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Dimension of metric spaces

by

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1. It is to be shown that a metric space has dimension $\leqslant n$ if and only if there exists a sequence $\{a_i\}$ of locally finite open coverings, each of order $\leqslant n$, with mesh tending to zero as $i \to \infty$, such that

(a) the closure of each member of a_{i+1} is contained in some member of a_i.

For a compact metric space, every sequence of coverings of order $\leqslant n$ with mesh tending to zero contains a subsequence satisfying condition (a). But condition (a) can not in general be omitted, as is shown by K. Sitnikov's example [8] of a two-dimensional metric separable space which has a sequence of coverings, each of order one, with mesh tending to zero.

In the course of proving the above proposition, we incidentally give a new proof of the theorem of M. Katětov (see [4]; also [5], theorem 3.4 and also K. Morita [7], theorem 8.6) that for an arbitrary metric space X the covering dimension ($\dim X$) is equal to the dimension ($\operatorname{Ind} X$) defined inductively in terms of the separation of closed sets.

2. By a *covering* of a topological space X we mean a collection of open sets of X whose union is X. A covering β is called a *refinement* of a covering α if each member of β is contained in some member of α.

The *order* of a collection of subsets of X is the largest integer n such that some point of X is contained in $n+1$ members of the collection, or is ∞ if there is no such largest integer.

Definition 1. The *dimension of a space* X ($\dim X$) is the least integer n such that every finite covering of X has a refinement of order $\leqslant n$, or the dimension is ∞ if there is no such integer.

A collection of subsets of X is called *locally finite* if every point of X has a neighborhood meeting at most a finite number of members of the collection. If X is a metric space, it is known ([9], corollary 1, and [3], theorem 3.5) that $\dim X \leqslant n$ if and only if every covering of X has a locally finite refinement of order $\leqslant n$.

The *mesh* of a collection of subsets of a metric space is the upper bound of the diameters of the members of the collection.

Definition 2. The *sequential dimension of a metric space* X ($\operatorname{ds} X$) is the least integer n such that there exists a sequence $\{\alpha_i\}$ of locally finite coverings, each of order $\leqslant n$, with mesh $\alpha_i \to 0$ as $i \to \infty$, such that

(a) the closure of each member of α_{i+1} is contained in some member of α_i.

If there is no such integer, $\operatorname{ds} X = \infty$.

Lemma 1. *If X is a metric space,* $\operatorname{ds} X \leqslant \dim X$.

Proof. It is sufficient to show that if $\dim X \leqslant n$ then $\operatorname{ds} X \leqslant n$. Let $\dim X \leqslant n$ and suppose that the locally finite coverings $\alpha_1, \ldots, \alpha_{i-1}$ of order $\leqslant n$ have been constructed so that mesh $\alpha_k \leqslant 2^{-k}$ and, for $1 < k < i$, the closure of each member of α_k is contained in some member of α_{k-1}. We now construct the covering α_i.

It follows from [9], corollary 1, that α_{i-1} has a locally finite refinement β_i of mesh $\leqslant 2^{-i}$. By [3], theorem 3.5, since $\dim X \leqslant n$, β_i has a locally finite refinement $\gamma_i = \{U_{i\lambda}\}$ of order $\leqslant n$. By [6], p. 26, (33.4), the covering γ_i can be shrunk to a covering $\alpha_i = \{V_{i\lambda}\}$ such that each $\bar{V}_{i\lambda} \subset U_{i\lambda}$. Then α_i is locally finite and of order $\leqslant n$, and mesh $\alpha_i \leqslant 2^{-k}$. And, since γ_i is a refinement of α_{i-1}, each $\bar{V}_{i\lambda}$ is contained in some member of α_{i-1}. Thus the required sequence $\{\alpha_i\}$ (see definition 2) can be constructed, and hence $\operatorname{ds} X \leqslant n$ as was to be shown.

Definition 3. The *inductive dimension of a space* X ($\operatorname{Ind} X$) is defined inductively as follows: If X is empty, $\operatorname{Ind} X = -1$. For $n = 0, 1, \ldots$, $\operatorname{Ind} X \leqslant n$ means that for each closed set E and open set G with $E \subset G$ there exists an open set U with $E \subset U \subset G$ and $\operatorname{Ind}(\bar{U} - U) \leqslant n-1$.

$\operatorname{Ind} X = \infty$ means that there is no integer n for which $\operatorname{Ind} X \leqslant n$.

It is known ([1], § 18) that, if X is a normal space, $\operatorname{Ind} X \leqslant n$ if and only if, for each pair E, F of disjoint closed sets, X is the union of three disjoint sets U, V and K with U and V open, $E \subset U$, $F \subset V$ and $\operatorname{Ind} K \leqslant n-1$.

Lemma 2. *If X is a metric space,* $\operatorname{Ind} X \leqslant \operatorname{ds} X$.

Proof. It is sufficient to show that if $\operatorname{ds} X \leqslant n$ then $\operatorname{Ind} X \leqslant n$. The proof is by induction. It is clear that if $\operatorname{ds} X = -1$ then X is empty and hence $\operatorname{Ind} X = -1$. We assume it proved that $\operatorname{ds} X \leqslant n-1$ implies $\operatorname{Ind} X \leqslant n-1$.

Let X be a metric space for which $\operatorname{ds} X \leqslant n$. That is, let there exist a sequence $\{\alpha_i\}$ of locally finite coverings as in definition 2 above. We are to prove that $\operatorname{Ind} X \leqslant n$. Let E and F be an arbitrary pair of disjoint closed sets of X.

For each $i = 0, 1, \ldots$ we define a decomposition of X into the union of three disjoint sets M_i, N_i and K_i, of which M_i and N_i are closed and hence K_i is open. Let $M_0 = N_0 = 0$; for $i \geqslant 1$ the decompositions (M_i, N_i, K_i) are defined inductively as follows.

Let the members of α_i be put in the following three classes: α_{i1} consists of those members of α_i whose closures do not meet $F \cup N_{i-1}$, α_{i2} consists of those members of α_i whose closures meet $F \cup N_{i-1}$ but do not meet $E \cup M_{i-1}$, and α_{i3} consists of those members of α_i whose closures meet both $F \cup N_{i-1}$ and $E \cup M_{i-1}$. Let G_i be the union of the open sets which are elements of α_{i1}, let H_i be the union of α_{i2} and let J_i be the union of α_{i3}. Then G_i, H_i and J_i are open sets and their union is X. Let $M_i = X - H_i - J_i$, $N_i = X - G_i - J_i$ and $K_i = X - M_i - N_i = (G_i \cap H_i) \cup J_i$. Then M_i and N_i are closed and K_i is open. Since (G_i, H_i, J_i) covers X, therefore $M_i \cap N_i = 0$ and hence (M_i, N_i, K_i) is a decomposition of X into disjoint sets. Thus the sequence of decompositions is defined.

If $U \in \alpha_{i+1}$ then, for some $V \in \alpha_i$, $\bar{U} \subset V$. We will verify that

(1) $\quad V \in \alpha_{i1} \Rightarrow \bar{U} \cap N_i = 0, \quad \bar{U} \cap F = 0,$

(2) $\quad V \in \alpha_{i2} \Rightarrow \bar{U} \cap M_i = 0, \quad \bar{U} \cap E = 0,$

(3) $\quad V \in \alpha_{i3} \Rightarrow \bar{U} \cap M_i = 0, \quad \bar{U} \cap N_i = 0.$

For, if $V \in \alpha_{i1}$, then $\bar{V} \cap (F \cup N_{i-1}) = 0$ and hence $\bar{V} \cap F = 0$. Also $V \subset G_i = \bigcup \alpha_{i1}$ and hence $V \cap N_i = 0$. Since $\bar{U} \subset V$, therefore $\bar{U} \cap N_i = 0$ and $\bar{U} \cap F = 0$. Similarly, if $V \in \alpha_{i2}$, then $V \cap E = 0$ and $V \subset H_i$ and hence $V \cap M_i = 0$, from which (2) follows. And, if $V \in \alpha_{i3}$, then $V \subset J_i$ and hence $V \cap M_i = 0$ and $V \cap N_i = 0$, from which (3) follows.

By (1) and (3), if $\bar{U} \cap N_i \neq 0$ then $V \in \alpha_{i2}$ and hence, by (2), $\bar{U} \cap M_i = 0$ and $\bar{U} \cap E = 0$. Similarly, if $\bar{U} \cap M_i \neq 0$ then $V \in \alpha_{i1}$ and hence $\bar{U} \cap N_i = 0$ and $\bar{U} \cap F = 0$. Hence, if $U \in \alpha_{i+1,3}$, that is if $\bar{U} \cap (E \cup M_i) \neq 0$ and $\bar{U} \cap (F \cup N_i) \neq 0$, then $\bar{U} \cap N_i = 0$ and $\bar{U} \cap M_i = 0$. Thus

(4) $$U \in \alpha_{i+1,3} \Rightarrow \bar{U} \cap N_i = 0, \quad \bar{U} \cap M_i = 0, \quad \bar{U} \cap E \neq 0, \quad \bar{U} \cap F \neq 0.$$

Since the closure of the union of a locally finite collection of sets is the union of the closures of the sets, $\bar{J}_{i+1}$ is the union of the sets $\bar{U}$ with $U \in \alpha_{i+1,3}$. Hence, by (4), $\bar{J}_{i+1} \cap M_i = 0$ and $\bar{J}_{i+1} \cap N_i = 0$. Also $\bar{G}_{i+1}$ is the union of all $\bar{U}$ with $U \in \alpha_{i+1,1}$; hence $\bar{G}_{i+1} \cap N_i = 0$. Similarly $\bar{H}_{i+1}$ is the union of all $\bar{U}$ with $U \in \alpha_{i+1,2}$; hence $\bar{H}_{i+1} \cap M_i = 0$. Therefore

(5) $$M_i \subset X - \bar{H}_{i+1} - \bar{J}_{i+1} = \operatorname{Int} M_{i+1},$$

(6) $$N_i \subset X - \bar{G}_{i+1} - \bar{J}_{i+1} = \operatorname{Int} N_{i+1}.$$

Let $M=\bigcup_{i=1}^{\infty} M_i$ and $N=\bigcup_{i=1}^{\infty} N_i$. It follows from (5) and (6) that M and N are open sets. And, since, for each i, $M_i \cap N_i=0$, it follows that $M\cap N=0$. Let $K=X-M-N=\bigcap_{i=1}^{\infty} K_i$.

If $x \in E$ then the distance $\varrho(x,F)>0$ and hence, for some i, $\varrho(x,F)>\operatorname{mesh}\alpha_i$. For any $U \in \alpha_i$ with $x \in U$, we have $\overline{U}\cap E\neq 0$ and $\overline{U}\cap F=0$. Thus, since $\overline{U}\cap E\neq 0$, $U \notin \alpha_{i2}$. And, since $\overline{U}\cap F=0$, therefore (see (4)) $U\notin \alpha_{i3}$. Hence $x\notin H_i$ and $x \notin J_i$. Therefore $x \in M_i$. Thus $E\subset M$ and similarly $F\subset N$.

Thus X is decomposed into three disjoint sets M, N and K with M and N open and $E\subset M$ and $F\subset N$. To show that $\operatorname{Ind} X\leqslant n$ it is sufficient to show that $\operatorname{Ind} K\leqslant n-1$.

Let $C_i=K-J_i$; then, since J_i is open, C_i is closed. If $U \in \alpha_{i+1,3}$ then $\overline{U}\subset V$ with $V \in \alpha_i$. It follows from (4) that $V\cap E\neq 0$ and $V\cap F\neq 0$, and hence that $V \in \alpha_{i3}$. Therefore $J_{i+1}\subset J_i$ and hence $C_i\subset C_{i+1}$. Thus $\{C_i\}$ is an ascending sequence of closed sets.

For each point $x \in X$, either $\varrho(x,E)>0$ or $\varrho(x,F)>0$. Hence, for sufficiently large i, if $x \in U \in \alpha_i$ then either $\overline{U}\cap E=0$ or $\overline{U}\cap F=0$. Hence, by (4), $U\notin \alpha_{i3}$ and hence $x\notin J_i$. Thus $\bigcap_{i=1}^{\infty} J_i=0$ and therefore $\bigcup_{i=1}^{\infty} C_i=K$.

We now show that $\operatorname{ds} C_i\leqslant n-1$ for each $i=1,2,\ldots$ Let β_{ij} be the family of open subsets $U\cap C_i$ of C_i with $U \in \alpha_{i+j,2}$. Since $C_i\subset K\subset K_{i+j}$ $=(G_{i+j}\cap H_{i+j})\cup J_{i+j}$ and $C_i\subset C_{i+j}=K-J_{i+j}$, therefore $C_i\subset G_{i+j}\cap H_{i+j}$. Thus each point x of C_i is contained in some element of $\alpha_{i+j,1}$ and also in some element of $\alpha_{i+j,2}$ and, since β_{i+j} is of order $\leqslant n$, x is in at most n elements of $\alpha_{i+j,2}$. Hence β_{ij} is a covering of C_i and is of order $\leqslant n-1$. Since α_{i+j} is locally finite, so is β_{ij}. Also $\operatorname{mesh}\beta_{ij}\leqslant \operatorname{mesh}\alpha_{i+j}$ and hence $\operatorname{mesh}\beta_{ij}\to 0$ as $j\to\infty$.

Let $U \in \alpha_{i+j+1,2}$ and $U\cap C_i\neq 0$ so that $U\cap C_i$ is a non-empty member of the covering $\beta_{i,j+1}$. Then $\overline{U}\subset V$ for some $V \in \alpha_{i+j}$. Since $V\cap C_i\neq 0$, V non $\subset J_i$ and hence $V\notin \alpha_{i+j,3}$. Also, if V were an element of $\alpha_{i+j,1}$ then, by (1), $\overline{U}\cap(F\cup N_{i+j})=0$ and hence $U \in \alpha_{i+j+1,1}$ contrary to assumption. Therefore $V \in \alpha_{i+j,2}$ and hence

$$\overline{U\cap C_i}\subset \overline{U}\cap C_i\subset V\cap C_i \in \beta_{ij}.$$

Thus we have $\operatorname{ds} C_i\leqslant n-1$. Hence, by the induction hypothesis, $\operatorname{Ind} C_i\leqslant n-1$ and hence, by the sum theorem ([1], § 19) for inductive dimension, $\operatorname{Ind} K\leqslant n-1$. Therefore $\operatorname{Ind} X\leqslant n$ as was to be shown. This completes the proof of Lemma 2.

The inequality $\dim X\leqslant \operatorname{Ind} X$ was proved by E. Čech ([1], § 26) for perfectly normal spaces and later by N. Vedenissoff [10] for arbitrary normal spaces. For completeness we include a proof of this result.

LEMMA 3. (Vedenissoff) *If X is a normal space,* $\dim X\leqslant \operatorname{Ind} X$.

Proof. Let $\operatorname{Ind} X\leqslant n$; it is to be shown that $\dim X\leqslant n$. The proof is by induction, the case $n=-1$ being trivial.

Let $\{U_1,\ldots,U_k\}$ be a finite covering of X. Since X is normal there exists a covering $\{V_1,\ldots,V_k\}$ of X with $\overline{V}_i\subset U_i$. Since $\operatorname{Ind} X\leqslant n$ there exist open sets W_i with boundaries $B_i=\overline{W}_i-W_i$ such that $\overline{V}_i\subset W_i\subset U_i$ and $\operatorname{Ind} B_i\leqslant n-1$. Let $Y_i=W_i-\bigcup_{j<i}\overline{W}_j$; then $\{Y_i\}$ is a collection of disjoint open sets. Each point $x \in X$ is in some W_i, hence in a first W_i and hence, unless $x \in B_j$ for some $j<i$, we have $x \in Y_i$. Thus, if $B=\bigcup_{j=1}^{k} B_j$ and $Y=\bigcup_{i=1}^{k} Y_i$, we have $X=B\cup Y$.

By the induction hypothesis, since $\operatorname{Ind} B_i\leqslant n-1$, $\dim B \leqslant n-1$. The closed set B is normal and hence by the sum theorem ([2], § 23), since each B_i is closed, $\dim B=\dim\bigcup_j B_j\leqslant n-1$. Hence the covering $\{B\cap U_i\}$ of B has a refinement $\{G_j\}$ of order $\leqslant n-1$, where the sets G_j are open in B. Let each G_j be associated with one of the sets U_i containing it, and let H_i be the union of the sets G_j associated with U_i. Then $\{H_i\}$ is a covering of B of order $\leqslant n-1$ and $H_i\subset U_i$.

The covering $\{H_i\}$ of the normal space B can be shrunk ([6], p. 26, (33.4)) to a covering K_i with $\overline{K}_i\subset H_i$. The family $\{\overline{K}_i\}$ of closed sets of X can be extended to a system $\{L_i\}$ of open sets of X similar to $\{\overline{K}_i\}$ and hence of order $\leqslant n-1$ ([2], § 12). If $M_i=L_i\cap U_i$ then M_i is open, $\{M_i\}$ is of order $\leqslant n-1$, $M_i\subset U_i$ and, since $\overline{K}_i\subset M_i$, $\{M_i\}$ covers B.

Adding the collection $\{Y_i\}$ of disjoint open sets, we get a covering $\{M_i, Y_j\}$ of X which is a refinement of $\{U_i\}$ and which is of order $\leqslant n$. Thus $\dim X\leqslant n$ as was to be shown.

THEOREM 1. *If X is a metric space,* $\dim X=\operatorname{ds} X=\operatorname{Ind} X$.

Proof. This is an immediate consequence of Lemmas 1, 2 and 3.

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Remarques sur un théorème de F. J. Dyson relatif à la sphère

par

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1. F. J. Dyson [2] a démontré le théorème suivant:

Si $f(x)$ est une fonction à valeurs réelles, définie sur la sphère (à deux dimensions) S^2, on peut toujours trouver un carré inscrit dans un grand cercle de S^2, de sommets a,b,a^*,b^*, tels que

$$f(a)=f(b)=f(a^*)=f(b^*).$$

Ce théorème a été généralisé presque simultanément par Zarankiewicz [9] et Livesay [5]. Ils ont montré que le théorème reste valable, même si l'on remplace le carré par un rectangle quelconque, dont le rapport des côtés peut être fixé d'avance.

Nous nous proposons de montrer que, en combinant la démonstration de Zarankiewicz avec celle de Livesay, on aboutit à un théorème encore plus général.

Soit E un continu (supposé un espace métrique) localement connexe et unicohérent [1]). Soit encore T: $E \to E$ une involution topologique (c. à d. une transformation topologique de E en lui-même, dont le carré est l'identité: $T(T(x))=x$). Nous supposons toujours que T n'a pas de point fixe. Alors $\inf \varrho(x,T(x))=\delta>0$ car E est compact ($\varrho(x,y)$ est la distance des points x,y dans la métrique de E). Nous convenons de dire que δ est le *diamètre* de l'involution T. La généralisation annoncée du théorème de Dyson a alors l'énoncé suivant:

Quel que soit le nombre d, $0<d\leqslant\delta$, on peut toujours trouver deux points $a,b\in E$, tels que $\varrho(a,b)=d$, et que $f(a)=f(b)=f(a^*)=f(b^*)$.

Nous avons désigné par a^*,b^*, les „antipodes" des points a et b par l'involution T, c. à d.

$$a^*=T(a), \quad b^*=T(b).$$

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[1]) Un espace E connexe s'appelle *unicohérent* si, pour chaque décomposition $E=F_1\cup F_2$, où F_1 et F_2 sont fermés et connexes, $F_1\cap F_2$ est connexe.