Some remarks on η_α-sets *

by

L. Gillman (Lafayette, Ind.)

An ordered set S was called by Hausdorff ([6], p. 180-185) an *η_α-set* provided that:

(i) S has no "neighboring" subsets of power $<\aleph_\alpha$, that is, if A and B are subsets of S of power $<\aleph_\alpha$, with $A<B$ [1]), then there is an $x \in S$ such that $A<x<B$ [1]); and

(ii) S is neither coinitial nor cofinal with any subset of power $<\aleph_\alpha$.

Clearly, every (nonempty) open interval of an η_α-set is itself an η_α-set.

On the basis of his general theory of ordered sets, Hausdorff established the following facts:

I. *Any η_α-set S is a universal order set for the cardinal $\aleph_\alpha$*, that is, for every ordered set M of power $\aleph_\alpha$, S has a subset similar to M.

II. *There exists an $\eta_{\beta+1}$-set $S_{\beta+1}$ of power $2^{\aleph_\beta}$.*

III. *The set $S_{\beta+1}$ of* II *has no well-ordered subset (increasing or decreasing) of power $>\aleph_{\beta+1}$.*

IV. *Every $\eta_{\beta+1}$-set S has a subset similar to $S_{\beta+1}$.* (Hence, from II, every $\eta_{\beta+1}$-set is of power $\geqslant 2^{\aleph_\beta}$.)

V. *Any η_α-set for singular $\aleph_\alpha$ is also an $\eta_{\alpha+1}$-set.*

Sierpiński [8] has shown how to derive Hausdorff's results by direct methods. Let U_Θ denote the lexicographically ordered set of all sequences $z=(z_\xi)_{\xi<\Theta}$ of 0's and 1's; and let H_α denote the subset of U_{ω_α} consisting of all $x=(x_\xi)_{\xi<\omega_\alpha}$ in U_{ω_α} for which there exists an ordinal $\varphi(x)<\omega_\alpha$, such that $x_{\varphi(x)}=1$, while $x_\xi=0$ for all ξ with $\varphi(x)<\xi<\omega_\alpha$. Sierpiński shows that:

I'. *U_{ω_α} is a universal order set for $\aleph_\alpha$.*

II'. *$H_{\beta+1}$ is an $\eta_{\beta+1}$-set of power $2^{\aleph_\beta}$* (hence, by I, a universal order set for $\aleph_{\beta+1}$).

---

* The preparation of this paper was sponsored (in part) by the National Science Foundation (U. S. A.).

[1]) By $A<B$ (resp. $A<x<B$), we mean that $a<b$ (resp. $a<x<b$) for all $a \in A$ and all $b \in B$.

III'. *U_Θ has no well-ordered subset (increasing or decreasing) of power $>\overline{\Theta}$.*

As a particular consequence of III', we have

(1) *H_α has no well-ordered subset (increasing or decreasing) of power $\aleph_{\alpha+1}$.*

IV'. *Every $\eta_{\beta+1}$-set has a subset similar to U_{ω_β}.*

The object of this paper is to extend some of these results. Following this, we shall make some remarks about similarity of η_α-sets.

THEOREM 1. *If α is a limit ordinal, then H_α is a universal order set for the cardinal $\aleph_\alpha$.*

Proof. The result is well known for $\alpha=0$, so we assume $\alpha>0$.

For every $\beta<\alpha$, we denote by K_β the set of all elements $x=(x_\xi)_{\xi<\omega_\alpha}$ of H_α for which $\varphi(x)<\omega_\beta$. Obviously, $K_\beta\simeq H_\beta$; and $K_\gamma\subset K_\beta$ for $\gamma<\beta$. Furthermore, it is evident that $H_\alpha=\bigcup_{\beta<\alpha}K_{\beta+1}$; it follows (see II') that

(2) $$\overline{\overline{H}}_\alpha=\sum_{\beta<\alpha}2^{\aleph_\beta}\geqslant\aleph_\alpha.$$

Let M be any ordered set of power $\aleph_\alpha$. Since $\aleph_\alpha=\sum_{\beta<\alpha}\aleph_{\beta+1}$, there exist disjoint subsets $M_{\beta+1}$ $(\beta<\alpha)$ of M, whose union is M, and such that $\overline{\overline{M}}_{\beta+1}=\aleph_{\beta+1}$ $(\beta<\alpha)$. For every $\sigma<\alpha$, define $N_\sigma=\bigcup_{\beta<\sigma}M_{\beta+1}$.

Consider any ordinal $\tau<\alpha$, and suppose that for every $\sigma<\tau$, there has been defined a subset N'_σ of K_σ such that (a) $N'_\varrho\subset N'_\sigma$ for all $\varrho<\sigma$, and (b) there is a similarity f_σ of N_σ upon N'_σ that agrees with f_ϱ on N_ϱ $(\varrho<\sigma)$.

If τ is a limit ordinal, define $N'_\tau=\bigcup_{\sigma<\tau}N'_\sigma$.

Define $\pi=\tau$ or $\tau-1$, according as τ is or is not a limit ordinal. In either case, we have

(3) $$N_\pi=\bigcup_{\beta<\pi}M_{\beta+1}=\bigcup_{\beta<\tau}N_\sigma,$$

and $N'_\pi=\bigcup_{\sigma<\tau}N'_\sigma$. Since $K_\pi=\bigcup_{\sigma<\tau}K_\sigma$, we have $N'_\pi\subset K_\pi$. It is seen without difficulty that there is a similarity f_π of N_π upon N'_π that agrees with f_σ on N_σ $(\sigma<\pi)$. Using (3), we find that

$$\overline{\overline{N}}{}'_\pi=\sum_{\beta<\pi}\aleph_{\beta+1}=\aleph_\pi.$$

We shall now imbed the set $M_{\pi+1}$ in the set $K_{\pi+1}$. For every Dedekind cut $[L,R]$ of N_π, we proceed as follows. Let $[L',R']$ denote the corresponding cut of N'_π. Since $\overline{\overline{N}}{}'_\pi<\aleph_{\pi+1}$, $N'_\pi\subset K_{\pi+1}$, and $K_{\pi+1}$ is an $\eta_{\pi+1}$-set (II'), there is an $\eta_{\pi+1}$-set $P\subset K_{\pi+1}$ such that $L'<P<R'$ [2]). Consequently, since $\overline{\overline{M}}_{\pi+1}=\aleph_{\pi+1}$, the set of all elements of $M_{\pi+1}$ that lie between L and R can be imbedded in P.

By an evident modification of this procedure, we see that the set of all elements of $M_{\pi+1}$ that precede L (resp. follow R) can be imbedded in $K_{\pi+1}$ so as to precede L' (resp. follow R') therein.

In this way, we imbed $M_{\pi+1}$ in $K_{\pi+1}$ so as to preserve, not only the order of $M_{\pi+1}$ itself, but also all its order relations with N_π. We have thus constructed a subset $N'_{\pi+1}$ of $K_{\pi+1}$ that contains N'_π, and is similar to $N_{\pi+1}$; moreover, there is a similarity $f_{\pi+1}$ of $N_{\pi+1}$ upon $N'_{\pi+1}$ that is an extension of the mapping f_π of N_π upon N'_π.

This completes the induction step. Finally, we define $M'=\bigcup_{\sigma<\alpha}N'_\sigma$. Obviously, $M'\subset H_\alpha$; and, clearly, $M'\simeq M$.

THEOREM 2. *If α is a limit ordinal, then H_α is an η_α-set if and only if $\aleph_\alpha$ is regular.*

Proof. Suppose, first, that $\aleph_\alpha$ is singular, and assume that H_α is an η_α-set. Then, by V, H_α is an $\eta_{\alpha+1}$-set, hence, by I, a universal order set for $\aleph_{\alpha+1}$. But this is impossible, since, by (1), H_α cannot have a subset of type $\omega_{\alpha+1}$ (cf. Sierpiński [8], Théorème II).

Suppose, now, that $\aleph_\alpha$ is regular. Assume $\alpha>0$, the case $\alpha=0$ being well known. Let A, B be any two subsets of H_α, whose union is of power $\leqslant\aleph_\beta<\aleph_\alpha$, and such that $A<B$. Since $\aleph_\alpha$ is regular, the set of ordinals $\{\varphi(x):\ x\in A\cup B\}$ has an upper bound that is $<\omega_\alpha$. Since α is a limit ordinal, there is, in fact, an initial ordinal $\omega_\gamma<\omega_\alpha$ that is an upper bound.

Let $\delta=\max\{\beta,\gamma\}$. Then $\delta+1<\alpha$, and both A and B are subsets, of power $<\aleph_{\delta+1}$, of the $\eta_{\delta+1}$-set $K_{\delta+1}\subset H_\alpha$. Therefore there are elements u, v, w of H_α such that $u<A<v<B<w$. Therefore H_α is an η_α-set.

Remark. Alternative proof of Theorem 1 for *regular* cardinals: I and Theorem 2.

THEOREM 3. *Let α be a limit ordinal. Then an η_α-set of power $\aleph_\alpha$ exists if and only if $\aleph_\alpha$ is a regular cardinal such that*

(4) $$2^{\aleph_\beta}\leqslant\aleph_\alpha\quad\textit{for every}\quad\beta<\alpha.$$

When this condition is fulfilled, then the set H_α is an example of an η_α-set of power $\aleph_\alpha$. In particular, this is the case whenever $\aleph_\alpha$ is strongly inaccessible.

Proof. If $\aleph_\alpha$ is singular, then an η_α-set is also an $\eta_{\alpha+1}$-set (V), hence must be of power $\geqslant\aleph_{\alpha+1}$. (This also proves Theorem 2 for those singular cardinals $\aleph_\alpha$ such that $\overline{\overline{H}}_\alpha=\aleph_\alpha$ (see (2).)

---

[2]) Alternatively, we could reach this conclusion by utilizing Sierpiński [8], Lemme I.

Suppose, now, that $\aleph_\alpha$ is regular (hence inaccessible). Then H_α is an η_α-set (Theorem 2); and if (4) holds, then, from (2), we have $\overline{\overline{H}}_\alpha=\aleph_\alpha$. If $\aleph_\alpha$ is strongly inaccessible, then (4) holds (in fact, with the strict inequality — cf. Tarski [9]); however, if $\aleph_\alpha$ is weakly, but not strongly inaccessible, then condition (4) is in doubt.

For the remainder of the proof, consider any $\beta<\alpha$. Then $\beta+1<\alpha$. Therefore, any η_α-set S is also an $\eta_{\beta+1}$-set, hence of power $\geqslant 2^{\aleph_\beta}$ (IV); so if (4) fails, then $\overline{\overline{S}}>\aleph_\alpha$.

THEOREM 4. *Every η_α-set S has a subset similar to the set H_α.*

Proof. Case 1: $\alpha=\beta+1$. For every $\sigma<\omega_{\beta+1}$, denote by V_σ the set of all elements $z=(z_\xi)_{\xi<\omega_{\beta+1}}$ of $U_{\omega_{\beta+1}}$ such that $z_\xi=0$ for all ξ with $\omega_\beta\sigma\leqslant\xi<\omega_{\beta+1}$. Obviously, $V_\sigma\simeq U_{\omega_\beta\sigma}$; and $V_\varrho\subset V_\sigma$ for $\varrho<\sigma$.

Consider any ordinal τ, with $0<\tau<\omega_{\beta+1}$, and suppose that for every $\sigma<\tau$, there has been defined a subset V'_σ of S such that (a) $V'_\varrho\subset V'_\sigma$ for all $\varrho<\sigma$, and (b) there is a similarity f_σ of V_σ upon V'_σ that agrees with f_ϱ on V_ϱ ($\varrho<\sigma$).

In case τ is a limit ordinal, define $\pi=\tau$, and put

$$W_\pi=\bigcup_{\sigma<\pi}V_\sigma\,,\qquad W'_\pi=\bigcup_{\sigma<\pi}V'_\sigma\,.$$

Then $W'_\pi\subset S$. It is seen without difficulty that there is a similarity F_π of W_π upon W'_π that agrees with f_σ on V_σ ($\sigma<\pi$).

It is easily verified (cf. Sierpiński [7]) that W_π is dense in V_π. Now by III′, every well-ordered subset (increasing or decreasing) of V_π is of power $<\aleph_{\beta+1}$. The same then holds for W'_π. Hence, since S is an $\eta_{\beta+1}$-set, there is, for every gap $[L,R]$ of W'_π, an element $s\in S$ such that $L<s<R$ — as is seen by a cofinality argument [3]). Likewise, there are elements $a,b\in S$ such that $a<W'_\pi<b$. It follows that S has a subset V'_π such that

(5) $V'_\sigma\subset V'_\pi$ for all $\sigma<\pi$,

and

(6) there is a similarity f_π of V_π upon V'_π that agrees with the mapping F_π of W_π upon W'_π.

If τ is not a limit ordinal, we define $\pi=\tau-1$, $W_\pi=V_\pi$, $W'_\pi=V'_\pi$, and $f_\pi=F_\pi$. Evidently, (5) and (6) hold in this case as well.

We shall now imbed the set $V_{\pi+1}$ in S. For each element $x=(x_\xi)_{\xi<\omega_{\beta+1}}$ of V_π, define $V_{\pi+1}(x)$ to be the set of all elements $u=(u_\xi)_{\xi<\omega_{\beta+1}}$ of $V_{\pi+1}$ for which the segment $(u_\xi)_{\xi<\omega_\beta\pi}$ coincides with the corresponding segment $(x_\xi)_{\xi<\omega_\beta\pi}$ of x. Obviously, $V_{\pi+1}(x)\simeq U_{\omega_\beta}$. Also, we have $x\leqslant v<y$

[3]) Replace L (resp. R) by a well-ordered cofinal (resp. coinitial) subset.

for all $v\in V_{\pi+1}(x)$, and all $y\in V_\pi$ with $y>x$. Furthermore, it is evident that

$$\bigcup_{x\in V_\pi}V_{\pi+1}(x)=V_{\pi+1}\,.$$

We pass now to the $\eta_{\beta+1}$-set S, and recall once more that every (nonempty) open interval of an $\eta_{\beta+1}$-set is itself an $\eta_{\beta+1}$-set. By the cofinality device used before [3]), we find, referring to IV′, that S has a subset $V'_{\pi+1}$ that contains V'_π, and is similar to $V_{\pi+1}$; moreover, there is a similarity $f_{\pi+1}$ of $V_{\pi+1}$ upon $V'_{\pi+1}$ that is an extension of the mapping f_π of V_π upon V'_π.

This completes the induction step. Finally, we define $W'=\bigcup_{\sigma<\omega_{\beta+1}}V'_\sigma$. Obviously, $W'\subset S$. And it is evident that W' has a subset similar to H_α.

Case 2: *α is a limit ordinal.* We dismiss the trivial case $\alpha=0$, and assume $\alpha>0$. For every $\sigma<\alpha$, we denote by X_σ the set of all elements $z=(z_\xi)_{\xi<\omega_\alpha}$ of U_{ω_α} such that $z_\xi=0$ for all ξ with $\omega_\sigma\leqslant\xi<\omega_\alpha$. We observe that S is an $\eta_{\sigma+1}$-set for every $\sigma<\alpha$. The proof now continues much like that of Case 1.

We turn now to the question of similarity. Hausdorff ([6], p. 180-185) proved that

VI. Any two η_α-sets of power $\aleph_\alpha$ are similar.

VII (from II and IV). There exists an $\eta_{\beta+1}$-set of power $\aleph_{\beta+1}$ if and only if $2^{\aleph_\beta}=\aleph_{\beta+1}$.

The problem arises as to whether one can prove that any two $\eta_{\beta+1}$-sets of power $2^{\aleph_\beta}$ are similar — without using the hypothesis $2^{\aleph_\beta}=\aleph_{\beta+1}$. I have shown (cf. [3]) that this is not the case. More generally, let $\aleph_{\mu_\alpha}$ denote the smallest cardinal $\mathfrak{p}$ such that there exists an η_α-set of power $\mathfrak{p}$; and define $\nu_\alpha=\alpha$ if $\aleph_\alpha$ is regular, $\nu_\alpha=\alpha+1$ if $\aleph_\alpha$ is singular. Then, on combining various of the above results, we find that

(7) H_{ν_α} is an η_α-set of power $\aleph_{\mu_\alpha}$.

We have $\aleph_{\mu_{\beta+1}}=2^{\aleph_\beta}$, $\aleph_{\mu_\alpha}=2^{\aleph_\alpha}$ for singular $\aleph_\alpha$, and $\aleph_{\mu_\alpha}=\aleph_\alpha$ for strongly inaccessible $\aleph_\alpha$; the values of $\aleph_{\mu_\alpha}$ for the other inaccessible numbers $\aleph_\alpha$, however, remain in doubt.

THEOREM 5. *If α and δ are such that $\delta\geqslant\mu_\alpha$ and $\delta>\nu_\alpha$, then there exist two η_α-sets of power $\aleph_\delta$ that are not similar.*

Proof. Let P and Q be sets whose order types are

$$\overline{P}=\overline{H}_{\nu_\alpha}(\omega_\delta+1)\,,\qquad \overline{Q}=\overline{H}_{\nu_\alpha}(\omega_\delta+1)^*\,.$$

Then (see (7)) $\overline{\overline{P}}=\overline{\overline{Q}}=\aleph_{\mu_\alpha}\aleph_\delta=\aleph_\delta$. The conclusion that P is an η_α-set follows easily from the facts that P is both coinitial and cofinal with $\overline{H}_{\nu_\alpha}$, that $\bar{H}_{\nu_\alpha}$ is an η_α-type, and that $\omega_\delta+1$ is an ordinal. Likewise, Q is an η_α-set.

Now P obviously has a subset of type $\omega_\delta \geqslant \omega_{\nu_\alpha+1}$. On the other hand, since H_{ν_α} has no subset of type $\omega_{\nu_\alpha+1}$ (1), it follows at once that Q has no such subset either. Therefore P and Q are not similar.

COROLLARY. *If $2^{\aleph_\beta} \neq \aleph_{\beta+1}$, then there exist two $\eta_{\beta+1}$-sets of power $2^{\aleph_\beta}$ that are not similar.*

Proof. The hypothesis implies that $2^{\aleph_\beta} > \aleph_{\beta+1}$. Hence the theorem applies with $\nu_\alpha = \alpha = \beta+1$, and $\aleph_\delta = \aleph_{\mu_\alpha} = 2^{\aleph_\beta}$.

Added in proof. Remark 1. For the inaccessible $\aleph_\alpha$ that satisfy (4), I propose the term *semi-strongly* inaccessible. As I have observed elsewhere ([2], Lemma 3.2 ff.), the semi-strongly inaccessible cardinals are precisely those limit cardinals $\aleph_\alpha$ for which $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$ for all $\beta < \alpha$. I have encountered these cardinals again in another paper on ordered sets [4]. One may note that under the *Hypothesis of inaccessible numbers* proposed by Erdös and Tarski [1], *every* inaccessible cardinal would be semi-strongly inaccessible.

Remark 2. Since H_α is dense in U_{ω_α} (cf. [7]), U_{ω_α} has no gaps ([8], Lemme I), and $U_{\omega_\alpha} = 2^{\aleph_\alpha}$ (obviously), we obtain the following result of Hausdorff ([5], Satz XXI) as an immediate corollary to our Theorem 4:

Every continuous η_α-set is of power at least $2^{\aleph_\alpha}$.

References

[1] P. Erdös and A. Tarski, *On families of mutually exclusive sets*, Ann. of Math. 44 (1943), p. 315-329.

[2] L. Gillman, *On intervals of ordered sets*, Ann. of Math. 56 (1952), p. 440-459.

[3] — *Remarque sur les ensembles η_α*, C. R. Acad. Sci. Paris 241 (1955), p. 12-13.

[4] — *On anti-homogeneous sets*, Michigan Math. J., to appear.

[5] F. Hausdorff, *Grundzüge einer Theorie der geordneten Mengen*, Math. Ann. 65 (1908), p. 435-505.

[6] — *Grundzüge der Mengenlehre*, Leipzig 1914.

[7] W. Sierpiński, *Sur un problème concernant les sous-ensembles croissants du continu*, Fund. Math. 3 (1922), p. 109-112.

[8] — *Sur une propriété des ensembles ordonnés*, Fund. Math. 36 (1949), p. 56-67.

[9] A. Tarski, *Über unerreichbar Kardinalzahlen*, Fund. Math. 30 (1938), p. 68-89.

PURDUE UNIVERSITY

Reçu par la Rédaction le 17.5.1955

Dimension of metric spaces

by

C. H. Dowker (London) and **W. Hurewicz** (Cambridge, Mass.)

1. It is to be shown that a metric space has dimension $\leqslant n$ if and only if there exists a sequence $\{\alpha_i\}$ of locally finite open coverings, each of order $\leqslant n$, with mesh tending to zero as $i \to \infty$, such that

(a) the closure of each member of α_{i+1} is contained in some member of α_i.

For a compact metric space, every sequence of coverings of order $\leqslant n$ with mesh tending to zero contains a subsequence satisfying condition (a). But condition (a) can not in general be omitted, as is shown by K. Sitnikov's example [8] of a two-dimensional metric separable space which has a sequence of coverings, each of order one, with mesh tending to zero.

In the course of proving the above proposition, we incidentally give a new proof of the theorem of M. Katětov (see [4]; also [5], theorem 3.4 and also K. Morita [7], theorem 8.6) that for an arbitrary metric space X the covering dimension ($\dim X$) is equal to the dimension ($\operatorname{Ind} X$) defined inductively in terms of the separation of closed sets.

2. By a *covering* of a topological space X we mean a collection of open sets of X whose union is X. A covering β is called a *refinement* of a covering α if each member of β is contained in some member of α.

The *order* of a collection of subsets of X is the largest integer n such that some point of X is contained in $n+1$ members of the collection, or is ∞ if there is no such largest integer.

Definition 1. The *dimension of a space* X ($\dim X$) is the least integer n such that every finite covering of X has a refinement of order $\leqslant n$, or the dimension is ∞ if there is no such integer.

A collection of subsets of X is called *locally finite* if every point of X has a neighborhood meeting at most a finite number of members of the collection. If X is a metric space, it is known ([9], corollary 1, and [3], theorem 3.5) that $\dim X \leqslant n$ if and only if every covering of X has a locally finite refinement of order $\leqslant n$.