de deux ensembles de classe $M_4$ est aussi de classe $M_4$. Pourtant j'ai démontré plus haut que l'ensemble $D_m$ ne peut appartenir à la classe $M_4$, l'hypothèse mène donc à une contradiction.

Travaux cités


Recu par la Révision le 17. 11. 1954

Derivate planes of continuous functions of two real variables

by

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1. Introduction

We study here the problem of sufficient conditions for the existence of a derivate plane to a continuous function of two real variables. The work is based on the paper [2] (henceforth referred to as D. A.) in which are contained all the basic definitions. Speaking loosely, we find that there are two related sufficient conditions for the existence of a derivative plane: that of continuity with respect to $x, y$ of a pair of derivate in two continuously varying directions, and equality of upper and lower derivate in two continuously varying directions. These two conditions combine to yield sufficient conditions for the existence of a derivative plane both at a point (Theorem 2) and in a "global" sense (Theorem 1).

2. Notation

From Theorem 1 of D. A. we obtain the following theorem, which will be basic for the present work:

THEOREM A. If $f(z)$ has finite derivate on a region $R$, then there is a region $R'$ everywhere dense on $R$, such that at every point $z_0$ of $R'$, $D^f(z_0) = D^f(z_0)$, $D_0^f(z_0) = D_0^f(z_0)$, all $\theta$,
and $D^f(z_0)$ and $D_0^f(z_0)$ are continuous functions of $\theta$.
If the derivate of $f(z)$ is bounded on $R$, then $R' = R$.

This result is of twofold significance: first, we may ignore the distinction between $D^f(z_0)$ and $D_0^f(z_0)$; and second, more importantly, we may use the continuity of $D^f(z_0)$ as an essential step of all "category" arguments.

The statement "there exists a derivate plane to $f(z)$ at $z_0$" is defined by the equation

$$ D^f(z_0) = D_0^f(z_0) = \cos \theta \cdot D^f(z_0) + \sin \theta \cdot D_0^f(z_0) $$
for all $\theta$. Given two directions $\lambda$ and $\mu$ not diametrically opposed, such that $D^\lambda f(z_0)$ and $D^\mu f(z_0)$ are finite, a more general form of (1) is

$$D^\lambda f(z_0) = D_0 f(z_0) - \frac{\sin(\mu - \lambda)}{\sin(\lambda - \mu)} [\sin(\theta - \lambda) D^\lambda f(z_0) - \sin(\theta - \mu) D^\mu f(z_0)].$$

We denote this last expression by $P^{\lambda\mu}(z_0)$, and call it the upper $\lambda, \mu; \theta$ plane derivative. Then the definition of the existence of a derivative plane becomes

$$D^\lambda f(z_0) = D_0 f(z_0) - P^{\lambda\mu}(z_0), \quad \forall \theta, \lambda, \mu,$$

for any pair of directions $\lambda, \mu$ not diametrically opposed. We note that by using lower derivates throughout, we may define $P_{\lambda, \mu}(z_0)$, the lower $\lambda, \mu; \theta$ plane derivative.

It is convenient to define another type of derivative,

$$E^{\lambda\mu}(z_0) = D^\lambda f(z_0) - P^{\lambda\mu}(z_0) = D_0 f(z_0) - P_{\lambda, \mu}(z_0),$$

the upper and lower $\lambda, \mu; \theta$ error derivates, respectively. Then the existence of a derivative plane is equivalent to the pair of inequalities

$$E^{\lambda\mu}(z_0) < 0, \quad E_{\lambda, \mu}(z_0) > 0, \quad \forall \theta.$$  

We see this readily by observing first that the inequalities (4) imply $E^{\lambda\mu}(z_0) = P^{\lambda\mu}(z_0) > 0, \forall \theta$. This is possible only in the case of equality, since the left hand side is equal to $A \sin(\theta - B)$ for some constant $A$ and $B$. Then from the equality of the plane derivatives we easily deduce $D^\lambda f(z_0) = D_0 f(z_0), \forall \theta$, and (3) is immediate.

Throughout the paper, we shall use the symbols $a(z)$ and $b(z)$ to denote directions varying continuously with $z$. Expressions like $D^\lambda f(z)$ will sometimes be abbreviated to $D^\lambda f(z)$; it will always be understood that the direction $a(z)$ is evaluated at the point $z$ appearing as the argument of $f(z)$.

Finally, we recall the definitions $D^\lambda f(z_0) = \sup_{\theta \in [0,2\pi]} D^\lambda f(z_0)$, and $D_0 f(z_0) = \inf_{\theta \in [0,2\pi]} D^\lambda f(z_0)$, from D.A.

3. Preliminary Lemmas

**Lemma 1.** (a) For a continuous function $f(z)$ and a continuously varying direction $a(z)$, $D^a f(z_0)$ is a Borel-measurable function of $z$.

(b) Let $f(z) = f_1(z) + f_2(z)$, and suppose that for some point $z^*$ and direction $\mu$, $+\infty > D^\mu f(z^*) = D_0 f(z^*) > -\infty$. Then

$$D^\mu f(z^*) = D^\mu f(z^*) + D^\mu f(z^*).$$

(c) For a fixed point $z_0$ and three directions $\lambda, \mu, \nu$ such that $D^\lambda f(z_0)$, $D^\mu f(z_0)$ and $D^\nu f(z_0)$ are finite, we have

$$\sin(\mu - \lambda) E^{\lambda\mu}(z_0) = \sin(\nu - \mu) E^{\nu\mu}(z_0) = \sin(\lambda - \nu) E^{\lambda\nu}(z_0).$$

(d) Suppose that $D^\mu f(z_0) > M$. Then given any pair of directions $\lambda, \mu$ not diametrically opposed, and any pair of directions $\theta, \sigma$, we have

$$|\sin(\lambda - \mu)| |P^{\lambda\mu}(z_0) - P^{\sigma\mu}(z_0)| < 2M|0 - \sigma|.$$ 

**Proofs.** (a) It will be sufficient to show that for any $K$, the set $T^K = \{z; D^\mu f(z) > K\}$ is a $G_\delta$ set. We consider the set $\mathcal{C} = \{z, 0; D^\mu f(z) > K\}$ as a set of points in Euclidean three-space, as in D.A. § 2, and let $\mathcal{Q} = \{z, 0; \theta = a(z)\}$. Since $a(z)$ is continuous, then $\mathcal{Q}$ is closed, and so $\mathcal{C} \cap \mathcal{Q}$ is a $G_\delta$ set in $E_2$. Since $a(z)$ is single-valued, then $T^K = \{z; \mu = a(z)\}$, and it remains to show that the projection takes $G_\delta$ sets into $G_\delta$ sets. This follows readily if we consider $\mathcal{Q}$ as a space with a metric defined by $q(p_1, p_2) = |z_1 - z_2|$ for any points $p_i = (z, a(z_i))$, $i = 1, 2$, of $\mathcal{Q}$; the projection is then a homeomorphism.

(b) Let $(h_{n, j}) (n = 1, 2, \ldots)$, for $j = 0, 1, 2, \ldots$, be three sequences of complex numbers such that $|h_{n, j}| \to 0$ and $\arg h_{n, j} \to \mu$ as $n \to \infty$, and

$$D^\mu f(z) = \lim_{n \to \infty} |\sum j \in |1/2| \{f(z + h_{n, j}) - f(z^*)\}|.$$  

For any function $k(z)$ denote by $d_k k(z)$ the difference quotient $d_k z = k(z + h_{n, j}) - k(z)$. By the definition of the directed derivative, we have

$$D^\mu f(z) = \lim_{n \to \infty} D_k f(z^*), \quad j = 0, 1, 2, \ldots, \quad \lim_{n \to \infty} D_k f(z^*) = \lim_{n \to \infty} d_k f(z^*).$$

On the other hand, we have

$$\Delta_{\mu} f(z^*) = \Delta_{\mu} f(z^*) + d_k f(z^*)$$

for any $\mu$. Taking upper limits, we have

$$\lim_{n \to \infty} d_k f(z^*) = \lim_{n \to \infty} D_k f(z^*) + \lim_{n \to \infty} \Delta_{\mu} f(z^*),$$

equivalent to $D^\mu f(z^*) = D^\mu f(z^*) + D^\mu f(z^*)$, as desired.

(c) and (d) are elementary.

**Lemma 2.** For a continuous function $f(z)$ and a point $z_0$, suppose that $D^\lambda f(z_0)$ and $D^\mu f(z_0)$ are finite, where $\lambda$ and $\nu$ are directions such that $\lambda < \nu < \lambda + \pi$. Let $\epsilon$ and $\theta_0$ be arbitrary positive numbers.

Then for any direction $\theta$ in the area $(\nu, \lambda + \pi)$, the inequalities

$$D^\lambda f(z_0) > P^{\lambda\mu}(z_0) + \epsilon,$$

$$D_0 f(z_0) < P^{\lambda\mu}(z_0) + \epsilon$$

are satisfied by points $z = z(\theta)$, $z = z(\theta)$ lying in $[z - \zeta] < \theta$.
Without loss of generality, we assume \( z_0 = 0 \), \( f(0) = 0 \). Let \( \mu \) be an arbitrary direction in the arc \((\nu, \lambda + \pi)\). Let \( g(z) = g(x, y) = Ax + By \) \((A, B \text{ real constants})\) be such that \( D^\nu g(z) = D^\nu f(0) \), \( D^\lambda g(z) = D^\lambda f(0) \), for all \( z \).

Then for any \( z \) and \( \theta \),
\[
D^{\phi \nu \lambda \theta} g(z) = D^{\phi \nu \lambda \theta} f(0) + \nabla^\phi g(0) \cdot \nabla^\theta f(0).
\]
Let \( h(z) = f(z) - g(z) \). Then by Lemma 1(b), for any \( z \) and \( \theta \),
\[
D^\nu h(z) = D^\nu f(z) - D^\nu g(z) = D^\nu f(0) - D^\nu g(0) = D^\nu f(0) - \nabla^\nu g(0) \cdot \nabla^\nu f(0).
\]
Hence \( D^\nu h(0) = D^\nu f(0) = 0 \).

We consider pairs of points \( z, x^* \) with the following properties:
(i) \( \arg z = \lambda \),
(ii) \( \min \left| f_2 - f_1 \right| < \min \left| f_2 - f_1 \right| / \sin \theta \),
(iii) \( \arg(x^* - z) = \mu \).

We note that to each \( x^* \) there corresponds just one \( z \). It is clear that the upper bounds of the ratios
\[
\frac{|z|}{|x^*|}, \frac{|x^*|}{|z|}
\]
are finite. Denote them by \( M_1, M_2, M_3 \) respectively.

Since \( D^\nu h(0) = 0 \), there exists a positive \( \rho_4 \) such that for \( |z| < \rho_4 \), \( h(z) < \rho_2 / 2M_2 \). By the construction, there exists a positive \( \rho_3 \) such that for any \( z^* \) in \( |z| < \rho_3 \), the segment \( z^* \) lies in the region \( |z| < \min (\rho_2, \rho_3) \). Since \( D^\nu h(0) = 0 \), there exist points \( z^* \) such that \( |z^*| < \rho_2 \) and \( h(z^*) > -|z^*| / 2M_2 \). We choose such a point \( z^* \), denote it by \( z_1 \), and let \( z_1 \) be the unique point \( z \) corresponding to it.

Then we have
\[
\frac{h(z_1) - h(z)}{|z_1 - z|} > \frac{-|z_1|}{2M_2 |z_1 - z|} > \frac{|z_1|}{2M_2 |z_1 - z|} > -\varepsilon.
\]
Hence on the segment \( z_1 z_2 \), there exists a point \( z' \) such that
\[
-\varepsilon < D^\nu h(z') = D^\nu f(z') - D^\nu g(z') = D^\nu f(z') - \nabla^\nu g(0),
\]
as desired. Changing signs in (7) gives us \( |z_1 - z_2| > h(z_1) - h(z_2) \); hence on \( z_1 z_2 \) there exists a point \( z'' \) such that
\[
\varepsilon > D_{\lambda \nu \theta} h(z'') = D_{\lambda \nu \theta} f(z'') - \nabla^\lambda \nu \theta \nu \lambda \theta f(0).
\]

Thus we have established (5) for \( \theta \) in \((\nu, \nu + \pi)\) and (6) for \( \theta \) in \((\nu + \pi, \lambda)\); the constructions are the same. Since \( z' \) and \( z'' \) are constructed to lie in \( |z| < \rho_4 \), the proof is complete.

Note. If in the conditions of the Lemma, we have \( D^\nu f(z_0) \) and \( D^\nu f(z_0) \) finite, then inequalities (5) and (6) become
\[
D^\nu f(z') > P_{\lambda \nu \theta} f(z_0) - \varepsilon,
\]
\[
D^\nu f(z') < P_{\lambda \nu \theta} f(z_0) + \varepsilon
\]
respectively, with the same conditions on \( \theta \).

**Lemma 3.** Let \( \nu \) be a point in \( N(z_0) \) of a point \( \nu \), \( D^\nu f(z) < M \), then for any pair of directions \( \theta \),
\[
|D^\nu f(z) - D^\nu f(z_0)| < M |\theta - \nu|.
\]

Suppose the lemma to be false, and that
\[
\sup \left| |\theta - \nu|^{-1} |D^\nu f(z) - D^\nu f(z_0)| \right| > \eta M,
\]
for some \( \eta > 1 \). Then there exists a pair of directions \( \nu, \nu \) such that
\[
|\nu - \nu| < 2 \cos \eta^{-1}
\]
and
\[
D^\nu f(z) - D^\nu f(z_0) > \eta M |\nu - \nu|.
\]
Without loss of generality, we may assume \( \nu < \nu < \nu + \pi \).

Let \( \xi \) be the midpoint of the arc \((\nu, \nu)\), and let \( \mu = \xi - \pi \). We have
\[
\sin(\mu - \lambda) = \sin(\mu - \nu) = -\cos(\nu - \lambda),
\]
and so
\[
P_{\lambda \nu \theta} f(z_0) = \cos(\nu - \lambda) \frac{|D^\nu f(z_0) - D^\nu f(z_0)|}{\sin(\nu - \lambda)} > \eta M \frac{|\nu - \nu|}{\sin(\nu - \lambda)} > \eta^2 M.
\]
Observing that \( \mu \) is in the arc \((\nu + \pi, \lambda)\), we apply Lemma 2, with \( N(z_0) \) as the given neighbourhood and \( \varepsilon = (\eta^2 - \eta - \eta^2) \). Then in \( N(z_0) \) we obtain points \( z' \) such that
\[
\eta M - (\eta^2 - \eta - \eta^2) M = \eta M < D^\nu f(z') < D^\nu f(z'),
\]
The desired contradiction.

**Note.** Since \( D^\nu f(z) < M \) for \( z \in N(z_0) \) trivially implies \( D^\nu f(z) < M \) for \( z \in N(z_0) \), and conversely, we see that upper derivates may be replaced by lower either in the conditions or conclusions of the Lemma.

**Lemma 4.** Let \( f(z) \) be a continuous function defined on a region \( R \), such that \( D^\nu f(z) < M \), for \( z \in R \). Let \( \nu(z) \) be a direction defined on \( R \), varying continuously with \( z \).

Then
\[
\sup_{z \in R} D^\nu f(z) = \sup_{z \in R} D^\nu f(z).
\]

Hence, under the same conditions, if \( D^\nu f(z) \) is continuous at \( z_0 \), then \( D^\nu f(z_0) = D^\nu f(z_0) \), and \( D^\nu f(z) \) is also continuous at \( z_0 \).
By Lemma 1(a), we know that $D^j f(z)$ and the other derivatives with continuously varying direction are Borel-measurable functions of $z$. Hence (Kuratowski [1], p. 201) there exists a set $E_0$ residual on $E$, such that all these derivatives (considered as functions of $z$) are continuous on $E_0$ with respect to $E$. Let $E^* = \{E = E_0\}$ then a fortiori we have that the derivatives are continuous on $E^*$ with respect to $E^*$. Also, $E^*$ is residual on $X^{(M)}$.

We recall that the existence of a derivative plane at a point $z_0$ is equivalent to

$$E^{h \mu k}(z_0) < 0 < E_{h \mu k}(z_0)$$

for some pair of fixed directions $\lambda$ and $\mu$ not diametrically opposed, and all directions $\theta$. Let $\Theta(z)$ be an arc of directions, varying with $z$, defined by

$$\Theta(z) = (\theta; \Theta(z)/\pi - \pi < \theta < 2\pi).$$

We prove the theorem by studying sets of points related to the inequality

$$(11) \quad E^{h \mu k}(z_0) < 0 < E_{h \mu k}(z_0).$$

Let $G$ be the set of points $z_0$ for which (11) is satisfied for all $\theta \in \Theta(z_0)$. We shall show that $G$ is residual on $E^{(M)}$. The proof is by contradiction. We suppose that $G$ is not residual on $E^{(M)}$; then there is a set $U$ of the second category consisting of points $z_1$ for which either

$$\sup_{\theta \in \Theta(z_1)} E^{h \mu k}(z_1) > 0 \quad \text{or} \quad \inf_{\theta \in \Theta(z_1)} E_{h \mu k}(z_1) < 0.$$ 

We suppose the first inequality to hold at all points $z_1$ of $U$.

Since $U \subset E^{(M)}$ is included in the set $E^*$ of Theorem A, then $D^j f(z_0)$ is a continuous function of $\theta$ for fixed $z_1$. Let $(\theta)$ be the rational directions. Then $U = \bigcup_{\theta \in \Theta} A_\theta$, where

$$A_\theta = \{z \in E^{(M)}; \theta \in \Theta(z), E^{h \mu k}(z) > j^{-1}\},$$

and one of the sets $A_\theta$ is of the second category. Abbreviating the notation, we set this set $A_\theta = B$, where

$$B = \{z \in E^{(M)}; \mu \in \Theta(z), E^{h \mu k}(z) > j^{-1}\},$$

with $\mu = \theta$, $\eta = j^{-1}$. The set $E^*$ being residual on $E^{(M)}$, we have that $E^* \cup B$ is a set of the second category on $E^{(M)}$.

Let $A$ be the subset of $E^{(M)}$ consisting of points $z_0$ such that for every $\theta, \eta > 0$, the set $\{z \in E^{(M)}; E^* \cup B\}$ is of the second category. Then there exists an open set $\mathcal{D}$ such that $A$ is everywhere dense on $\mathcal{D}$. If not, then $A$ is a nowhere-dense set; let $E^{(M)} - A = \bigcup_{i=1}^{\infty} O_i$, where each $O_i$ is
open. Then \( E^* \cdot B \cdot E^{(0)} = E^* \cdot B \cdot \overline{A} + \bigcup_{i=1}^{m} E^* \cdot B \cdot O_i \). For each point \( x_0 \) of \( O_i \), there exists a positive \( \varepsilon_0 \) such that \( E^* \cdot B \cdot |x - x_0| < \varepsilon_0 \) is of the first category. We may cover \( O_i \) by a denumerable set of such neighbourhoods, denote them by \( O_{ij} \), and then we have

\[
E^* \cdot B \cdot E^{(0)} = E^* \cdot B \cdot \overline{A} + \bigcup_{i=1}^{m} E^* \cdot B \cdot O_{ij}.
\]

Each of the sets in the denumerable union is of the first category; hence \( E^* \cdot B \cdot E^{(0)} \) is as well. This contradicts the original assumption; hence an open set \( D \) exists on which \( A \) is everywhere dense.

We choose a point \( z^* \) in \( D \), and let \( \varepsilon_1 \) be such that \( |x - z^*| < \varepsilon_1 \) is included in \( D \). From the definitions, we have \( \mu = \theta(z^*) \), and \( \mu \neq a(z^*) \). Hence \( \mu \) lies in one of the open arcs \( (b(z^*) - \pi, a(z^*)) \), \( (a(z^*), b(z^*)) \), \( (b(z^*), a(z^*) + \pi) \). We assume that it lies in the first, and that for all points \( x \) in \( R_2 = |x - z^*| < \varepsilon_1 \), \( \mu \) lies in \( (b(x) - \pi, a(x)) \). The subsequent constructions depend in detail upon the choice of \( \mu \), but they are all substantially the same.

There exist positive \( \varepsilon \) and \( \varepsilon_1 \) such that for \( z \) in \( R_1 = |x - z^*| < \varepsilon_1 \),

\[
\sin (a(z) - \mu) > 4\varepsilon, \quad \sin (b(z) - a(z)) > 4\varepsilon.
\]

We already have that \( D \) has no limit for \( z \in R_1 \). There exists a positive \( \varepsilon_2 > 0 \) such that the region \( R_3 = |x - z^*| < \varepsilon_2 \) has the following properties:

(i) The variation of \( D_{z_0}^* \) over all points of \( E^* \cdot R_2 \) is less than \( \varepsilon_3 \).

(ii) For \( z \in R_3 \), the set of values \( b(z) \) consists of an arc \( \Phi \) such that

\[
\Phi < 2K\varepsilon M.
\]

Applying (13) to Lemma 1(d), we obtain the result that for any point \( z_0 \in R_3 \) and any pair of directions \( \beta, \gamma = \varepsilon \),

\[
|\rho^{(a_0\beta\gamma)}(z_0) - \rho^{(a_0\beta\gamma)}(z_0)| < K_7.
\]

We now show that the assumption that \( E^* \cdot B \) is everywhere dense on \( R_2 \) leads to a contradiction. This will imply that \( B \) is of the first category, hence that \( U \) is of the first category, and finally that \( G \) is residual, as desired.

Let \( B' = E^* \cdot B \cdot B_0 \). By Lemma 1(c), we have

\[
\sin (b - a) E^* \cdot B \cdot \rho^{(a_0\beta\gamma)}(z) = \sin (a - \mu) E^* \cdot B \cdot \rho^{(a_0\beta\gamma)}(z),
\]

and hence by the definition of \( B_0 \),

\[
\eta < \rho^{(a_0\beta\gamma)}(z) = \frac{\sin (a(z) - \mu)}{\sin (b(z) - a(z))} E^* \cdot B \cdot \rho^{(a_0\beta\gamma)}(z) \quad \text{for} \quad z \in B'.
\]

Since \( B' \subseteq B \), we have by (12),

\[
0 < \sin (a - \mu) < 1, \quad 0 < 4K < \sin (b - a),
\]

and so from (15),

\[
4K < \rho^{(a_0\beta\gamma)}(z) = D^f(z) = D_{z_0}^f(z) - \rho^{(a_0\beta\gamma)}(z_0) = D_{z_0}^f(z) - \rho^{(a_0\beta\gamma)}(z_0),
\]

for \( z \in B' \). Hence, for \( z \in B' \),

\[
\rho^{(a_0\beta\gamma)}(z) < D_{z_0}^f(z) - 4K_7.
\]

Let \( L = \sup_{z \in B'} D_{z_0}^f(z) \). Then by property (1) of \( R_0 \),

\[
L > D_{z_0}^f(z) > L - K_7,
\]

and

\[
\rho^{(a_0\beta\gamma)}(z) < L - 4K_7 \quad \text{for} \quad z \in B'.
\]

Let \( \beta_i \) be a set of directions everywhere dense on \( \Phi \). Then for any \( i \) and any point \( z \in B' \), we have by (18),

\[
|\rho^{(a_0\beta_i\gamma)}(z) - \rho^{(a_0\beta_i\gamma)}(z_0)| < K_7
\]

and so

\[
\rho^{(a_0\beta_i\gamma)}(z) < L - 3K_7 \quad \text{for} \quad z \in B'.
\]

Let \( C_i = \{ z \in R_4, D_{z_0}^f(z) < L - 2K_7 \} \). We apply Lemma 2 to (19), with \( \theta = K_7 \), with \( \beta_0 \) lies in the arc \( a(z) < \theta < a(z) + \pi \), for \( z \in R_4 \), then in every neighbourhood of \( B' \) there is a point of \( C_i \). Hence each of the sets \( C_i \) is everywhere dense on \( R_4 \); being \( G \) sets, each of them is residual on \( R_4 \), and so their intersection \( C \) is also residual on \( R_4 \). For all points \( z \) of \( C \), we have \( D_{z_0}^f(z) < L - 2K_7 \) \((i=1,2,\ldots) \). By the continuity of the derivatives at a fixed point, we also have \( D_{z_0}^f(z) < L - 2K_7 \) for \( \theta \in \Phi \). In particular,

\[
D_{z_0}^f(z_0) < L - 2K_7, \quad z \in C.
\]

Since \( B' \) is of the second category on \( R_0 \), and \( C \) is residual on \( R_0 \), then \( B' \cdot C \) is not empty. At a point \( z_0 \) of \( B' \cdot C \), we have by (17) and (19)

\[
L - 2K_7 > D_{z_0}^f(z) > L - K_7;
\]

this is the desired contradiction.

We now extend inequality (11) to the full circle of directions. Let \( H \) be the set of points \( z \) in \( R^{(0)} \) at which either \( E^{(a_0\beta\gamma)}(z) > 0 \) or \( E^{(a_0\beta\gamma)}(z) < 0 \) for some \( \lambda \) such that \( a(z) + \pi < \lambda < b(z) - \pi \). We will show that \( H \) is a set of the first category; this will complete the proof.

If \( H \) is a set of the second category on \( R^{(0)} \), then \( G \) is residual on \( R^{(0)} \); \( G \cdot H \) is also of the second category on \( R^{(0)} \). Let \( \beta_0 \) be the rational directions, and let \( \Phi(z) \) denote the arc \( a(z) + \pi < \theta < b(z) - \pi \).
Denote by \( P' \) the set \( \{ z; z \in \mathcal{G} \cdot H, \theta \in \mathcal{G}'(z), \mathcal{P}' / \mathcal{P}_i(z) > 0 \} \), and by \( P_i \) the corresponding set involving lower derivates. By the continuity of the derivates at a fixed point, we have \( H = \bigcup_{i=1}^n (P' + F_i) \), and so one of the sets \( P' \) or \( P_i \) must be of the second category. Let it be one of the sets \( P' \); abbreviating \( \theta \) to \( \mu \) we have at all points of \( P' \):

\[
0 < E_{\mu}P_i(j) = D'P_i(j) - P_{\mu}P_i(j).
\]

Since \( P' \subset \mathcal{G} \), and \( \mu + \pi \in \Theta(z) \), we have

\[
0 = E_{\mu + \pi}P_i(j) = D_{\mu + \pi}P_i(j) - P_{\mu + \pi}P_i(j) = D_{\mu + \pi}P_i(j) - P_{\mu + \pi}P_i(j) = D_{\mu + \pi}P_i(j) + P_{\mu + \pi}P_i(j),
\]

resulting from \( P_{\mu + \pi}P_i(j) = D_{\mu + \pi}P_i(j) \) for \( z \in \mathcal{G} \cdot \mathcal{E}' \), and (23) from the definition of \( P_{\mu + \pi}P_i(j) \).

Hence, by (21) and (23), we have at all points of \( P' \):

\[
0 < D'P_i(j) + P_{\mu + \pi}P_i(j).
\]

Since the derivates are bounded on \( P' \), then by Theorem 4 of D. A., \( P' \) must be a set of the first category, in contradiction to the assumption that it is of the second category. Therefore, \( H \) is of the first category on \( E_{\mu}P_i(j) \). Then \( H \) is residual on \( R_{\mu}P_i(j) \), and similarly \( \mathcal{G} \cdot \mathcal{E}' \cdot H \). Denoting this last set by \( F_i \), we have that \( F_i \) is residual on \( R_{\mu}P_i(j) \), and (11) is satisfied at each point of \( F_i \) for all directions \( \theta \). Thus the proof is complete.

**Theorem 2.** Let \( f(z) \) be a continuous function, \( z_0 \) a fixed point, and \( a(z) \) and \( b(z) \) two continuously varying directions such that \( a(z_0) \) and \( b(z_0) \) are not diametrically opposed.

If the derivates of \( f(z) \) are bounded in some neighborhood of \( z_0 \), and \( D'f(z) \) and \( D''f(z) \) are continuous functions of \( z \) at \( z_0 \) then there exists a derivate plane to \( f(z) \) at \( z_0 \).

Without loss of generality, we let \( a_0 = 0 \) and \( f(0) = 0 \), and assume that \( a(0) < b(0) < a(0) + \pi \). For some integer \( M \) there exists a positive \( \theta_1 \) such that at all points of the region \( |z| < \theta_1 \),

\[
D'f(z) < M, \quad \sin(b(z) - a(z)) > M^{-1}.
\]

From Lemma 4, we have

\[
D'f(z) = D_0f(z), \quad D''f(z) = D_{0}f(0),
\]

and all four derivates are continuous at \( z = 0 \). We denote \( a(0) \) and \( b(0) \) by \( \lambda \) and \( \nu \), respectively.

By the arguments of § 2, it will be sufficient to establish the inequalities

\[
E_{\mu}E_{\nu}f(0) < 0, \quad E_{\mu}E_{\nu}f(0) > 0,
\]

for all directions \( \theta \). The proof is in two stages: first we establish (25) and (26) for \( \theta \) in \( (\lambda, \nu) \), and then we use this result to prove (25) and (26) for directions \( \theta \) in the remaining arc. No single construction suffices for the first stage of the proof, separate constructions are necessary for (25) and (26) in each of the arcs \( (\lambda, \nu) \), \( (\nu, \lambda + \pi) \). However, these constructions are substantially the same; we shall prove (26) for \( \theta \) in \( (\lambda, \nu) \).

Suppose that for a direction \( \mu \) in the open arc \( (\lambda, \nu) \),

\[
E_{\mu}f(0) = -h < 0.
\]

Let \( |z| < \theta_1 \) be a region at whose points

\[
|D_f(z) - D_f(0)| < h/4M
\]

and

\[
|a(z) - \lambda| < h/6M^2.
\]

Applying Lemma 1(c), we have

\[
\sin(\nu - \lambda) E_{\nu}f(0) = \sin(\mu - \nu) E_{\nu}f(0);
\]

this together with (24) and (27) gives

\[
E_{\nu}f(0) = \frac{\sin(\nu - \lambda)}{\sin(\mu - \nu)} E_{\nu}f(0) = \frac{\sin(\nu - \lambda)}{\sin(\mu - \nu)} h > h/M.
\]

Hence

\[
P_{\nu}f(0) < D_f(0) - h/M.
\]

Observing that \( \lambda \) is in the arc \( (\lambda + \pi, \nu) \), we apply Lemma 2 (inequality (10)) to obtain a point \( z' \) in \( |z| < \theta_1 \) such that

\[
D_f(z') < P_{\nu}f(0) < h/2M < D_f(0) - h/2M,
\]

the last inequality following from (30). By Lemma 3 and (29), we have

\[
D_f(z') - D_f(0) < M|a(z') - \lambda| < h/6M.
\]

Combining (31) and (32), we get

\[
D_f(z') < D_f(0) - h/3M
\]

for a point \( z' \) in \( |z| < \theta_1 \). This is in contradiction to (28). Thus we have proved that for \( \theta \) in \( (\lambda, \nu) \),

\[
E_{\mu}E_{\nu}f(0) > 0.
\]
Assuming that (25) and (26) have been established for all directions \( \theta \) in \( (\pi - \lambda, \lambda + \pi) \), we now complete the proof. We choose two directions \( \xi, \eta \) with the following properties:

(i) \( \lambda + \pi > \xi > \eta > -\pi < \eta < \lambda \).

(ii) The smaller arc with endpoints \( \xi, \eta \) includes the directions \( \lambda + \pi \) and \( -\pi \).

(iii) \( \sin(\xi - \eta) > 1/2M, \quad \sin(\lambda - \eta) > 1/2M \).

By the previous proof, we have

\[
D^0(0) = D_{\xi}(0) = -P^{\text{arc}}(0),
\]

\[
D^0(0) = D_{\eta}(0) = -P^{\text{arc}}(0),
\]

and hence for any direction \( \theta \),

\[
P^{\text{arc}}(0) = \frac{P^{\text{arc}}(0)}{D^0(0)}, \quad \text{etc.}
\]

Considering directions \( \theta \) in the arc \( (\xi, \eta) \), there are four cases for the proof: \( \theta \) lying outside the arc \( (\xi, \eta) \), lying outside \( (-\pi, \pi) \), and for each of these, proofs of (25) and (26). As before, we discuss one case only.

Suppose that for some direction \( \mu \) outside \( (-\pi, \pi) \), and hence in \( (\xi - \pi, \eta - \pi) \), we have

\[
P^{\text{arc}}(0) = h > 0.
\]

By (34) we then have

\[
P^{\text{arc}}(0) = h > 0.
\]

Applying Lemma 1(c), we have

\[
P^{\text{arc}}(0) = \sin(\xi - \eta) \sin(\lambda - \eta) \sin(\mu - \eta) \sin(\mu - \pi) > 0,
\]

the last inequality following from condition (iii) above. We notice that \( \theta \) is in the arc \( (\mu - \pi, \xi) \), and so we may apply Lemma 2, choosing constants as before, to obtain the desired contradiction. We complete the proof.

References


*Reçu pour la Rédaction le 31.10.1955*