On the extending of models (III)

Extensions in equationally definable classes of algebras

by

J. Słomiński (Toruń)

In his paper [4] V. Pták deals with the problem of extending semigroups to groups. This problem was first discussed and solved by Malcev [2, 3], who constructed a set of general sentences such that their validity in a semigroup S forms a necessary and sufficient condition for the existence of a group G which is an extension of S.

Another solution of this problem is given by V. Pták in the above mentioned paper [4]. Let S be a semigroup generated by the set A_0, $\gamma(A_0)$ and $\mathfrak{G}(A_0)$ the respective semigroup and group freely generated by the set A_0. Obviously $\gamma(A_0)$ is a subsemigroup of $\mathfrak{G}(A_0)$. Let $h(\gamma(A_0))=S$ be a homomorphism with $h(a)=a$ for $a \epsilon A_0$ and N the least normal subgroup of $\mathfrak{G}(A_0)$, such that $h(s_1)=h(s_2)$ implies $s_1N=s_2N$ for s_1,s_2 in $\gamma(A_0)$. Pták has shown that for the existence of a group G which is an extension of S it is necessary and sufficient that for s_1,s_2 in $\gamma(A_0)$, $s_1N=s_2N$ imply $h(s_1)=h(s_2)$.

It may seem that the solution of Malcev is of a "logical" character whereas that of Pták is more "algebraical" and more closely connected with the normal methods of group-theoretical researches. The purpose of this paper is to show that this is not true. The construction of Pták may be generalized to the case of equationally definable classes of algebras which fulfil some additional conditions (see main theorem on p. 72) without introducing any new idea and therefore the construction itself is not connected with groups.

On the other hand, although it is true that the solution of Malcev is of a logical character, it follows from the results obtained by Łoś (see [1], theorem 1, p. 45) that this is the correct manner of solving the problem in question. Moreover, it is easy to see that this solution is closely connected with groups: the sentences found by Malcev express the specific properties of those semigroups which may be extended.

§ 1. Terms and notations. By a *k-ary operation* on the set A we understand a function $o(x_1,x_2,\ldots,x_n)$ defined on A and with values in A. A system $\langle A,o_1,o_2,\ldots,o_n\rangle$, where A is a non-empty set and o_i are

k_i-ary operations on A, is called an *algebra of the type* $\langle k_1, k_2, \ldots, k_n\rangle$. Two algebras of the same type are called *similar*. In the sequel we shall denote algebras by the same letter as their sets. Thus for example the algebra $\langle A, o_1, o_2, \ldots, o_n\rangle$ will be denoted by A. Let $\mathfrak{A}$ be the class of all similar algebras of the type $\langle k_1, k_2, \ldots, k_n\rangle$.

The notion of an $\mathfrak{A}$-*term* is defined by induction:

1° each variable $x_1, x_2, \ldots$ is an $\mathfrak{A}$-term;

2° if $\tau_1, \ldots, \tau_{k_i}$ are $\mathfrak{A}$-terms, then $O_i(\tau_1, \ldots, \tau_{k_i})$ is also an $\mathfrak{A}$-term.

An expression of the form "$\tau = \vartheta$" where τ and ϑ are $\mathfrak{A}$-terms is called an $\mathfrak{A}$-equation. The set of all $\mathfrak{A}$-*equations* is denoted by $E_{\mathfrak{A}}$.

There is no need to explain what we understand by the validity of an $\mathfrak{A}$-equation in an algebra. For a given class $\mathfrak{A}_0 \subset \mathfrak{A}$ we denote by $E_{\mathfrak{A}}(\mathfrak{A}_0)$ the set of all $\mathfrak{A}$-equations which are valid in every algebra of $\mathfrak{A}_0$. For a given set E_0 of $\mathfrak{A}$-equations we denote by $\mathfrak{A}(E_0)$ the class of all algebras in $\mathfrak{A}$ in which every equation of the set E_0 is valid. A class $\mathfrak{A}_0 \subset \mathfrak{A}$ is called *equationally definable* if $\mathfrak{A}_0 = \mathfrak{A}\big(E_{\mathfrak{A}}(\mathfrak{A}_0)\big)$.

We shall now consider two classes of similar algebras. The first — $\mathfrak{A}$, consists of all algebras of the type $\langle k_1, k_2, \ldots, k_n\rangle$, the second — $\mathfrak{B}$, of all algebras of the type $\langle k_1, k_2, \ldots, k_n, l_1, l_2, \ldots, l_m\rangle$. An algebra $A \in \mathfrak{A}$ is called *subalgebra* of the algebra $B \in \mathfrak{A} \cup \mathfrak{B}$ if A is a subset of B and moreover, if the operations o_i of A and B are, for $i \leqslant n$, identical on A. If A is a subalgebra of B, then B is also called an *extension* of A.

By a *homomorphism* of an algebra $A \in \mathfrak{A}$ in an algebra $B \in \mathfrak{A} \cup \mathfrak{B}$ we understand each mapping $h(A) \subset B$, such that $h\big(o_i(a_1, \ldots, a_{k_i})\big) = o_i\big(h(a_1), \ldots, h(a_{k_i})\big)$ for $a_1, \ldots, a_{k_i}$ in A and $i \leqslant n$.

A binary relation $\sim$ defined in $A \in \mathfrak{A} \cup \mathfrak{B}$ is called a *congruence* in A, if it is symmetric, reflexive and transitive, and if $a_1, a_2', \ldots, a_{k_i}, a_1', a_2', \ldots, a_{k_i}' \in A$, $a_1 \sim a_1', a_2 \sim a_2', \ldots, a_{k_i} \sim a_{k_i}'$ implies $o_i(a_1, a_2, \ldots, a_{k_i}) \sim o_i(a_1', a_2', \ldots, a_{k_i}')$ for every $i \leqslant n$ or $i \leqslant n+m$.

If $\sim$ is a congruence in A, then by $A/\!\sim$ we denote the class of all partition sets of $\sim$ in A. By $a/\!\sim$ is denoted that partition set to which a belongs.

Let $\sim_1$ and $\sim_2$ be congruences in an algebra A. We say that the congruence $\sim_1$ is *smaller* than the congruence $\sim_2$: $\sim_1 \leqslant \sim_2$, if $a \sim_1 b$ implies $a \sim_2 b$ for $a; b \in A$.

Let $\mathfrak{C} \subset \mathfrak{A}$ or $\mathfrak{C} \subset \mathfrak{B}$ be an equationally definable class of algebras and C an arbitrary set. We shall say that the algebra $A \in \mathfrak{C}$ is *freely generated* by the set C if

1° C is a set of generators for A;

2° every mapping $h(C) \subset B \in \mathfrak{C}$ may be extended to a homomorphism h of A in B.

§ 2. Lemmas. The following lemmas either are known or result from the definitions by simple verification.

(2.1) *If A and B are two similar algebras, A_0 and B_0 sets of generators of A and B respectively, h — a homomorphism of A in B with $h(A_0) = B_0$, then $h(A) = B$. Moreover, if $h_1(A) = B$ is a homomorphism such that $h_1(a) = h(a)$ for a in A_0, then $h_1 = h$.*

(2.2) *If $h(A) = B$ is a homomorphism, A and $B \in \mathfrak{A}$, then the relation $a \sim a'$ if and only if $h(a) = h(a')$, is a congruence in A.*

(2.3) *For every binary relation $R(a, a')$ defined in $A \in \mathfrak{A} \cup \mathfrak{B}$, there exists a least congruence $\sim$ in A such that $R(a, a')$ implies $a \sim a'$.*

(2.4) *If $\sim$ is a congruence in an algebra A, then $A/\!\sim$ may be considered as an algebra similar to A, with operations defined by the formulas: $o_i(a_1/\!\sim, \ldots, a_{k_i}/\!\sim) = o_i(a_1, \ldots, a_{k_i})/\!\sim$, $(i = 1, 2, \ldots, n)$.*

(2.5) *If $h(A) = B$ is a homomorphism, $A, B \in \mathfrak{A}$ and $\sim$ the congruence defined in (2.2), then $A/\!\sim$ is isomorphic to B.*

(2.6) *If $A \in \mathfrak{A}$, $B \in \mathfrak{B}$, A is a subalgebra of B, $\sim$ a congruence in A and $\sim_1$ the least congruence in B, such that $a \sim a'$ implies $a \sim_1 a'$, then the mapping $f(a/\!\sim) = a/\!\sim_1$ is a homomorphism of $A/\!\sim$ in $B/\!\sim_1$. This homomorphism is reversible if and only if for $a, a' \in A$, $a \sim_1 a'$ implies $a \sim a'$.*

(2.7) *If $\sim_1$ and $\sim_2$ are two congruences in an algebra A, and if for every $a, a' \in A$ it follows from $a \sim_1 a'$ that $a \sim_2 a'$, then the mapping $h(a/\!\sim_1) = a/\!\sim_2$ is a homomorphism of $A/\!\sim_1$ on $A/\!\sim_2$.*

(2.8) *If $\mathfrak{C}$ is an equationally definable class of algebras, then:*
1° $A/\!\sim \in \mathfrak{C}$, for every $A \in \mathfrak{C}$ and every congruence $\sim$ in A;
2° every subalgebra of an algebra in $\mathfrak{C}$ is also in $\mathfrak{C}$.

(2.9) *If $\mathfrak{C}$ is an equationally definable class of algebras, then for every set C there exists a $\mathfrak{C}$-free algebra, freely generated by C; two such algebras are always isomorphic.*

(2.10) *If $\mathfrak{A}_0 \subset \mathfrak{A}$ and $\mathfrak{B}_0 \subset \mathfrak{B}$ are two equationally definable classes of algebras, $\boldsymbol{A}$ and $\boldsymbol{B}$ algebras $\mathfrak{A}_0$-free and $\mathfrak{B}_0$-free generated by the set C and if moreover $E_{\mathfrak{A}} \cap E_{\mathfrak{B}}(\mathfrak{B}_0) = E_{\mathfrak{A}}(\mathfrak{A}_0)$, then there exists a reversible homomorphism $h(\boldsymbol{A}) \subset \boldsymbol{B}$ with $h(c) = c$, for c in C; therefore $\boldsymbol{A}$ may always be considered as contained in $\boldsymbol{B}$.*

(2.11) *Let $\mathfrak{A}_0 \subset \mathfrak{A}$ and $\mathfrak{B}_0 \subset \mathfrak{B}$ be two equationally definable classes of algebras, A and B two algebras in $\mathfrak{A}_0$ and $\mathfrak{B}_0$ respectively, such that A is a subalgebra of B. If A_0 generates both the algebra A and the algebra B, $\boldsymbol{A}$ and $\boldsymbol{B}$ are respectively $\mathfrak{A}_0$-free and $\mathfrak{B}_0$-free algebras generated by A_0, and if moreover $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$, then the homomorphism $h^*(\boldsymbol{B}) = B$ with $h^*(a) = a$ for a in A_0 is an extension of the homomorphism $h(\boldsymbol{A}) = A$ for a in A_0, i. e. $h^*(a) = h(a)$ for a in $\boldsymbol{A}$.*

§ 3. The main theorem. Let $\mathfrak{A}_0 \subset \mathfrak{A}$ and $\mathfrak{B}_0 \subset \mathfrak{B}$ be two equationally definable classes. Under what conditions is it possible to extend an algebra $A \in \mathfrak{A}_0$ to an algebra $B \in \mathfrak{B}_0$, *i. e.* when does there exist for an algebra $A \in \mathfrak{A}_0$ an algebra $B \in \mathfrak{B}_0$ such that $h(A) \subset B$ with a reversible homomorphism h?

Let A_0 be a set of generators of the algebra A and let $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively be an $\mathfrak{A}_0$-free and a $\mathfrak{B}_0$-free algebra, both generated by the same set A_0 (the existence of these algebras follows from (2.9)).

If $E_{\mathfrak{A}} \cap E_{\mathfrak{B}}(\mathfrak{B}_0) = E_{\mathfrak{A}}(\mathfrak{A}_0)$, then we may assume, following (2.10), that $\boldsymbol{A} \subset \boldsymbol{B}$.

Let $h(\boldsymbol{A}) = A$ be a homomorphism with $h(a) = a$, for a in A_0 (it follows from (2.1) that such a homomorphism is unique) and let $\sim$ be the congruence in $\boldsymbol{A}$ defined in (2.2). This congruence is of course a binary relation in $\boldsymbol{B}$, hence, following (2.3), there exists a least congruence $\sim^*$ in $\boldsymbol{B}$ such that $a \sim a'$ implies $a \sim^* a'$ for a, a' in $\boldsymbol{B}$. If

(P) *$a \sim^* a'$ implies $a \sim a'$, for a, a' in $\boldsymbol{A}$,*

then we shall say that A satisfies the condition (P).

It should be noted that ($\mathfrak{A}_0$ and $\mathfrak{B}_0$ being fixed) the condition (P) is imposed upon A and A_0 but we easily see that the choice of A_0 is irrelevant in this case.

THE MAIN THEOREM. *If $\mathfrak{A}_0 \subset \mathfrak{A}$ and $\mathfrak{B}_0 \subset \mathfrak{B}$ are equationally definable classes of algebras, and if moreover $E_{\mathfrak{A}} \cap E_{\mathfrak{B}}(\mathfrak{B}_0) = E_{\mathfrak{A}}(\mathfrak{A}_0)$, then an algebra $A \in \mathfrak{A}_0$ may be extended to an algebra $B \in \mathfrak{B}_0$ if and only if A fulfils the condition* (P).

Proof. Let us assume that (P) is fulfilled by A. From (2.5) it follows that $\boldsymbol{A}/\sim$ is isomorphic to A. From (2.6) it follows that $\boldsymbol{B}/\sim^*$ is an extension of $\boldsymbol{A}/\sim$ and therefore also of A. From (2.8) it follows $\boldsymbol{B}/\sim^* \in \mathfrak{B}_0$, therefore the sufficiency of (P) is proved.

We shall now prove the necessity of (P). Let $B \in \mathfrak{B}_0$ and let A_0 be a set of generators for A. Without limiting the generality of our considerations we may assume that A_0 is also a set of generators for B. Let $h^0(\boldsymbol{B}) = B$ be a homomorphism with $h^0(a) = a$ for a in A_0 and $\sim_0$ a congruence defined by the relation $h^0(b) = h^0(b')$ in $\boldsymbol{B}$. It follows from (2.11) that the homomorphism $h^0(\boldsymbol{B}) = B$ is an extension of the homomorphism $h(\boldsymbol{A}) = A$ with $h(a) = a$ for a in A_0, and therefore $h^0(a) = h(a)$ for a in $\boldsymbol{A}$. Hence $a \sim_0 a'$ is equivalent to $a \sim a'$ for a, a' in $\boldsymbol{A}$. The congruence $\sim^*$ is the least congruence such that $a \sim a'$ implies $a \sim^* a'$, therefore if $a \sim^* a'$ then $a \sim_0 a'$ for a, a' in $\boldsymbol{A}$. Finally, $a \sim^* a'$ implies $a \sim a'$, for a, a' in $\boldsymbol{A}$, and thus the necessity of (P) is also proved.

In conclusion we note, that the extension B of A is a homomorphic image of the extension $\boldsymbol{B}/\sim^*$. Indeed, $a \sim^* a'$ implies $a \sim a_0$ and therefore, following (2.7), $h(a/\sim^*) = a/\sim_0$ is a homomorphic mapping with $h(\boldsymbol{B}/\sim^*) = \boldsymbol{B}/\sim_0$. However, as $\boldsymbol{B}/\sim_0$ is isomorphic to B, therefore $h(\boldsymbol{B}/\sim^*) = B$ with $h(a) = a$ for a in A_0.

§ 4. The theorem of Pták. The assumptions of the main theorem are fulfilled when $\mathfrak{A}_0$ is the class of multiplicatively written semigroups, and $\mathfrak{B}_0$ the class of multiplicatively written groups supplemented by the operation of forming the inverse element "-1". As we know, every congruence in a group is of the form $aN = a'N$, where N is a normal subgroup, therefore from the main theorem we obtain precisely the theorem of Pták.

§ 5. $\mathfrak{B}_0$-free extensions. We say, that the algebra $W \in \mathfrak{B}_0$ is a *$\mathfrak{B}_0$-free extension* of the algebra $A \in \mathfrak{A}_0$ if

1° W is an extension of A and W is generated by A;

2° if $B \in \mathfrak{B}_0$ is an extension of A and B is generated by A, then the function $f(a) = a$ for a in A may be extended to a homomorphism $h(W) = B$.

It is easy to verify that two $\mathfrak{B}_0$-free extensions of an algebra $A \in \mathfrak{A}_0$ are always isomorphic.

The proof of the main theorem yields at once the following corollary:

(5.1) *If $\mathfrak{A}_0 \subset \mathfrak{A}$ and $\mathfrak{B}_0 \subset \mathfrak{B}$ are equationally definable classes and if moreover $E_{\mathfrak{A}} \cap E_{\mathfrak{B}}(\mathfrak{B}_0) = E_{\mathfrak{A}}(\mathfrak{A}_0)$, then for every algebra $A \in \mathfrak{A}_0$ which fulfils* (P), *$\boldsymbol{B}/\sim^*$ is a $\mathfrak{B}_0$-free extension.*

Thus we see that the criterion of the existence of an extension based upon the condition (P) is of an entirely tautological character. The whole construction of Pták, generalized above, should therefore be considered not as a criterion but as a general method of forming $\mathfrak{B}_0$-free extensions.

If we want to limit ourselves only to stating the existence of an extension it is more convenient to make use of a weaker theorem, namely:

(5.2) *If $\mathfrak{A}_0 \subset \mathfrak{A}$ and $\mathfrak{B}_0 \subset \mathfrak{B}$ are equationally definable classes of algebras, and $E_{\mathfrak{A}} \cap E_{\mathfrak{B}}(\mathfrak{B}_0) = E_{\mathfrak{A}}(\mathfrak{A}_0)$, then an algebra $A \in \mathfrak{A}_0$ may be extended to an algebra $B \in \mathfrak{B}_0$ if, and only if, there exists in $\boldsymbol{B}$ a congruence $\sim_1$ such that*

(P′) *$a \sim_1 a'$ is equivalent to $a \sim a'$, for a, a' in $\boldsymbol{A}$,*

where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\sim$ have the same meaning as in the main theorem.

Proof. Let us assume that there exists a congruence which fulfils (P′); then A also fulfils (P), since $a \sim^* a'$ implies $a \sim_1 a'$, for a, a' in $\boldsymbol{B}$. Therefore, on the basis of the main theorem, A has an extension in $\mathfrak{B}_0$. Evidently, from (2.6) it follows that $\boldsymbol{B}/\sim_1$ is an extension in $\mathfrak{B}_0$ of the algebra A and moreover, from (2.7) it follows that $\boldsymbol{B}/\sim_1$ is a homomorphic image of $\boldsymbol{B}/\sim^*$.

§ 6. Examples. By applying (5.2) it is possible to solve at once the problem of extension in the following examples:

(6.1) *Each distributive structure may be extended to a Boolean algebra.*

Without limiting the generality of our considerations we may discuss only distributive structures of 0 and 1.

Let $\mathfrak{A}_0 \subset \mathfrak{A}$ and $\mathfrak{B}_0 \subset \mathfrak{B}$ denote the class of distributive structures with 0 and 1, and the class of Boolean algebras.

Let $A \in \mathfrak{A}_0$ be a distributive structure generated by A_0, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively be an $\mathfrak{A}_0$-free distributive structure with 0 and 1, and a $\mathfrak{B}_0$-free Boolean algebra generated by A_0. Each element of $\boldsymbol{A}$ and $\boldsymbol{B}$ may be written in the form

(i) $$\sum_{\langle k_1,k_2,\ldots,k_n\rangle} x_{k_1}\cdot x_{k_2}\ldots x_{k_n},$$

(ii) $$\sum_{\langle k_1,k_2,\ldots,k_n\rangle} x_{k_1}^{i_1}\cdot x_{k_2}^{i_2}\ldots x_{k_n}^{i_n}$$

where the variables $x_{k_1},\ldots,x_{k_n}$ run through the set A_0, $i_k=0$ or 1, and $x_k^0=x_k$, $x_k^1=1-x_k=x_k'$.

The formula (ii) may also be written in the form

(iii) $$\sum_{\langle k_1,k_2,\ldots,k_n\rangle} x_{k_{l_1}}\cdot x_{k_{l_2}}\ldots x_{k_{l_r}}\cdot(x_{k_{l_{r+1}}}+x_{k_{l_{r+2}}}+\ldots+x_{k_{l_n}})'$$

where $l_1,l_2,\ldots,l_n$ denotes a permutation of the numbers $1,2,\ldots,n$ such that

$$i_{l_s}=\begin{cases}0 & \text{for } s=1,2,\ldots,r,\\ 1 & \text{for } s=r+1,r+2,\ldots,n.\end{cases}$$

From (i) and (ii) it follows that $\boldsymbol{A}\subset\boldsymbol{B}$, hence $E_{\mathfrak{A}}\cap E_{\mathfrak{B}}(\mathfrak{B}_0)=E_{\mathfrak{A}}(\mathfrak{A}_0)$.

We extend the congruence $\sim$ in A (induced by a homomorphism $h(A)=A$, $h(a)=a$, for $a\in A_0$) to the congruence $\sim_1$ in $\boldsymbol{B}$ in the following manner:

(6.1.1) $a,b\in\boldsymbol{A}$, $a\sim_1 b$ *if and only if* $a\sim b$.

(6.1.2) $a,b\in\boldsymbol{B}-\boldsymbol{A}$, $a\sim_1 b$ *if and only if there exist such representations* $a=\sum_{k=1}^{n}u_{a_k}v'_{a_k}$, $b=\sum_{k=1}^{m}u_{b_k}v'_{b_k}$, *where* $u_{a_k},u_{b_k},v_{a_k},v_{b_k}\in\boldsymbol{A}$ *(the existence of such representations is ascertained by* (iii)), *that* $n=m$, $u_{a_k}\sim u_{b_k}$, $v_{a_k}\sim v_{b_k}$.

(6.1.3) $a\in\boldsymbol{A}$, $b\in\boldsymbol{B}-\boldsymbol{A}$, $a\sim_1 b$ *if and only if there exist representations* $a=\sum_{k=1}^{n}u_{a_k}$, $b=\sum_{k=1}^{m}u_{b_k}v'_{b_k}$, $u_{a_k},u_{b_k},v_{b_k}\in\boldsymbol{A}$ *such that* $n=m$, $u_{a_k}\sim u_{b_k}$, $v_{b_k}\sim 0$.

It is easy to verify that the relation $\sim_1$ is a congruence in $\boldsymbol{B}$. This congruence fulfils of course (P′), hence it follows from (5.2) that $\boldsymbol{B}/\sim_1$ is an extension of A.

(6.2) *Each semigroup may be extended to a ring.*

Let $\mathfrak{A}_0\subset\mathfrak{A}$ denote the class of multiplicatively written semigroups, and $\mathfrak{B}_0\subset\mathfrak{B}$ the class of rings written as usually with help of addition and multiplication and moreover with the operation of substraction. In these operations $\mathfrak{B}_0$ is an equationally definable class.

Let $A\in\mathfrak{A}_0$ be a semigroup generated by the set A_0, $\boldsymbol{A}$ and $\boldsymbol{B}$ a free semigroup and a free ring, respectively, generated by A_0. The elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ may be written in the form

(i) $$x_{k_1}x_{k_2}\ldots x_{k_n}$$

(ii) $$\sum_{\langle k_1,k_2,\ldots,k_n\rangle} h^{\langle k_1,k_2,\ldots,k_n\rangle}x_{k_1}x_{k_2}\ldots x_{k_n}$$

where the variables $x_{k_1},x_{k_2},\ldots,x_{k_n}$ run through the set A_0, $h^{\langle k_1,k_2,\ldots,k_n\rangle}=1$ or -1. We shall denote the elements of $\boldsymbol{A}$ of the form $x_{k_1}x_{k_2}\ldots x_{k_n}$ by the letter t, hence the formula (ii) may be written in the form

(iii) $$\sum_{i=1}^{n}h_it_i,\quad \textit{where}\quad t_i\in\boldsymbol{A},\ h_i=\begin{cases}1 & \text{for } i=1,2,\ldots,r,\\ -1 & \text{for } i=r+1,r+2,\ldots,n.\end{cases}$$

Obviously $\boldsymbol{A}\subset\boldsymbol{B}$, therefore we have $E_{\mathfrak{A}}\cap E_{\mathfrak{B}}(\mathfrak{B}_0)=E_{\mathfrak{A}}(\mathfrak{A}_0)$.

The congruence $\sim$ in A (induced by the homomorphism $h(\boldsymbol{A})=A$, $h(a)=a$, $a\in A_0$) may be extended to the congruence $\sim_1$ in $\boldsymbol{B}$ as follows:

(6.2.1) $a\sim_1 b$ *if and only if there exist such representations* $a=\sum_{i=1}^{n}h_{a_i}t_{a_i}$, $b=\sum_{i=1}^{m}h_{b_i}t_{b_i}$ *(the existence of these representations is ascertained by* (iii)), *that* $n=m$, $h_{a_i}=h_{b_i}=1$ *for* $i=1,2,\ldots,s$, $h_{a_i}=h_{b_i}=-1$ *for* $i=s+1,\ldots,n$ *and* $t_{a_i}\sim t_{b_i}$.

It is easy to verify that the relation $\sim_1$ is a congruence in B which fulfils (P′) and therefore, following (5.2), $\boldsymbol{B}/\sim_1$ is an extension of the semigroup A.

References

[1] J. Łoś, *On the extending of models (I)*, Fund. Math. 42 (1955), p. 38-54.

[2] А. Мальцев, *О включении ассоциативных систем в группы*, Мат. сб. (Rec. Math.) 6 (48) (1939), p. 331-336.

[3] — *О включении ассоциативных систем в группы*, Мат. сб. (Rec. Math.) 8 (50) (1940), p. 251-264.

[4] В. Птак, *О включении семигрупп*, Чехословацкий мат. журнал 2 (77) (1952), p. 247-271.

Reçu par la Rédaction le 24.7.1955

Some remarks on η_α-sets *

by

L. Gillman (Lafayette, Ind.)

An ordered set S was called by Hausdorff ([6], p. 180-185) an *η_α-set* provided that:

(i) S has no "neighboring" subsets of power $<\aleph_\alpha$, that is, if A and B are subsets of S of power $<\aleph_\alpha$, with $A<B$ [1]), then there is an $x \in S$ such that $A<x<B$ [1]); and

(ii) S is neither coinitial nor cofinal with any subset of power $<\aleph_\alpha$.

Clearly, every (nonempty) open interval of an η_α-set is itself an η_α-set.

On the basis of his general theory of ordered sets, Hausdorff established the following facts:

I. *Any η_α-set S is a universal order set for the cardinal $\aleph_\alpha$*, that is, for every ordered set M of power $\aleph_\alpha$, S has a subset similar to M.

II. *There exists an $\eta_{\beta+1}$-set $S_{\beta+1}$ of power $2^{\aleph_\beta}$.*

III. *The set $S_{\beta+1}$ of* II *has no well-ordered subset (increasing or decreasing) of power $>\aleph_{\beta+1}$.*

IV. *Every $\eta_{\beta+1}$-set S has a subset similar to $S_{\beta+1}$.* (Hence, from II, every $\eta_{\beta+1}$-set is of power $\geqslant 2^{\aleph_\beta}$.)

V. *Any η_α-set for singular $\aleph_\alpha$ is also an $\eta_{\alpha+1}$-set.*

Sierpiński [8] has shown how to derive Hausdorff's results by direct methods. Let U_Θ denote the lexicographically ordered set of all sequences $z=(z_\xi)_{\xi<\Theta}$ of 0's and 1's; and let H_α denote the subset of U_{ω_α} consisting of all $x=(x_\xi)_{\xi<\omega_\alpha}$ in U_{ω_α} for which there exists an ordinal $\varphi(x)<\omega_\alpha$, such that $x_{\varphi(x)}=1$, while $x_\xi=0$ for all ξ with $\varphi(x)<\xi<\omega_\alpha$. Sierpiński shows that:

I′. *U_{ω_α} is a universal order set for $\aleph_\alpha$.*

II′. *$H_{\beta+1}$ is an $\eta_{\beta+1}$-set of power $2^{\aleph_\beta}$* (hence, by I, a universal order set for $\aleph_{\beta+1}$).

---

* The preparation of this paper was sponsored (in part) by the National Science Foundation (U. S. A.).

[1]) By $A<B$ (resp. $A<x<B$), we mean that $a<b$ (resp. $a<x<b$) for all $a \in A$ and all $b \in B$.