3(p,q,n)\big) = A\,(p,q)\cdot\sum_{m=0}^{n-1}\big(1 - A(p,p)\big);$$

$$\mathfrak{C}_9\colon\ g_9(u) = g_9\big(G_3(n,p,q)\big) = A_n(p,q)\cdot A_n(p,p)\cdot\big(1 - A^{n+1}(p,q)\big)\,.$$

Clearly each $g_i$ is countably recursively defined, and $R_i(A)$ is equivalent to $g_i \equiv 0$. Hence each $R_i(A)$ is decidable $D$.

Now let us consider Dienes's problem. For this we take $\mathcal{C}$ to be the class of all Borel sets over the complete separable metric space $\boldsymbol{A} = \boldsymbol{A}(\Delta)$ of Section 6. A given Borel set $X$ can be expressed in many ways in the form (5.8) and so $X$ will have associated with it many different maps $f\colon \Delta \to \mathcal{G}$ where $\Delta$ is in some $I^a$ for sufficiently high $a$. We therefore use the axiom of choice, and choose a definite map $f_X\colon \Delta \to \mathcal{G}$ for which $M(f_X) = X$. Then define $h\colon \mathcal{C} \to \mathcal{M}$ by $h(X) = F\colon \Delta \times I^4 \to \mathfrak{P}$, where $F(\boldsymbol{u} \times \boldsymbol{v}) = f_X(\boldsymbol{u})(\boldsymbol{v})$. Define $k\colon h(\mathcal{C}) \to \mathcal{C}$ by $k(F) = M(f_X)$, so that, as in (1.3), $kh = 1$.

Consider first on $\mathcal{C}$ the relation $R(X)$: "$X = 0$". To shew that this is decidable $D$ we construct the $\mathfrak{P}$-function $Q$ of (6.14), and then use the result at the beginning of this section. For, $X = 0$ if and only if $M(f_X) = 0$, and by (6.15), this is equivalent to $R_0(Q)$, where $R_0$ is the relation of having no thread, as in (8.1). It remains to shew that the map $g$ in (8.1) is countably recursively defined rel $h(X)$; and for this it suffices to shew that $Q$ is recursively defined in terms of $f_X$ and $\Delta$. But by (6.14) $Q$ is recursively defined if and only if the same is true of the set of maps $\mu_{qj}\colon X_{qj} \to I^a$ of (6.14); and this follows in obvious fashion by induction on (6.8), since the functions $F_q$ of Section 1 belong to the class of primitive functions. Hence, since $R_0(Q)$ was shewn in (8.1) to be decidable $D$, so is "$X = 0$"[17]).

As Dienes points out in [1], to decide which of the possibilities (i) of the Introduction holds can be reduced to deciding whether or not a certain set $Z$ is empty. This is obvious for the last two of the possibilities, and for the others it follows from the fact that $X \subseteq Y$ if and only if $X \cap CY = 0$. Therefore given the countable specifications $h(X)$, $h(Y)$ with the Borel maps $f_X, f_Y$, then by (7.3), the map $f_X \circ f_Y$ is recur-

[17]) In a different context, we have the following application of the method. For each $n \in \mathfrak{J}$, let $G_n = \{a_{n0}, a_{n1}, \ldots, a_{nm}, \ldots\}$ be a group and $h_n\colon G_{n+1} \to G_n$ a homomorphism. Let $G$ be the inverse limit of the system $\{G_n, h_n\}$. Define $A(n,i,m,j)$ to be 1 if and only if $a_{ni} = h_n h_{n+1} \ldots h_{m-j}(a_{mj})$. Then the above argument (with a slight and obvious modification) shews that "$G$ is trivial" is decidable $D$.

sively defined rel $f_X, f_Y$, and $X \frown Y = M(f_X) \frown M(f_Y) = M(f_X \circ f_Y)$; hence, by the above, "$X \frown Y = 0$" is decidable $D$. To investigate $X \frown \complement Y$, we have, by (7.6),

$$X \frown \complement Y = M(f_X) \frown \complement M(f_Y) = \bigcup_{i=0}^{\infty} MK_i(f_X, f_Y),$$

so that $X \frown \complement Y = 0$ if and only if for each $i \in \mathfrak{I}$, $MK_i(f_X, f_Y) = 0$. But we have shewn, above, that there is a map $g_i: \mathfrak{I} \to \mathfrak{P}$, countably recursively defined rel $K_i(f_X, f_Y)$ such that [18])

$$MK_i(f_X, f_Y) = 0 . \leftrightarrow . g_i \equiv 0 .$$

Define $g: \mathfrak{I} \to \mathfrak{P}$ by

$$g(n) = g_i(j),$$

where $C_2(i,j) = n$. Then $g \equiv 0$ if and only if for each $i, g_i \equiv 0$. But $g$ is clearly countably recursively defined rel $f_X, f_Y$ (and so rel $h(X), h(Y)$), whence "$X \frown \complement Y = 0$" is decidable $D$. In this way all the possibilities (i) of Dienes's problem are decidable $D$.

We conclude by indicating some unsolved problems. Certain topological properties of a set $X$ (given by means of an $A: I^4 \to \mathfrak{P}$) are obviously decidable $D$ (*e. g.* connectedness, having trivial Čech homology groups, etc.); but an obvious one which seems very difficult, is to decide by an operation $D$, whether or not a given pair $X, Y$, of subsets of $A$ are homeomorphic; or even whether or not two $\mathfrak{P}$-functions satisfying axioms $\mathfrak{C}_1$-$\mathfrak{C}_9$ represent homeomorphic spaces. Are there objects with a countable specification, for which certain predicates are undecidable $D$? A more purely topological problem is this: — Suppose that $A$ is a homogeneous space, with the property that, say, all the neighbourhoods $(U_{00}, U_{01}, \ldots)$ are homeomorphic. Then the diagram of the $\mathfrak{P}$-function $A(A)$, regarded as a graph, has the property that, if the part of the graph below each node is $G(P)$, then for all $P, Q$, $G(P)$, $G(Q)$ are isomorphic. Is the number of graphs with this property countable? An affirmative answer might throw light on the conjecture that all (locally connected) homogeneous, connected, locally compact metric spaces are locally Euclidean. What is the number of graphs $G$, if each $G(P)$ is isomorphic to $G$? If we regard a graph as an algebra, this leads us to the following problem. Let $\Sigma$ be a class of algebras (*e. g.* of groups, of rings, ...). How many algebras $X$ are there in $\Sigma$ with the property that all the proper sub-algebras of $X$ are isomorphic (i) to each other, (ii) to $X$?

[18]) We are implicitly assuming here, that if $f \in \mathcal{G}$, then the sequence $K_n(f)$ of (7.4) is recursively definable in terms of $f$. From the definition of $K_n(f)$ given after (7.4), it is clear that $K_n(f)$ is so definable, but for brevity we omit explicit display of a functional representation of $K_n(f)$ of the sort (8.1).



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*Reçu par la Rédaction le 19. 11. 1955*