Derivate planes of continuous functions of two real variables

by

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1. Introduction

We study here the problem of sufficient conditions for the existence of a derivate plane to a continuous function of two real variables. The work is based on the paper [2] (henceforth referred to as D. A.) in which are contained all the basic definitions. Speaking loosely, we find that there are two related sufficient conditions for the existence of a derivate plane: that of continuity with respect to x,y of a pair of derivates in two continuously varying directions, and equality of upper and lower derivates in two continuously varying directions. These two conditions combine to yield sufficient conditions for the existence of a derivate plane both at a point (Theorem 2) and in a "global" sense (Theorem 1).

2. Notation

From Theorem 1 of D. A. we obtain the following theorem, which will be basic for the present work:

THEOREM A. *If $f(z)$ has finite derivates on a region R, then there is a region R^+ everywhere dense on R, such that at every point z_0 of R^+,*

$$D^\theta f(z_0) = \partial^\theta f(z_0), \qquad D_\theta f(z_0) = \partial_\theta f(z_0), \qquad \textit{all } \theta ,$$

and $D^\theta f(z_0)$ and $D_\theta f(z_0)$ are continuous functions of θ.

If the derivates of $f(z)$ are bounded on R, then $R^+ = R$.

This result is of twofold significance: first, we may ignore the distinction between $D^\theta f(z_0)$ and $\partial^\theta f(z_0)$; and second, more important, we may use the continuity of $D^\theta f(z_0)$ as an essential step of all "category" arguments.

The statement "there exists a derivate plane to $f(z)$ at z_0" is defined by the equation

$$(1) \qquad D^\theta f(z_0) = D_\theta f(z_0) = \cos\theta \cdot D^0 f(z_0) + \sin\theta \cdot D^{\pi/2} f(z_0) ,$$

for all θ. Given two directions λ and μ not diametrically opposed, such that $D^{\lambda}f(z_0)$ and $D^{\mu}f(z_0)$ are finite, a more general form of (1) is

(2) $$D^{\theta}f(z_0)=D_{\theta}f(z_0)=[\sin(\mu-\lambda)]^{-1}[\sin(\theta-\lambda)D^{\mu}f(z_0)-\sin(\theta-\mu)D^{\lambda}f(z_0)].$$

We denote this last expression by $P^{\lambda,\mu;\theta}f(z_0)$, and call it the *upper* $\lambda,\mu;\theta$ *plane derivate.* Then the definition of the existence of a derivate plane becomes

(3) $$D^{\theta}f(z_0)=D_{\theta}f(z_0)=P^{\lambda,\mu;\theta}f(z_0), \quad \text{all } \theta,$$

for any pair of directions λ,μ not diametrically opposed. We note that by using lower derivates throughout, we may define $P_{\lambda,\mu;\theta}f(z_0)$, the *lower* $\lambda,\mu;\theta$ *plane derivate.*

It is convenient to define another type of derivate,

$$E^{\lambda,\mu;\theta}f(z_0)=D^{\theta}f(z_0)-P^{\lambda,\mu;\theta}f(z_0), \qquad E_{\lambda,\mu;\theta}f(z_0)=D_{\theta}f(z_0)-P_{\lambda,\mu;\theta}f(z_0),$$

the *upper and lower* $\lambda,\mu;\theta$ *error derivates,* respectively. Then the existence of a derivate plane is equivalent to the pair of inequalities

(4) $$E^{\lambda,\mu;\theta}f(z_0)\leqslant 0, \quad E_{\lambda,\mu;\theta}f(z_0)\geqslant 0, \quad \text{for all } \theta.$$

We see this readily by observing first that the inequalities (4) imply $P^{\lambda,\mu;\theta}f(z_0)-P_{\lambda,\mu;\theta}f(z_0)\geqslant 0$, all θ. This is possible only in the case of equality, since the left hand side is equal to $A\sin(\theta-B)$ for some constant A and B. Then from the equality of the plane derivates we easily deduce $D^{\theta}f(z_0)=D_{\theta}f(z_0)$, all θ, and (3) is immediate.

Throughout the paper, we shall use the symbols $a(z)$ and $b(z)$ to denote directions varying continuously with z. Expressions like $D^{a(z)}f(z)$ will sometimes be abbreviated to $D^{a}f(z)$; it will always be understood that the direction $a(z)$ is evaluated at the point z appearing as the argument of $f(z)$.

Finally, we recall the definitions $\mathfrak{D}^{*}f(z_0)=\sup_{0\leqslant\theta\leqslant 2\pi}D^{\theta}f(z_0)$, and $\mathfrak{D}_{*}f(z_0)=\inf_{0\leqslant\theta\leqslant 2\pi}D_{\theta}f(z_0)$, from D. A.

3. Preliminary Lemmas

LEMMA 1. (a) *For a continuous function $f(z)$ and a continuously varying direction $a(z)$, $D^{a}f(z)$ is a Borel-measurable function of z.*

(b) *Let $f_0(z)=f_1(z)+f_2(z)$, and suppose that for some point z^* and direction μ,*

$$+\infty>D^{\mu}f_2(z^*)=D_{\mu}f_2(z^*)>-\infty.$$

Then
$$D^{\mu}f_0(z^*)=D^{\mu}f_1(z^*)+D^{\mu}f_2(z^*).$$

(c) *For a fixed point z_0 and three directions λ,μ,ν such that $D^{\lambda}f(z_0)$, $D^{\mu}f(z_0)$ and $D^{\nu}f(z_0)$ are finite, we have*

$$\sin(\mu-\lambda)E^{\lambda,\mu;\nu}f(z_0)=\sin(\nu-\mu)E^{\mu,\nu;\lambda}f(z_0)=\sin(\lambda-\nu)E^{\nu,\lambda;\mu}f(z_0).$$

(d) *Suppose that $\mathfrak{D}^{*}f(z_0)\leqslant M$. Then given any pair of directions λ,μ not diametrically opposed, and any pair of directions θ,φ, we have*

$$|\sin(\lambda-\mu)|\,|P^{\lambda,\mu;\theta}f(z_0)-P^{\lambda,\mu;\varphi}f(z_0)|<2M|\theta-\varphi|.$$

Proofs. (a) It will be sufficient to show that for any K, the set $T^{K}=\{z;\ D^{a(z)}f(z)\geqslant K\}$ is a G_δ set. We consider the set $\mathfrak{C}^{K}=\{z,\theta;\ D^{\theta}f(z)\geqslant K\}$ as a set of points in Euclidean three-space, as in D. A. § 2, and let $\mathfrak{Q}=\{z,\theta;\ \theta=a(z)\}$. Since $a(z)$ is continuous, then $\mathfrak{Q}$ is closed, and so $\mathfrak{C}^{K}\cdot\mathfrak{Q}$ is a G_δ set in E_3. Since $a(z)$ is single-valued, then $T^{K}=(\mathfrak{C}^{K}\cdot\mathfrak{Q})_z$, and it remains to show that the projection takes G_δ sets into G_δ sets. This follows readily if we consider $\mathfrak{Q}$ as a space with a metric defined by $\varrho(p_1,p_2)=|z_2-z_1|$ for any points $p_i=(z_i,a(z_i))$ $(i=1,2)$ of $\mathfrak{Q}$; the projection is then a homeomorphism.

(b) Let $\{h_{3n+j}\}$ $(n=1,2,\ldots)$ for $j=0,1,2$, be three sequences of complex numbers such that $|h_{3n+j}|\to 0$ and $\arg h_{3n+j}\to\mu$ as $n\to\infty$, and

$$D^{\mu}f_j(z^*)=\lim_{n\to\infty}|h_{3n+j}|^{-1}[f_j(z^*+h_{3n+j})-f_j(z^*)].$$

For any function $k(z)$ denote by $\Delta_n k(z)$ the difference quotient $|h_n|^{-1}[k(z+h_n)-k(z)]$. By the definition of the directed derivate, we have

$$D^{\mu}f_j(z^*)=\overline{\lim_{n\to\infty}}\Delta_n f_j(z^*), \quad j=0,1, \quad D^{\mu}f_2(z^*)=\lim_{n\to\infty}\Delta_n f_2(z^*).$$

On the other hand, we have

$$\Delta_n f_0(z^*)=\Delta_n f_1(z^*)+\Delta_n f_2(z^*)$$

for any n. Taking upper limits, we have

$$\overline{\lim_{n\to\infty}}\Delta_n f_0(z^*)=\overline{\lim_{n\to\infty}}\Delta_n f_1(z^*)+\lim_{n\to\infty}\Delta_n f_2(z^*),$$

equivalent to $D^{\mu}f_0(z^*)=D^{\mu}f_1(z^*)+D^{\mu}f_2(z^*)$, as desired.

(c) and (d) are elementary.

LEMMA 2. *For a continuous function $f(z)$ and a point z_0, suppose that $D^{\lambda}f(z_0)$ and $D^{\nu}f(z_0)$ are finite, where λ and ν are directions such that $\lambda<\nu<\lambda+\pi$. Let ε and ϱ_0 be arbitrary positive numbers.*

Then for any direction θ in the arcs $(\nu,\lambda+\pi)$, $(\nu+\pi,\lambda)$, the inequalities

(5) $$D^{\theta}f(z')>P^{\lambda,\nu;\theta}f(z_0)-\varepsilon,$$

(6) $$D_{\theta}f(z'')<P^{\lambda,\nu;\theta}f(z_0)+\varepsilon$$

are satisfied by points $z'=z'(\theta)$, $z''=z''(\theta)$ lying in $|z-z_0|<\varrho_0$.

Without loss of generality, we assume $z_0=0$, $f(0)=0$. Let μ be an arbitrary direction in the arc $(\nu,\lambda+\pi)$. Let $g(z)=g(x,y)=Ax+By$ $(A,B$ real constants) be such that $D^\lambda g(z)=D^\lambda f(0)$, $D^\nu g(z)=D^\nu f(0)$, for all z. Then for any z and θ,

$$P^{\lambda,\nu;\theta}g(z)=D^\theta g(z)=D_\theta g(z)=P^{\lambda,\nu;\theta}f(0).$$

Let $h(z)=f(z)-g(z)$. Then by Lemma 1 (b), for any z and θ,

$$D^\theta h(z)=D^\theta f(z)-D^\theta g(z)=D^\theta f(z)-P^{\lambda,\nu;\theta}f(0).$$

Hence $D^\lambda h(0)=D^\nu h(0)=0$.

We consider pairs of points $\tilde{z},z^*$ with the following properties:

(i) $\arg\tilde{z}=\lambda$,

(ii) $|\arg z^*-\nu|<\frac{1}{6}\min[|\mu-\nu|,\ |\nu-\lambda|]$,

(iii) $\arg(z^*-\tilde{z})=\mu$.

We note that to each z^* there corresponds just one $\tilde{z}$. It is clear that the upper bounds of the ratios

$$\frac{|\tilde{z}|}{|z^*-\tilde{z}|},\quad \frac{|z^*|}{|z^*-\tilde{z}|},\quad \frac{|\tilde{z}|}{|z^*|}$$

are finite. Denote them by M_1, M_2, M_3 respectively.

Since $D^\lambda h(0)=0$, there exists a positive ϱ_1 such that for $|\tilde{z}|<\varrho_1$, $h(\tilde{z})<\varepsilon|\tilde{z}|/2M_1$. By the construction, there exists a positive ϱ_2 such that for any z^* in $|z|<\varrho_2$, the segment $\tilde{z}z^*$ lies in the region $|z|<\min(\varrho_0,\varrho_1)$. Since $D^\nu h(0)=0$, there exist points z^* such that $|z^*|<\varrho_2$ and $h(z^*)>-\varepsilon|z^*|/2M_2$. We choose such a point z^*, denote it by z_2, and let z_1 be the unique point $\tilde{z}$ corresponding to it.

Then we have

$$(7)\qquad \frac{h(z_2)-h(z_1)}{|z_2-z_1|}>\frac{-\varepsilon|z_2|}{2M_2|z_2-z_1|}-\frac{\varepsilon|z_1|}{2M_1|z_2-z_1|}>-\varepsilon.$$

Hence on the segment z_1z_2, there exists a point z' such that

$$(8)\qquad -\varepsilon<D^\mu h(z')=D^\mu f(z')-D^\mu g(z')=D^\mu f(z')-P^{\lambda,\nu;\mu}f(0),$$

as desired. Changing signs in (7) gives us $\varepsilon|z_1-z_2|>h(z_1)-h(z_2)$; hence on z_1z_2 there exists a point z'' such that

$$\varepsilon>D_{\mu+\pi}h(z'')=D_{\mu+\pi}f(z'')-P^{\lambda,\nu;\mu+\pi}f(0).$$

Thus we have established (5) for θ in $(\lambda,\nu+\pi)$ and (6) for θ in $(\nu+\pi,\lambda)$. We complete the proof by choosing a direction μ in $(\nu+\pi,\lambda)$; the constructions are the same. Since z' and z'' are constructed to lie in $|z|<\varrho_0$, the proof is complete.

Note. If in the conditions of the Lemma, we have $D_\lambda f(z_0)$ and $D_\mu f(z_0)$ finite, then inequalities (5) and (6) become

$$(9)\qquad D^\theta f(z')>P_{\lambda,\nu;\theta}f(z_0)-\varepsilon,$$

$$(10)\qquad D_\theta f(z'')<P_{\lambda,\nu;\theta}f(z_0)+\varepsilon$$

respectively, with the same conditions on θ.

Lemma 3. *If in a neighbourhood $N(z_0)$ of a point z_0, $\mathfrak{D}^*f(z)\leqslant M$, then for any pair of directions θ,φ,*

$$|D^\theta f(z_0)-D^\varphi f(z_0)|\leqslant M|\theta-\varphi|.$$

Suppose the lemma to be false, and that

$$\sup\{|\theta-\varphi|^{-1}[D^\theta f(z_0)-D^\varphi f(z_0)]\}\geqslant\eta^4 M,$$

for some $\eta>1$. Then there exists a pair of directions λ,ν such that

$$|\nu-\lambda|<2\arccos\eta^{-1}$$

and

$$D^\lambda f(z_0)-D^\nu f(z_0)>\eta^3 M|\nu-\lambda|.$$

Without loss of generality, we may assume $\lambda<\nu<\lambda+\pi$.

Let ξ be the midpoint of the arc (λ,ν), and let $\mu=\xi-\frac{1}{2}\pi$. We have

$$\sin(\mu-\lambda)=\sin(\mu-\nu)=-\cos\tfrac{1}{2}(\nu-\lambda),$$

and so

$$P^{\lambda,\nu;\mu}f(z_0)=\frac{\cos\frac{1}{2}(\nu-\lambda)}{\sin(\nu-\lambda)}[D^\lambda f(z_0)-D^\nu f(z_0)]>\eta^2 M\frac{|\nu-\lambda|}{\sin(\nu-\lambda)}>\eta^2 M.$$

Observing that μ is in the arc $(\nu+\pi,\lambda)$, we apply Lemma 2, with $N(z_0)$ as the given neighbourhood and $\varepsilon=(\eta^2-\eta)M$. Then in $N(z_0)$ we obtain points z' such that

$$\eta^2M-(\eta^2-\eta)M=\eta M<D^\mu f(z')\leqslant\mathfrak{D}^*f(z'),$$

the desired contradiction.

Note. Since $\mathfrak{D}^*f(z)\leqslant M$ for z in $N(z_0)$ trivially implies $\mathfrak{D}_*f(z)\geqslant -M$ for z in $N(z_0)$, and conversely, we see that upper derivates may be replaced by lower either in the conditions or conclusions of the Lemma.

Lemma 4. *Let $f(z)$ be a continuous function defined on a region R, such that $\mathfrak{D}^*f(z)\leqslant M$, for $z\in R$. Let $a(z)$ be a direction defined on R, varying continuously with z.*

Then

$$\sup_{z\in R}D^{a(z)}f(z)=\sup_{z\in R}D_{a(z)}f(z).$$

Hence, under the same conditions, if $D^af(z)$ is continuous at z_0, then $D^af(z_0)=D_af(z_0)$, and $D_af(z)$ is also continuous at z_0.

The region R is included in the open set R^+ of Theorem A. Let $N=\sup_{z\in R} D^a f(z)$. Then for any $\varepsilon>0$, there is a point z_0 such that $D^a f(z_0)>N-\varepsilon$. There exists a positive ϱ such that for $|z-z_0|<\varrho$, $|a(z)-a(z_0)|<\varepsilon/M$. Denote $a(z_0)$ by μ, and $f(z_0+te^{i\mu})$ by $g(t)$, a continuous function of a single real variable. Since $D^+g(0)>N-\varepsilon$, then in the interval $|t|<\varrho$, there is a point t' such that

$$N-\varepsilon<D_+g(t')=\partial_\mu f(z_0+t'e^{i\mu})=D_\mu f(z_0+t'e^{i\mu}).$$

Denote $z_0+t'e^{i\mu}$ by z'. We have, by Lemma 3,

$$D_a f(z')>D_\mu f(z')-M|a(z')-\mu|.$$

Since $|z'-z_0|<\varrho$, then $|a(z')-\mu|<\varepsilon/M$, and so

$$D_a f(z')>D_\mu f(z')-\varepsilon>N-2\varepsilon.$$

Since ε is arbitrary, this is the desired result.

The second part of the lemma follows directly.

4. Derivate planes

THEOREM 1. *Let $f(z)$ be a continuous function having finite derivates on a region R, and let $a(z)$ and $b(z)$ be continuously varying directions such that $a(z)<b(z)<a(z)+\pi$.*

Suppose that there exists a set E residual on R, at all of whose points z_0,

$$D^a f(z_0)=D_a f(z_0),\qquad D^b f(z_0)=D_b f(z_0).$$

Then at all points of a set F residual on R, there exists a derivate plane to $f(z)$.

As a preliminary, we show that without loss of generality we may assume $f(z)$ to have derivates uniformly bounded on R. For, let $R^{(M)}$ be the interior of the set $\{z; \mathfrak{D}^*f(z)<M\}$; we have $R'=\bigcup_{M=1}^{\infty}R^{(M)}$ is everywhere dense on R. If this were not so, then there would be a region $R''\subset R-R'$ on which each of the sets $P^M=\{z; \mathfrak{D}^*f(z)\geqslant M\}$ would be everywhere dense. Since the P^M are G_δ sets, they would all be residual on R'', and so their intersection $\{z; \mathfrak{D}^*f(z)=+\infty\}$ would be non-empty. Hence R' is everywhere dense on R.

On each of the sets $R^{(M)}$, we have $\mathfrak{D}_*f(z)\geqslant -M$. If we prove the theorem for each of the sets $R^{(M)}$, then the set $R-F=\bigcup_{M=1}^{\infty}(R^{(M)}-F)+(R-R')$ is still of the first category, and so F is residual on R. We assume henceforth that z is in $R^{(M)}$, for some M.

By Lemma 1(a), we know that $D^a f(z)$ and the other derivates with continuously varying direction are Borel-measurable functions of z. Hence (Kuratowski [1], p. 191) there exists a set E_0 residual on $R^{(M)}$, such that all these derivates (considered as functions of z) are continuous on E_0 with respect to E_0. Let $E^*=E\cdot E_0$; then *a fortiori* we have that the derivates are continuous on E^* with respect to E^*. Also, E^* is residual on $R^{(M)}$.

We recall that the existence of a derivate plane at a point z_0 is equivalent to

$$E^{\lambda,\mu;\theta}f(z_0)\leqslant 0\leqslant E_{\lambda,\mu;\theta}f(z_0)$$

for some pair of fixed directions λ and μ not diametrically opposed, and all directions θ. Let $\Theta(z)$ be an arc of directions, varying with z, defined by

$$\Theta(z)=\{\theta; b(z)-\pi<\theta<a(z)+\pi\}.$$

We prove the theorem by studying sets of points related to the inequality

$$E^{a,b;\theta}f(z_0)\leqslant 0\leqslant E_{a,b;\theta}f(z_0). \tag{11}$$

Let G be the set of points z_0 for which (11) is satisfied for all θ in $\Theta(z_0)$. We shall show that G is residual on $R^{(M)}$. The proof is by contradiction. We suppose that G is not residual on $R^{(M)}$; then there is a set U of the second category consisting of points z_1 for which either

$$\sup_{\theta\in\Theta(z_1)}E^{a,b;\theta}f(z_1)>0\quad\text{or}\quad\inf_{\theta\in\Theta(z_1)}E_{a,b;\theta}f(z_1)<0.$$

We suppose the first inequality to hold at all points z_1 of U.

Since $U\subset R^{(M)}$ is included in the set R^+ of Theorem A, then $D^\theta f(z_1)$ is a continuous function of θ for fixed z_1. Let $\{\theta_i\}$ be the rational directions. Then $U=\bigcup_{i,j=1}^{\infty}A_{ij}$, where

$$A_{ij}=\{z; z\in R^{(M)},\ \theta_i\in\Theta(z),\ E^{a,b;\theta_i}f(z)>j^{-1}\},$$

and one of the sets A_{ij} is of the second category. Abbreviating the notation, we set this $A_{ij}=B$, where

$$B=\{z; z\in R^{(M)},\ \mu\in\Theta(z),\ E^{a,b;\mu}f(z)>\eta\},$$

with $\mu=\theta_i$, $\eta=j^{-1}$. The set E^* being residual on $R^{(M)}$, we have that $E^*\cdot B$ is a set of the second category on $R^{(M)}$.

Let A be the subset of $R^{(M)}$ consisting of points z_0 such that for every $\varrho>0$, the set $\{|z-z_0|<\varrho\}\cdot E^*\cdot B$ is of the second category. Then there exists an open set D such that A is everywhere dense on D. If not, then A is a nowhere-dense set; let $R^{(M)}-\bar{A}=\bigcup_{i=1}^{\infty}O_i$, where each O_i is open. Then $E^*\cdot B\cdot R^{(M)}=E^*\cdot B\cdot\bar{A}+\bigcup_{i=1}^{\infty}E^*\cdot B\cdot O_i$. For each point z_0 of O_i, there exists a positive ϱ_{0i} such that $E^*\cdot B\cdot\{|z-z_{0i}|<\varrho_{0i}\}$ is of the first category. We may cover O_i by a denumerable set of such neighbourhoods, denote them by O_{ij}, and then we have

$$E^*\cdot B\cdot R^{(M)}=E^*\cdot B\cdot\bar{A}+\bigcup_{i,j=1}^{\infty}E^*\cdot B\cdot O_{ij}.$$

Each of the sets in the denumerable union is of the first category; hence $E^*\cdot B\cdot R^{(M)}$ is as well. This contradicts the original assumption; hence an open set D exists on which A is everywhere dense.

We choose a point z^* in D, and let ϱ_1 be such that $\{|z-z^*|<\varrho_1\}$ is included in D. From the definitions, we have $\mu\in\Theta(z^*)$, and $\mu\neq a(z^*)$, $\mu\neq b(z^*)$. Hence μ lies in one of the open arcs $(b(z^*)-\pi, a(z^*))$, $(a(z^*), b(z^*))$, $(b(z^*), a(z^*)+\pi)$. We assume that it lies in the first, and that for all points z in $R_1=\{|z-z^*|<\varrho_1\}$, μ lies in $(b(z)-\pi, a(z))$. The subsequent constructions depend in detail upon the choice of arc for μ, but they are all substantially the same.

There exist positive K and $\varrho_2\leqslant\varrho_1$ such that for z in $R_2=\{|z-z^*|<\varrho_2\}$,

(12) $$\sin\big(a(z)-\mu\big)>4k,\qquad \sin\big(b(z)-a(z)\big)>4k.$$

We already have that $\mathfrak{D}^*f(z)\leqslant M$, for $z\in R_2$. There exists a positive $\varrho_3\leqslant\varrho_2$ such that the region $R_3=\{|z-z^*|<\varrho_3\}$ has the following properties:

(i) The variation of $D_bf(z)$ over all points of $E^*\cdot R_3$ is less than $K\eta$.

(ii) For $z\in R_3$, the set of directions $b(z)$ consists of an arc Φ such that

(13) $$|\Phi|<2K^2\eta/M.$$

Applying (13) to Lemma 1(d), we obtain the result that for any point $z_0\in R_3$ and any pair of directions $\beta,\gamma\in\Phi$,

(14) $$|P^{\mu,a;\beta}f(z_0)-P^{\mu,a;\gamma}f(z_0)|<K\eta.$$

We now show that the assumption that $E^*\cdot B$ is everywhere dense on R_3 leads to a contradiction. This will imply that B is of the first category, hence that U is of the first category, and finally that G is residual, as desired.

Let $B'=E^*\cdot B\cdot R_3$. By Lemma 1(c), we have

$$\sin(b-a)E^{a,b;\mu}f(z)=\sin(a-\mu)E^{\mu,a;b}f(z),$$

and hence by the definition of B,

$$\eta<E^{a,b;\mu}f(z)=\frac{\sin\big(a(z)-\mu\big)}{\sin\big(b(z)-a(z)\big)}E^{\mu,a;b}f(z)\quad\text{for}\quad z\in B'.$$

Since $B'\subset R$, we have by (12),

(15) $$0<\sin(a-\mu)<1,\qquad 0<4K<\sin(b-a),$$

and so from (15),

$$4K\eta<E^{\mu,a;b}f(z)=D^bf(z)-P^{\mu,a;b}f(z)=D_bf(z)-P^{\mu,a;b}f(z),$$

for $z\in B'\subset E^*$. Hence, for $z\in B'$,

(16) $$P^{\mu,a;b}f(z)<D_bf(z)-4K\eta.$$

Let $L=\sup_{z\in B'}D_bf(z)$. Then by property (i) of R_3,

(17) $$L\geqslant D_bf(z)\geqslant L-K\eta,$$

and

(18) $$P^{\mu,a;b}f(z)<L-4K\eta\quad\text{for}\quad z\in B'.$$

Let $\{\beta_i\}$ be a set of directions everywhere dense on Φ. Then for any i and any point $z\in B'$, we have by (14)

$$|P^{\mu,a;b}f(z)-P^{\mu,a;\beta_i}f(z)|<K\eta$$

and so

(19) $$P^{\mu,a;\beta_i}f(z)<L-3K\eta\quad\text{for}\quad z\in B'.$$

Let $C_i=\{z;\, z\in R_3, D_{\beta_i}f(z)\leqslant L-2K\eta\}$. We apply Lemma 2 to (19), with $\varepsilon=K\eta$; since β_i lies in the arc $a(z)<\theta<\mu+\pi$, for $z\in R_3$, then in every neighbourhood of every point of B' there is a point of C_i. Hence each of the sets C_i is everywhere dense on R_3; being G_δ sets, each of them is residual on R_3, and so their intersection C is also residual on R_3. For all points z_0 of C, we have $D_{\beta_i}f(z_0)\leqslant L-2K\eta$ $(i=1,2,\dots)$. By the continuity of the derivates at a fixed point, we also have $D_\theta f(z_0)\leqslant L-2K\eta$, $\theta\in\Phi$. In particular,

(20) $$D_bf(z_0)\leqslant L-2K\eta,\quad z_0\in C.$$

Since B' is of the second category on R_3, and C is residual on R_3, then $B'\cdot C$ is not empty. At a point z_1 of $B'\cdot C$, we have by (17) and (20)

$$L-2K\eta\geqslant D_bf(z_1)\geqslant L-K\eta;$$

this is the desired contradiction.

We now extend inequality (11) to the full circle of directions. Let H be the set of points z_0 in $R^{(M)}$ at which either $E^{a,b;\lambda}f(z_0)>0$ or $E_{a,b;\lambda}f(z_0)<0$ for some λ such that $a(z_1)+\pi\leqslant\lambda\leqslant b(z_1)-\pi$. We will show that H is a set of the first category; this will complete the proof.

If H is a set of the second category on $R^{(M)}$, then since G is residual on $R^{(M)}$, $G\cdot H$ is also of the second category on $R^{(M)}$. Let $\{\theta_i\}$ be the rational directions, and let $\Theta'(z)$ denote the arc $a(z)+\pi\leqslant\theta\leqslant b(z)-\pi$.

Denote by P^i the set $\{z;\ z\in G\cdot H,\ \theta_i\in\Theta'(z),\ E^{a,b;\theta_i}f(z)>0\}$, and by P_i the corresponding set involving lower derivates. By the continuity of the derivates at a fixed point, we have $H=\bigcup_{i=1}^{\infty}(P^i+P_i)$, and so one of the sets P^i or P_i must be of the second category. Let it be one of the sets P^i; abbreviating θ_i to μ, we have at all points of P^i,

$$(21)\qquad 0<E^{a,b;\mu}f(z)=D^{\mu}f(z)-P^{a,b;\mu}f(z)\,.$$

Since $P^i\subset G$, and $\mu+\pi\in\Theta(z)$, we have

$$0=E_{a,b;\mu+\pi}f(z)=D_{\mu+\pi}f(z)-P_{a,b;\mu+\pi}f(z)$$

$$(22)\qquad =D_{\mu+\pi}f(z)-P^{a,b;\mu+\pi}f(z)$$

$$(23)\qquad =D_{\mu+\pi}f(z)+P^{a,b;\mu}f(z)\,;$$

(22) resulting from $P^{a,b;\theta}f(z)=P_{a,b;\theta}f(z)$ for $z\in G\subset E^*$, and (23) from the definition of $P^{a,b;\theta}f(z)$.

Hence, by (21) and (23), we have at all points of P^i,

$$0<D^{\mu}f(z)+D_{\mu+\pi}f(z)\,.$$

Since the derivates are bounded on P^i, then by Theorem 4 of D. A., P^i must be a set of the first category, in contradiction to the assumption that it is of the second category. Therefore, H is of the first category on $R^{(M)}$. Then $R^{(M)}-H$ is residual on $R^{(M)}$, and similarly $G\cdot(R^{(M)}-H)$. Denoting this last set by F, we have that F is residual on $R^{(M)}$, and (11) is satisfied at each point of F for all directions θ. Thus the proof is complete.

THEOREM 2. *Let $f(z)$ be a continuous function, z_0 a fixed point, and $a(z)$ and $b(z)$ two continuously varying directions such that $a(z_0)$ and $b(z_0)$ are not diametrically opposed.*

If the derivates of $f(z)$ are bounded in some neighbourhood of z_0, and $D^af(z)$ and $D^bf(z)$ are continuous functions of z at $z=z_0$, then there exists a derivate plane to $f(z)$ at z_0.

Without loss of generality, we let $z_0=0$ and $f(0)=0$, and assume that $a(0)<b(0)<a(0)+\pi$. For some integer M there exists a positive ϱ_0 such that at all points of the region $|z|<\varrho_0$,

$$(24)\qquad \mathfrak{D}^*f(z)<M\,,\qquad \sin\big(b(z)-a(z)\big)>M^{-1}\,.$$

From Lemma 4, we have

$$D^af(0)=D_af(0)\,,\qquad D^bf(0)=D_bf(0)\,,$$

and all four derivates are continuous at $z=0$. We denote $a(0)$ and $b(0)$ by λ and ν respectively.

By the arguments of § 2, it will be sufficient to establish the inequalities

$$(25)\qquad E^{\lambda,\nu;\theta}f(0)\leqslant 0\,,$$

$$(26)\qquad E_{\lambda,\nu;\theta}f(0)\geqslant 0\,,$$

for all directions θ. The proof is in two stages: first we establish (25) and (26) for θ in $(\nu-\pi,\lambda+\pi)$, and then we use this result to prove (25) and (26) for directions θ in the remaining arc. No single construction suffices for the first stage of the proof; separate constructions are necessary for (25) and (26) in each of the arcs $(\nu-\pi,\lambda)$, (λ,ν), $(\nu,\lambda+\pi)$. However, these constructions are substantially the same; we shall prove (26) for θ in (λ,ν).

Suppose that for a direction μ in the open arc (λ,ν),

$$(27)\qquad E_{\lambda,\nu;\mu}f(0)=-h<0\,.$$

Let $|z|<\varrho_1\leqslant\varrho_0$ be a region at whose points

$$(28)\qquad |D_af(z)-D_\lambda f(0)|<h/4M$$

and

$$(29)\qquad |a(z)-\lambda|<h/6M^2\,.$$

Applying Lemma 1(c), we have

$$\sin(\nu-\lambda)E_{\lambda,\nu;\mu}f(0)=\sin(\mu-\nu)E_{\nu,\mu;\lambda}f(0)\,;$$

this together with (24) and (27) gives

$$E_{\nu,\mu;\lambda}f(0)=\frac{\sin(\nu-\lambda)}{\sin(\mu-\nu)}E_{\lambda,\nu;\mu}f(0)=\frac{\sin(\nu-\lambda)}{\sin(\nu-\mu)}h>h/M\,.$$

Hence

$$(30)\qquad P_{\nu,\mu;\lambda}f(0)<D_\lambda f(0)-h/M\,.$$

Observing that λ is in the arc $(\nu+\pi,\mu)$, we apply Lemma 2 (inequality (10)) to obtain a point z' in $|z|<\varrho_1$ such that

$$(31)\qquad D_\lambda f(z')<P_{\nu,\mu;\lambda}f(0)+h/2M<D_\lambda f(0)-h/2M\,,$$

the last inequality following from (30). By Lemma 3 and (29), we have

$$(32)\qquad D_af(z')-D_\lambda f(z')<M|a(z')-\lambda|<h/6M\,.$$

Combining (31) and (32), we get

$$D_af(z')<D_\lambda f(0)-h/3M$$

for a point z' in $|z|<\varrho_1$. This is in contradiction to (28). Thus we have proved that for θ in (λ,ν),

$$E_{\lambda,\nu;\theta}f(0)\geqslant 0\,.$$

Assuming that (25) and (26) have been established for all directions θ in $(\nu-\pi, \lambda+\pi)$, we now complete the proof. We choose two directions ξ, η with the following properties:

(i) $\lambda+\pi>\xi>\nu, \quad \nu-\pi<\eta<\lambda$.

(ii) The smaller arc with endpoints ξ, η includes the directions $\lambda+\pi$ and $\nu-\pi$.

(iii) $\sin(\xi-\nu)>1/2M, \quad \sin(\lambda-\eta)>1/2M$.

By the previous proof, we have

$$D^{\xi}f(0)=D_{\xi}f(0)=P^{\lambda,\nu;\xi}f(0)\,, \qquad D^{\eta}f(0)=D_{\eta}f(0)=P^{\lambda,\nu;\eta}f(0)\,, \tag{33}$$

and hence for any direction θ,

$$P^{\lambda,\nu;\theta}f(0)=P^{\lambda,\xi;\theta}f(0)=P^{\nu,\eta;\theta}f(0)\,, \quad \textit{etc.} \tag{34}$$

Considering directions θ in the arc (ξ,η), there are four cases for the proof: θ lying outside the arc $(\xi, \lambda+\pi)$, lying outside $(\nu-\pi, \eta)$, and for each of these, proofs of (25) and (26). As before, we discuss one case only.

Suppose that for some direction μ outside $(\nu-\pi, \eta)$, and hence in $(\xi, \nu-\pi)$, we have

$$E^{\lambda,\nu;\mu}f(0)=h>0\,.$$

By (34) we then have

$$E^{\xi,\nu;\mu}f(0)=h>0\,.$$

Applying Lemma 1(c), we have

$$E^{\mu,\xi;\nu}f(0)=\frac{\sin(\nu-\xi)}{\sin(\xi-\mu)}h=\frac{\sin(\xi-\nu)}{\sin(\mu-\xi)}h>h/2M\,,$$

the last inequality following from condition (iii) above.

We notice that ν is in the arc $(\mu-\pi, \xi)$, and so we may apply Lemma 2, choosing constants as before, to obtain the desired contradiction. In this fashion, we complete the proof.

References

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