On algebraically compact groups of I. Kaplansky

by

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In his paper [2] J. Łoś considers a class of abelian groups, that are direct summands of every abelian group which contains them as pure subgroups. This class is denoted by **D**. Łoś has proved the following propositions, giving an interesting characterization of this class:

(1) *A group G belongs to* **D** *if and only if G is a direct summand of a group which admits a compact (= bicompact) topology.*

(2) *A group G belongs to* **D** *if and only if G is a direct summand of group H of a form*

$$H=\sum_{p}^{*}\sum_{t_p\in T_p}^{*} C_{p^{a_{t_p}}}.$$

In his book [1] Kaplansky introduces the notion of algebraically compact (abelian) group.

The purpose of this paper is to prove that class **D** is identical with the class of algebraically compact groups. It gives another proof of Kaplansky's theorem, stating that every group which admits a compact topology is algebraically compact.

For a prime integer p let R_p be the ring of p-adic integers, and M a R_p-module with no element of infinite height, (*i. e.*, the module satisfies the condition $\bigcap_{n=1}^{\infty} p^n M=\{0\}$). Taking submodules $p^n M$ as neighbourhoods of 0 we get a p-adic topology in M.

Let us repeat Kaplansky's definition:

An abelian group G is *algebraically compact* if it has the form

$$G=C+\sum_{p}^{*} D_p,$$

C being a divisible group and D_p a module over R_p, with no element of infinite p-height and complete in its p-adic topology (complete direct sum over all prime integers).

The algebraical structure of algebraically compact groups is fully known.

The result of this paper is contained in the following

THEOREM. *A group G belongs to* **D** *if and only if it is algebraically compact.*

The following theorems will be used in the proof of the above theorem (see [1], p. 51-52).

(3) *Let M be a complete (in its p-adic topology) R_p-module and T a pure submodule of M. Then the closure of T is likewise pure.*

(4) *Let M be a complete R_p-module. Then M is the completion of a direct sum of cyclic R_p-modules.*

(5) *Let M be an R_p-module, S its pure submodule with no element of infinite height which is complete in its p-adic topology. Then S is a direct summand of M.*

It can easily be verified that

(6) *Every cyclic module over R_p is compact in its p-adic topology (it is finite or isomorphic to R_p).*

(7) *A complete direct sum of R_p-modules that are complete is complete.*

The following property of an algebraically compact group is given in [1], p. 56:

(8) *A direct summand of an algebraically compact group is an algebraically compact group.*

Proof of the theorem. Let G belong to **D**; by (2) it is a direct summand of a group H of a form as in (2). It is easy to see that group H is algebraically compact. Let T'_p be such a subset of T_p that $t_p \in T'_p$ if and only if $a_{t_p} < \infty$; then

$$H = C + \sum_p{}^* \sum_{t_p \in T'_p}{}^* C_{p^{a_{t_p}}}$$

C being a divisible group. Each of the groups $C_{p^{a_{t_p}}}$ is a complete module over R_p, and thus every group $\sum^*_{t_p \in T'_p} C_{p^{a_{t_p}}}$ is such a module; H is then an algebraically compact group, and by (8) G is such a group.

Let G be an algebraically compact group; by the definition G has the form $G = C + \sum^*_p D_p$, C and D_p having the properties stated in the definition. By a well known theorem C is a direct sum of groups of type C_{p^∞} and R^+ (additive group of rational numbers). Each of those groups is a subgroup of the (multiplicative) group K of all complex numbers z with $|z| = 1$, which admits a compact topology. By the known theorem of Baer C is a direct summand of group $\sum^*_{a \in A} K_a$ (each K_a being isomorphic to K) with A of sufficiently great cardinality.

By (4) every group D_p is a completion in the p-adic topology of a direct sum of cyclic R_p-modules $M = \sum_{\beta \in B} R_p x_\beta$. Let $M_1 = \sum^*_{\beta \in B} R_p x_\beta$; it is obvious that M is a pure submodule of M_1, and by (6), (7) M_1 is complete in its p-adic topology. The completion M^* of M (isomorphic to D_p) is the same as a closure of M in M_1, and by (3) M^* is pure in M_1. By (5) M^* is a direct summand of M_1. M_1 admits a compact topology, as a complete direct sum of groups admitting such a topology. Each of the groups D_p and C is a direct summand of a group which admits a compact topology, and their complete direct sum is such a group; by (1) everything is proved.

Let a group G admit a compact topology: then G is in D, and by our theorem it is algebraically compact. We get another proof of Kaplansky's theorem.

(9) *If an abelian group G admits a compact topology, then G is an algebraically compact group.*

From our theorem, (1) and (9) follows.

COROLLARY. *A group G is algebraically compact if and only if G is a direct summand of a group which admits a compact topology.*

References

[1] I. Kaplansky, *Infinite abelian groups*, Ann Arbor 1954.

[2] J. Łoś, *Abelian groups that are direct summands of every abelian group which contains them as pure subgroups*, this volume, p. 84-90.

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