Models of axiomatic theories admitting automorphisms

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The present paper is concerned with models of axiomatic theories based on the first order logic with identity and more specifically with automorphisms of such models. The main results of the paper are contained in section 5 and in particular in theorem 5.7 which says that if a theory possesses at least one infinite model, it also possesses a model with a "very large" automorphism group. It is a corollary to this theorem that axiomatic systems of arithmetic possess models which admit non-trivial automorphisms. This corollary solves a problem formulated by G. Hessenjaeger.

From the point of view of methods it may be interesting to note that the proofs of our fundamental results are not constructive and that for two reasons: First we use a theorem which states that if a theory is consistent, then the set of its axioms can be extended to a consistent and complete set. Secondly we use the so called ordering principle, i.e. an axiom stating that every set can be ordered. Since in the whole paper we are dealing with theories containing an arbitrary (not necessarily denumerable) number of constants, we see that the first non-constructive theorem mentioned above is equivalent to the so called fundamental theorem of the ideal theory in Boolean algebras (Henkin [2], especially p. 89 and Loś [4]). Since the ordering principle is known to follow from that theorem (Loś and Ryll-Nardzewski [6]), we conclude that the non-constructive tools used in the proofs of our principal theorems are all reducible to the fundamental theorem of the ideal theory in Boolean algebras.

It should also be mentioned that our proofs provide another instance of what has been called by Tarski [10] "the principle of condensation of singularities": The existence of a model admitting a large group of automorphisms is equivalent to the simultaneous satisfiability of an infinite number of sentences. We secure the satisfiability of these sentences by showing that the adjunction of an arbitrary finite number of them to the axioms does not render the theory inconsistent.

In order to make the paper self-contained we have collected in the introductory sections 1-3 all the notions and lemmas which are necessary to an exact formulation of the main theorems and to their proofs. Some of the sections contain new results: in sections 1 and 2 we lay down the terminology and recall some well-known facts concerning models. In section 3 we exposè the general method (due to Henkin [11], Novák [7], and Rasiowa [9]) of constructing models for arbitrary theories. In section 4 we recall some properties of automorphisms and prove a theorem stating that for each group G there is a theory some models of which possess an automorphism-group isomorphic with G (this is the only theorem in our paper in whose proof the full axiom of choice is used).

It seems to us that the automorphism-groups discussed in the present paper deserve a closer study. We intend publishing some of their applications in subsequent papers.

1. Axiomatic theories and their syntax. We consider axiomatic theories based on the functional calculus of the first order. Every such theory S is determined by three sets: 1) \( P(S) \), the set of functors (symbols for functions), 2) \( \mathcal{P}(S) \), the set of predicates (symbols for relations (i.e., for propositional functions)), 3) \( \mathcal{A}(S) \), the set of axioms. We make no assumptions as to the cardinal numbers of these sets, which may be finite or denumerable even non-denumerable. We assume however that \( P(S) \) contains at least one symbol, viz. the identity predicate \( i \). If \( \varphi \) is a functor or a predicate, then we denote by \( a(\varphi) \) the number of arguments of \( \varphi \). We do not exclude the case where \( a(\varphi)=0 \); in this case \( \varphi \) is called a constant. Of course we assume that \( a(\cdot)=2 \).

Finally we assume that all the theories which will be considered below contain the same individual variables and we denote these variables by \( \xi_1, \xi_2, \ldots \).

By \( W(S) \) we denote the class of terms of \( S \). Thus \( W(S) \) is the smallest class that contains all the variables and contains the expressions \( \varphi(a_0, \ldots, a_{n-1}) \) (where \( \varphi \in P(S) \)) whenever it contains \( a_0, a_1, \ldots, a_{n-1} \).

By \( Z(S) \) we denote the class of (sentential) matrices of \( S \). Thus \( Z(S) \) is the smallest class satisfying the following conditions: 1) If \( \pi \in P(S) \) and \( a_0, \ldots, a_{n-1} \in W(S) \), then \( \pi(a_0, \ldots, a_{n-1}) \in Z(S) \); 2) if \( \xi, \xi_1, \xi_2 \in Z(S) \), then \( \sim \xi, \xi_1, \xi_2 \in Z(S) \); if \( \xi \in Z(S) \), then \( (\Xi \xi_1\xi) \in Z(S) \) for \( n=1, 2, \ldots, 3 \).

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3) The letters \( \xi, \xi_1, \xi_2, \ldots \) are not variables but names for them. In a similar way we construct the symbols "~", "\&" etc. which we shall use below as names for symbols actually occurring in \( S \).

4) Other logical operations can be defined in an obvious way in terms of negation, conjunction, and existential quantifier.
An expression which results from an expression \( a \in W(S) \cup Z(S) \) by a substitution of a term \( a_j \) for a variable \( \xi_j \) (\( j = 1, 2, \ldots \)) will be denoted by \( \text{sub}_{a_j} \xi_1, \xi_2, \ldots \) or, in cases where no misunderstanding is possible, more simply by \( \text{sub}_{a_j}(\xi_1, \xi_2, \ldots) \). We omit the explicit formulation of the well-known conditions which must be satisfied in order that the operation \( \text{sub} \) be performable.

A matrix \( \xi \in Z(S) \) is open or closed according as it contains no bound or no free variables. A term \( a \in W(S) \) is called constant if it contains no variables. The set of constant terms will be denoted by \( W(S) \).

The class of theorems of \( S \) will be denoted by \( T(S) \). The following matrices are assumed to be contained in \( T(S) \) for each \( S \):

\[
\xi \xi_1, \xi \xi_2, \xi \xi_3, \xi \xi_4, \xi \xi_5, \xi \xi_6, \xi \xi_7, \xi \xi_8, \xi \xi_9, \xi \xi_{10},
\]

\[
=_{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9, \xi_{10}}
\]

or simply \( \text{val}_{\xi}(\xi_1, \xi_2, \xi_3, \ldots) \).

Let \( \pi \in P(S) \), \( a_1, \ldots, a_{10} \in W(S) \), and \( \xi = \pi(a_1, \ldots, a_{10}) \). We define \( \xi \)

\[
\text{val}_{\xi}\xi_1, \xi_2, \ldots \)

or simpler \( \text{val}_{\xi}(\xi_1, \xi_2, \ldots) \).

Instead of \( \text{val}_{\xi}(\xi) \) we shall usually write \( \text{val}_{\xi}(\xi_1, \xi_2, \ldots) \).

or simpler \( \text{val}_{\xi}(\xi) \).

Let \( \pi \in P(S) \), \( a_1, \ldots, a_{10} \in W(S) \), and \( \xi = \pi(a_1, \ldots, a_{10}) \). We define \( \xi \)

\[
\text{val}_{\xi}\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9, \xi_{10},
\]

or simpler \( \text{val}_{\xi}(\xi_1, \xi_2, \xi_3, \ldots) \).

We denote by \( F_M \) the set of matrices \( \xi \) which are valid in \( M \), i.e. are such that \( \text{val}_{\xi}(\xi) \) holds for all \( \xi \). If \( A(S) \subseteq F_M \), then we say that \( S \) is a model of \( M \).

Let \( S' \) be an extension of \( S \) and let \( M' \) and \( M \) be pseudo-models of \( S' \) and \( S \) over the same set \( X \). We call \( M' \) an extension of \( M \) of \( S' \) over \( S \).

A theory \( S \) is called open if all matrices that belong to \( A(S) \) are open.

The so-called second \( \varepsilon \)-theorem (Hilbert and Bernays [3], p. 18-33) follows:

**Theorem 1.1.** For every theory \( S \) there exists an open theory \( S' \) which is an inessential extension of \( S \).

**2. Models of Axiomatic Theories.** Let \( S \) be a theory and \( X \) a set. We consider a function \( M \) with the following properties:

1. \( M \) assigns a function \( M_a \) (with \( \theta \) arguments) defined in \( X \) and taking on values which are elements of \( X \) to each \( \varepsilon \in E(S) \);
2. \( M \) assigns a relation \( R \) (with \( \alpha \) arguments) defined in \( X \) to each \( \varepsilon \in E(S) \);
3. \( M \) assigns the relation of identity in \( X \) to the predicate \( \varepsilon \).

Every such function \( M \) we call a pseudo-model of \( S \) over \( X \).

Let \( M \) be a pseudo-model of \( S \) over \( X \). A function \( f(x_1, x_2, \ldots) \) which assigns an element of \( X \) to each variable \( x \) we call a valuation. We put \( \text{val}_{\xi} = x \) and extend this definition over the whole class \( W(S) \) by assuming

\[
\text{val}_{\xi}(\xi) = \pi(x_1, x_2, \ldots, x_{10}) = x_1, \ldots, x_{10}) = M_a(\text{val}_{\xi}(x_1), \ldots, \text{val}_{\xi}(x_{10})).
\]
LEMMA 3.1. The theory $S^*(X)$ is consistent.

Proof. Let $M$ be a model of $S$ over an infinite set $X$. Let us first assume that $X$ is a finite set consisting of the elements $x_1, \ldots, x_n$. Let $y_1, \ldots, y_n$ be different elements of $Y$. We extend the model $M$ of $S$ over $Y$ by a pseudo-model $M^*$ of $S^*(X)$ over $Y$ by putting

$$M^*_y = M_y \quad \text{for} \quad \varphi \in F(S), \quad M^*_y = y_j \quad \text{for} \quad j = 1, 2, \ldots, n,$$

$$M^*_y = M_y \quad \text{for} \quad \varphi \in F(S).$$

From the theorem 2.1 immediately follows that if $\xi \in \mathcal{A}(S)$, then $\xi \in \mathcal{M}$. Since the formula $\sim (a \leftrightarrow a') \in \mathcal{M}$ is evident, we conclude that $M^*$ is a model of $S^*(X)$ over $Y$. Hence $S^*(X)$ is consistent.

The general case can be reduced to the case of a finite $X$ by the observation that an inconsistency of $S^*(X)$ would entail the inconsistency of $S^*(X)$ where $X$ is a finite subset of $X$.

Now let $I$ be an arbitrary consistent and complete subset of $Z[S^*(X)]$ containing $A[S^*(X)]$. The existence of $I$ is secured by lemma 3.1. We denote by $S(X, I)$ a theory $S^*$ such that $F(S^*) = F[S^*(X)]$, $Z(S^*) = Z[S^*(X)]$ and $A(S^*) = I$. Two constant terms $a_0, a_1$ of the theory $S(X, I)$ will be called equivalent if $a_0 \equiv a_1 \in I$. We write then $a_0 \equiv a_1$. The following properties of the relation $\equiv$ are obvious:

LEMMA 3.2. $\equiv$ is an equivalence relation and $x_1 \equiv x_2$ for $x_1, x_2 \in X$.

LEMMA 3.3. If $\varphi \in F(S(X, I))$, $\xi_1 \equiv \xi_2 \in (S_1, S_2, \ldots, S_n)$ are constant terms of the theory $S(X, I)$ ($i < a(\varphi)$, $k < a(\varphi)$), and if $a_0 \equiv a_1$, $a_2 \equiv a_3$, then

$$\varphi(a_0, a_1, a_2) \equiv \varphi(a_1, a_2, a_3), \quad \pi(a_1, a_2, a_3) \equiv \pi(a_1, a_2, a_3).$$

We denote by $S_x$ the set of equivalence classes of $W^*[S(X, I)]$ under the relation $\equiv$. The equivalence class containing a constant term will be denoted by $[a]_0$.

We assign to a function $\varphi \in F(S)$ a function $M_\varphi$ such that

$$M_\varphi([a_0], \ldots, [a_n]) = [\varphi(a_0, \ldots, a_n)],$$

and to a predicate $\varphi \in F(S)$ a relation $M_\varphi$ such that

$$M_\varphi(a_0, \ldots, a_n) \equiv \pi(a_0, \ldots, a_n) \in I.$$

It follows from 3.3 that the values of $M_\varphi$ and of $M_\varphi$ do not depend on terms $a_0$, but on the equivalence classes $[a_0]$. Since

$$M_\varphi([a_0], [a_0]) = a_0 \Leftrightarrow a_0 \in I = a_0 \equiv a_0 \equiv [a_0] = [a_0],$$

we obtain

LEMMA 3.4. The function $M_\varphi$ is a pseudo-model of $S$ over the set $S_x$.

The pseudo-model $M_\varphi$ depends on the sets $X$ and $I$ and will therefore be denoted by $M_\varphi(X, I)$ if its dependence on $X$ and $I$ will have to be emphasized.

From the definitions we obtain by an easy induction

LEMMA 3.5. If $\varphi \in W(S)$ and $\tau_1, \tau_2, \ldots$ are constant terms of $S(X, I)$, then $\forall a_0 \in (\tau_1, \tau_2, \ldots) = \forall \mathcal{M} a_0(\tau_1, \tau_2, \ldots) = \forall \mathcal{M} a_0(\tau_1, \tau_2, \ldots)$.

LEMMA 3.6. If $\xi$ is an open matrix of $S$ and $\tau_1, \tau_2, \ldots$ are constant terms of $S(X, I)$, then $\mathcal{M} a_0(\tau_1, \tau_2, \ldots) = \mathcal{M} a_0(\tau_1, \tau_2, \ldots)$.

Since $I$ contains the axioms of $S$ and these axioms are open matrices, we obtain from lemma 3.6

THEOREM 3.7. $M_\varphi(X, I)$ is a model of $S$ over $S_x$.

Again let $S$ be an arbitrary theory and $X$ an arbitrary set. Let $M$ be a pseudo-model of $S$ over a set $Y$ and let $M^*$ be its extension to a pseudo-model of $S^*(X)$ over $Y$. The following theorem will be needed in section 5:

THEOREM 3.8. If $\varphi \in W(S)$, $\xi \in \mathcal{M}(S)$, and if $a_0, a_2, \ldots \in X$, then $\forall a_0 \in (a_0, a_2, \ldots)$ is a constant term of $S^*(X)$ and $\forall a_0 \in (a_0, a_2, \ldots)$ is a closed matrix of $S^*(X)$; moreover

$$(3.8.1) \quad \forall a_0 \in (a_0, a_2, \ldots) = \forall a_0 \in (M_{a_0}, M_{a_2}, \ldots)$$

$$(3.8.2) \quad \mathcal{M}(a_0 \in (M_{a_0}, M_{a_2}, \ldots)) = \mathcal{M}(a_0 \in (M_{a_0}, M_{a_2}, \ldots)).$$

Proof. If $a_0 = \xi$, then both the left and the right hand sides of (3.8.1) are equal to $M_{a_0}$. If $\varphi = \forall a_0 \in (a_0, a_2, \ldots)$, then $\forall a_0 \in (a_0, a_2, \ldots)$ where the accents denote operation $\mathcal{M}(a_0, a_2, \ldots)$. If (3.8.1) holds for the terms $a_0$ ($j < a(\varphi)$), then

$$(3.8.2) \quad \mathcal{M}(a_0 \in (M_{a_0}, M_{a_2}, \ldots)) = \mathcal{M}(a_0 \in (M_{a_0}, M_{a_2}, \ldots)).$$

Since $M_\varphi = M_a$, we obtain (3.8.1) for the term $a_0$.

Proof of (3.8.2) is similar.

4. Automorphisms of models. Let $M$ be a model of $S$ over $X$. A one-one mapping $f$ of $X$ onto itself is called an automorphism of $M$ if the following equations are satisfied for arbitrary $a_0 \in F(S), \pi \in F(S)$ and $a_1, a_2, \ldots \in X$:

$$f(M_a(a_0, \ldots, a_0)) = M_a(a_1, \ldots, a_0),$$

$$M_a(a_0, a_2, \ldots) = M_a(a_1, a_2, \ldots).$$
The group of automorphisms of $M$ is denoted by $G_M$.

In the following two lemmas we note some well-known properties of automorphisms:

**Lemma 4.1.** If $f \in G_M$, $\omega \in W(S)$, $\gamma \in Z(S)$, and $x_1, x_2, \ldots \in X$, then

$$f(\varphi_M(\omega(x_1, x_2, \ldots))) = \varphi_M(f(\omega), f(x_1), f(x_2), \ldots),$$

$$\text{staf}_M(\gamma(x_1, x_2, \ldots)) = \text{staf}_M(f(\gamma), f(x_1), f(x_2), \ldots).$$

**Lemma 4.2.** If $S'$ is an extension of $S$, $M$ is a model of $S'$ over $X$ and $M'$ is a model of $S$ which is an extension of $M$, then $G_{M'} \subseteq G_M$.

We shall now show that each group can be represented as $G_M$ for a suitably chosen model $M$ of a suitable theory $S$.

**Theorem 4.3.** For each group $G$ there is a theory $S$ and a model $M$ of $S$ such that the groups $G_M$ and $G$ are isomorphic.

Proof. We call, as a left translation of $G$ a mapping $\iota$ of $G$ onto itself defined by means of the formula $l(g) = g \cdot g$, where $g \in G$ runs over $G$ and $g_0$ is a fixed element of $G$.

We take as $P(S)$ the empty set and as $P(S)$ the set consisting of all $\pi$ and binary predicate $\gamma$ of $f$ one-one mappings of $G$ onto itself that are not left translations of $G$. The set $A(S)$ is to consist exclusively of the axioms of identity enumerated on p. 52.

If $f$ is a one-one mapping of $G$ onto itself that is not a left translation of $G$, then there are two elements $g_1, g_2$ of $G$ such that $f(g_1) \cdot g_1^{-1} = f(g_2) \cdot g_2^{-1}$. We select for each $f$ a pair $g_1, g_2$ of elements of $G$ satisfying this condition and denote by $\pi(f)$ the binary relation defined in $G$ such that

$$\text{(4.3.1)} \quad M_\pi(g', g'' = (g' \cdot g, (g'' = g_1 g_2), (g'' = g_2 g_1)).$$

Denoting by $M$, the relation of identity in $G$, we obtain a model of $S$ over $G$.

Let $I$ be a left translation of $G, l(g) = g \cdot g_0$ where $g_0 \in G$. From (4.3.1) we immediately obtain

$$\text{(4.3.2)} \quad M_\pi(g', g'' = (l(g'), l(g''))) \quad \forall \pi \in P(S)$$

and hence $\iota \in G_M$.

If $I$ is not a left translation of $G$, then (4.3.2) does not hold for all $\pi \in P(S)$. Indeed, suppose that (4.3.2) is true for $f = I$. Since $M_\pi(\varphi_M(\omega), \varphi_M(\gamma))$, we obtain $M_\pi(l(\varphi_M(\omega)), l(\varphi_M(\gamma)))$ and hence infer that there is a $g \in G$ such that $l(\varphi_M(\omega), g \cdot g_1)$ and $l(\varphi_M(\gamma), g \cdot g_2)$, i.e., $l(\varphi_M(\omega), g \cdot g_1) = l(\varphi_M(\gamma), g \cdot g_2)$, which contradicts the choice of the elements $g_1, g_2$. Hence $I$ is not an automorphism of $M$.

It follows that $G_M$ is identical with the group of all left translations of $G$ and hence isomorphic with $G$.

**3. Models with non-trivial automorphism groups.** The following theorem due to Ramsey ([8], theorem A on p. 384) is basic for all theorems given in this section:

**Theorem 5.1.** Let $Y$ be an infinite set and $Y'$ the set of subsets of $Y$ having exactly $n$ elements. If $Y' = C_1 \cup C_2 \cup \ldots \cup C_k$ is a partition of $Y'$ into mutually disjoint sets, then there is a $j < k$ and an infinite set $Y_j \subseteq Y$ such that $Y_j \subseteq C_j$.

In the sequel we consider an open theory $S$ and a model $\mathcal{M}(X, I)$ of $S$ (cf. section 3).

**Lemma 5.2.** A one-one mapping $h$ of $X$ onto itself determines at most one automorphism $f$ of $\mathcal{M}(X, I)$ satisfying the condition $f(x) = h(x)$ for $x \in X$.

Proof. A constant term $r$ of $S^*(X)$ has the form $r = \text{subt}(x_1, x_2, \ldots)$ where $\omega \in W(S)$ and $\gamma \in X$ for $j = 1, 2, \ldots$. Hence by 3.5 and 4.1 we obtain the formula $f(x) = \varphi_M(\omega, \gamma) = \varphi_M(\omega, \gamma, \ldots)$, which shows that the value of $f(x)$ is determined by the values of $f(x)$ for $x \in X$. This proves the lemma.

If $h$ is a one-one mapping of $X$ onto itself for which there is an automorphism $f$ with the properties described in lemma 5.2, then we shall say that $h$ induces an automorphism. The automorphism induced by $h$ will be denoted by $f_h$.

**Lemma 5.3.** If $h \neq h_0$ and the automorphisms $f_h, f_{h_0}$ exist, then $f_{h_0} \neq f_h$.

Proof follows immediately from lemma 3.2.

**Lemma 5.4.** A one-one mapping $h$ of $X$ onto itself induces an automorphism of $\mathcal{M}(X, I)$ if and only if the following condition is satisfied by each open matrix $\zeta$ and each assignment $(x_1, x_2, \ldots)$:

$$\text{(5.4.1)} \quad \text{subt}(\zeta, x_1, x_2, \ldots) = \text{subt}(h(x_1, x_2, \ldots), \zeta)$$. 

Proof. From the completeness of $I$ it follows that exactly one of the closed matrices $\text{subt}(\zeta, x_1, x_2, \ldots), \sim \text{subt}(x_1, x_2, \ldots)$ belongs to $I$. We can assume that it is the first.

By 3.6 we obtain the formula $\text{staf}_M(\zeta, x_1, x_2, \ldots)$, whence we infer by lemma 4.1 that if $h$ induces an automorphism of $\mathcal{M}(X, I)$, then $\text{staf}_M(\zeta, x_1, x_2, \ldots)$, $\zeta$ and $\sim \text{subt}(x_1, x_2, \ldots)$ belongs to $I$. This proves the formula (5.4.1).

Let us now assume that (5.4.1) holds for each open matrix $\zeta$ and let $r$ be a constant term of the theory $S^*(X)$. We assign variables of $S$
to the elements of the set \( X \) occurring in \( \tau \) in such a way that different variables are correlated with different elements. Let \( \overline{x} \) denote the variable assigned to \( x \) and let \( \overline{\tau} \) be a term of \( S \) obtained from \( \tau \) by replacing each \( x \) by the corresponding variable \( \overline{x} \).

Now let \( \tau_1, \tau_2 \) be two constant terms of the theory \( S^n(X) \) and let \( x_1, \ldots, x_n \) be all the elements of \( X \) which occur in \( \tau_1 \) or \( \tau_2 \) or in both of them. We shall show that

\[
(5.4.2) \quad \text{if } \tau_1 \equiv \tau_2 \text{ then } \text{subt}_1(\tau_1) \equiv \text{subt}_1(\tau_2),
\]

Indeed, \( \tau_1 \equiv \tau_2 \) means that \( \tau_1, \tau_2 \in I \), whence

\[
\text{subt}_1(\tau_1) = \text{subt}_1(\tau_2) = \left[ \overline{\tau_1}, \ldots, \overline{\tau_1}, \overline{x}_1, \ldots, \overline{x}_n \right] \in I.
\]

Now we use (5.4.1), in which we take \( \xi = \tau_1, \tau_2 \) and replace the variables \( \xi_1, \xi_2, \ldots, \xi_n \) by \( \overline{\xi_1}, \overline{\xi_2}, \ldots, \overline{\xi_n} \). In this way we obtain

\[
\text{subt}_1(\tau_1) = \left[ \overline{\xi_1}, \ldots, \overline{\xi_1}, \overline{x}_1, \ldots, \overline{x}_n \right] \in I,
\]

which proves (5.4.2).

From (5.4.2) it follows that defining \( f_\pi \) by means of the formula

\[
f_\pi(\tau_1) = \left[ \text{subt}_1(\tau_1) \right]
\]

(where \( x_1, \ldots, x_n \) are all the elements of \( X \) that occur in \( \tau_1 \)), we obtain a function defined on \( S^n \).

Each element \( [\tau] \) of \( S^n \) is the value of \( f_\pi \) for a suitable argument. Indeed, if

\[
\tau' = \text{subt}_1(\overline{\tau_1}, \ldots, \overline{\tau_n}),
\]

then

\[
\tau'' = \text{subt}_1(\overline{\tau_1}, \ldots, \overline{\tau_n})
\]

and hence

\[
f_\pi(\tau'') = \left[ \text{subt}_1(\overline{\tau_1}, \ldots, \overline{\tau_n}) \right]
\]

This is the required automorphism-property for \( \varphi \in P(S) \).

If \( \pi \in P(S) \), then

\[
\text{M}_\pi(\tau_1, \ldots, \tau_n) = \pi(\tau_1, \ldots, \tau_n) \in I,
\]

where \( \tau_1, \ldots, \tau_n \) are all the elements of \( X \) that occur in \( \pi(\tau_1, \ldots, \tau_n) \). Using (5.4.1) for \( \xi = \pi(\tau_1, \ldots, \tau_n) \) we obtain therefore

\[
\text{M}_\pi(\tau_1, \ldots, \tau_n) = \pi(\tau_1, \ldots, \tau_n) \in I.
\]

The right-hand side of this equivalence means precisely the same as \( \text{M}_\pi(\tau_1, \ldots, \tau_n) = f_\pi(\tau_1, \ldots, \tau_n) \). Lemma 5.4 is thus proved.

In order to express conveniently the content of lemmas 5.2, 5.3, and 5.4 we shall adopt the following

Definition. A group \( \Theta \) of transformations of a set \( X \) strongly contains a group \( \Theta \) of transformations of a set \( X \) if \( X \rightarrow X \) and each \( f \in \Theta \) can be extended to at least one function \( f_1 \in \Theta_1 \).
If $G$ is a cyclic group generated by a transformation $h$, then instead of saying that $G$ strongly contains $G$ we shall say that $G$ strongly contains $h$.

From lemmas 5.2, 5.3, and 5.4 we obtain

**Theorem 5.5.** Let $S$ be an open theory and $X$ a set. In order that there exist a model $M$ of $S$ over a set $X$ such that $G_M$ strongly contains a group $G$ of transformations of $X$ it is necessary and sufficient that the theory $S^*(X)$ remain consistent after the adjunction of all equivalences $(5.4.1)$ to its axioms where $h$ is an arbitrary element of $G$ and $G$ an arbitrary open matrix of $S$.

**Proof.** If the condition is satisfied, we can extend the set $T(S^*(X))$ to a complete set $I$ satisfying (5.4.1). On using lemma 5.4 we obtain a model $M(X, I)$ whose automorphism group strongly contains the group of transformations $[x] \rightarrow [h(x)]$ of the set $[X] = \bigcup_{i \in I} [x]$. Since there is a one-one correspondence between the elements of $X$ and those of $[X]$, we can exchange the classes $[x]$ for the elements $x$ and obtain thus from $M(X, I)$ (which is a model of $S$ over $E_X$) a model $M$ of $S$ over a set $X \cup X$ such that $G_M$ strongly contains the group $G$.

Conversely, if there is a model $M$ of $S$ over a set $X \cup X$ such that $G_M$ strongly contains $G$, then we use 4.1 and find that formulas (5.4.1) belong to $V_M$ for each open matrix $\xi \in Z(S)$ and each $h \in G$. Since the axioms of $S^*(X)$ are evidently elements of $V_M$, we obtain the desired consistency.

**Theorem 5.6.** Each theory $S$ (not necessarily open), which possesses at least one model over an infinite set, possesses a model $M_\infty$ such that the group $G_{M_\infty}$ strongly contains an infinite cyclic group.

**Proof.** Let us first assume that $S$ is open and consider an infinite set

$$X = \{\ldots, x_{-1}, x_0, x_1, x_2, \ldots\}$$

where $x_i \neq x_j$ for $i \neq j$. Let $h$ be the transformation $h(x_j) = x_{j+1}$ ($j = 0, \pm 1, \pm 2, \ldots$).

In order to prove our theorem we shall show that the adjunction of equivalences (5.4.1) (where $\xi \in Z(S)$ and $x_1, \ldots, x_n$ are to be replaced by arbitrary elements of $X$) does not render theory $S^*(X)$ inconsistent. It will of course be sufficient to show that no inconsistency occurs if we adjoin an arbitrary finite number of equivalences (5.4.1) to the axioms of $S^*(X)$.

We therefore consider a sequence $\xi_1, \ldots, \xi_n \in Z(S)$ and assume that no variable different from $\xi_1, \xi_2, \ldots, \xi_n$ occurs in any of these matrices. We consider further sequences of integers each containing exactly $t$ (not necessarily different) terms:

$$t_1; t_2; \ldots; t_t$$

where $t_i \in \mathbb{Z}$.

We may assume that the terms of these sequences lie in the interval $-n < \lambda < n$.

We extend $S$ to a theory $S^*$ such that $P(S^*) = P(S)$, $F(S^*) = F(S) + \{x_i = x_j, \ldots, x_{n+1}\}$ (where $\delta(x) = 0$ for $-n < j < n+1$) and $A(S^*)$ is obtained from $A(S)$ by adjunction of the matrices

$$\begin{align*}
\text{sub} \xi_1 (\xi_1, \xi_2, \ldots, \xi_n) &= \text{sub} \xi_1 (\xi_1, \xi_2, \ldots, \xi_n) \\
(\lambda = 1, 2, \ldots, t) \\
\sim (\xi_1, \xi_2) &\sim (\xi_1, \xi_2) \\
&\sim (\xi_1, \xi_2)
\end{align*}$$

Note that the only symbols of $S^*$ that do not occur in $S$ are $x_i, x_{i+1}, x_{i+2}, \ldots$

In order to prove the consistency of $S^*$ we shall construct a model for this theory. To this effect we first assign $2n+1$ different variables of $S$ to the elements $x_1, \ldots, x_{n+1}$ and denote by $x_0$ the variable corresponding to $x_0$. We consider further matrices

$$\begin{align*}
\text{sub} \xi_1 (\xi_1, \xi_2, \ldots, \xi_n) &= \text{sub} \xi_1 (\xi_1, \xi_2, \ldots, \xi_n) \\
(\lambda = 1, 2, \ldots, t) \\
\sim (\xi_1, \xi_2) &\sim (\xi_1, \xi_2) \\
&\sim (\xi_1, \xi_2)
\end{align*}$$

where $t_i = \pm 1$ for $p = 1, 2, \ldots, t$ and $t_\infty$ stands for $\xi$ or $\sim \xi$ according as $e = +1$ or $e = -1$. These matrices evidently belong to $Z(S)$; their free variables are $x_i, x_{i+1}$ or some of them. It is evident also that the matrices (5.6.3) possess the following properties:

$$\sim (\xi_1, \xi_2) \sim (\xi_1, \xi_2),$$

(5.6.5) the alternation of $t$ matrices (5.6.3) belongs to $T(S)$.

Now let $M$ be a model of $S$ over an infinite set $Y$. The existence of $M$ is assured by the assumptions of the theorem. We assume $Y$ to be ordered by an arbitrary relation $\ll$ which, in general, has nothing in common with relations definable in $S$. Let $Y_{x+1}$ be the set consisting of subsets of $Y$ with exactly $2n+1$ elements and let $G_{x+1, r}$ be the set containing as elements all those sets $(y, r) \in Y_{x+1}$ for which $\nu \ll \cdots \ll \nu$ and

$$\begin{align*}
\text{sub} \xi_1 (\xi_1, \xi_2, \ldots, \xi_n) &= \text{sub} \xi_1 (\xi_1, \xi_2, \ldots, \xi_n) \\
(\lambda = 1, 2, \ldots, t) \\
\sim (\xi_1, \xi_2) &\sim (\xi_1, \xi_2) \\
&\sim (\xi_1, \xi_2)
\end{align*}$$
From (5.6.4) and (5.6.5) it is clear that the sets \( C_{n,a} \) determine a partition of \( P^{n+1} \). Applying theorem 5.1 we infer that there is a fixed system of indices \( q_1, \ldots, q_n \) and an infinite set \( Y \subseteq X \) such that \( Y = C_{n,a} \). We choose from \( Y, 2n+2 \) elements \( y_1, \ldots, y_n, y_{n+1} \) such that \( y_1 < \ldots < y_n < y_{n+1} \). Hence we have the formula (5.6.6) and also the formula

\[
\text{staf}(y_{a,1}, \ldots, y_{a,n+1}) = (\bar{y}_{a,1}, \ldots, \bar{y}_{a,n+1}).
\]

We now define a pseudo-model \( M^* \) of \( S^* \) over \( Y \) by assuming

\[
M^* = M^*_a \quad \text{for} \quad \varphi \in P(S), \quad M^*_i = y_j \quad \text{for} \quad j = -n, \ldots, n, n+1,
\]

\[
M^*_a = M^*_a \quad \text{for} \quad \pi \in P(S).
\]

If \( \xi \in A(S) \), then \( \xi \in V_{M^*} \) and hence \( \xi \in V_{M^*} \) (cf. theorem 2.1). Axioms (5.6.2) of \( S^* \) are evidently contained in \( V_{M^*} \) because \( M^*_i = y_j \neq y_j = M^*_a \) for \( i \neq j \). Formulas (5.6.6) and (5.6.7) prove that

\[
\text{staf}(y_{a,1}, \ldots, y_{a,n+1}) \quad \text{and} \quad \text{staf}(y_{a,1}, \ldots, y_{a,n+1})
\]

for \( t = 1, 2, \ldots, 8 \) and hence, in accordance with theorem 3.8,

\[
\text{staf}(\text{sub}(\xi_1, \ldots, \xi_8)) \quad \text{and} \quad \text{staf}(\text{sub}(\xi_1, \ldots, \xi_8))
\]

From these two formulas it follows that axioms (5.6.1) are valid in \( M^* \), i.e. belong to \( V_{M^*} \). This proves the consistency of \( S^* \).

Theorem 5.6 is thus proved for the case of an open theory. The general case can be reduced to the case of an open theory by means of theorems 3.1 and 4.2.

The following example shows that theorem 5.6 ceases to be true if we replace in it the words “infinite cyclic group” by the words “an arbitrary transformation group”.

Assume that \( S \) is a consistent theory and that \( P(S) \) contains a bi-quantifier \( \pi \) such that the matrices

\[
(z_1 \pi z_2) \cup (z_1 \pi z_3), \quad (z_1 \pi z_3) \vee (z_1 \pi z_4) \vee (z_2 \pi z_4)
\]

belong to \( T(S) \).

If \( M \) is an arbitrary model of \( S \) over an arbitrary set \( X_i \), then \( G_M \) does not contain functions which, limited to a subset \( X \subseteq X_i \), are transformations of finite order different from identity. For assume that \( f \in G_M \), \( f(x) \neq x \), and \( f \), limited to a set \( X \subseteq X_i \), containing \( x \), is a transformation of order \( n \). Since \( X \) is ordered by the relation \( \leq \), we have either \( M(x, f(x)) \) or \( M(x, f'^{(1)}(x)) \). It will be sufficient to consider only the first case. We have evidently \( M_i(x, f'^{(1)}(x)) \) for \( i = 1, 2, \ldots, n-1 \) because \( f \) is an automorphism of \( M \). By the transitivity of \( M_a \) we obtain therefore \( M_a(x, f'^{(1)}(x)) \), i.e. \( M_a(x, f(x)) \), which is a contradiction.

In connection with these remarks we shall introduce the following definition.

Definition: For each set \( X \) ordered by a relation \( \leq \) we denote by \( G(X, \leq) \) the group of all transformations of \( X \) onto itself leaving invariant the relation \( \leq \) (i.e. satisfying the condition \( x \leq y \Rightarrow f(x) \leq f(y) \)) for \( f \in G(X, \leq) \).

Theorem 5.7. If a theory \( S \) has at least one model over an infinite set, then for each ordered set \( X \) there is a model \( M_a \) of \( S \) such that \( G_{M_a} \) strongly contains \( G(X, \leq) \).

Proof. As in the proof of theorem 5.6 we can limit ourselves to the case of an open theory \( S \). According to theorem 5.5 we have only to show that the theory \( S^*(X) \) remains consistent after the adjunction to its axioms of all matrices (5.6.1) where \( \zeta \) is an open matrix of \( S \) and \( k \in G(X, \leq) \). This again can be reduced to the proof that the theory \( S \) remains consistent after the adjunction of an arbitrary finite number of axioms of the form (5.6.1) and of a finite number of axioms of \( S^*(X) \) which are not already contained in \( A(S) \).

Accordingly we consider a finite number of open matrices \( \zeta_1, \zeta_2, \ldots, \zeta_s \) of \( S \) and assume that no variable different from \( x_1, \ldots, x_8 \) occurs in any of these matrices. We further consider \( s \) sequences each containing \( t \) elements of \( X \)

\[
x_{i,1}, x_{i,2}, \ldots, x_{i,t}, \quad i = 1, 2, \ldots, s
\]

(we do not assume that \( x_{i,j} \neq x_{i,k} \) for \( j \neq k \)). Finally we consider \( s \) functions \( g_1, \ldots, g_s \in G(X, \leq) \) and denote by \( X^* \) the set containing all the elements (5.7.1) and all the elements \( g_j(x_i) \) where \( i = 1, 2, \ldots, s \) and \( j = 1, 2, \ldots, s \). We extend \( S \) to a theory \( S^* \) assuming that \( P(S^*) = P(S) \cup X^* \). If \( P(S^*) = P(S) \cup X^* \) and letting \( A(S^*) \) consist of \( A(S) \) and of matrices

\[
(ax \pi ay) \quad a'x \pi a'y,
\]

\[
\text{sub}(\zeta_1, \ldots, \zeta_t) = \text{sub}(\zeta_1, \ldots, \zeta_t)
\]

\[
\hat{i} = 1, 2, \ldots, s.
\]

In order to prove the theorem it will be sufficient to define a model of \( S^* \).
We begin by assigning a variable \( z \) to each element \( x \) of \( X^* \) in such a way that \( x \neq z \). We further put \( p = st \) and introduce the following matrices:

\[
\begin{align*}
    y_i &= \text{sub}_x \left( z_i, \ldots, z_i \right)_{(z_i^0, \ldots, z_i^s)} , \quad i = 1, 2, \ldots, s , \\
    y_i' &= \text{sub}_x \left( z_i', \ldots, z_i' \right)_{(z_i'^0, \ldots, z_i'^s)} , \quad i = 1, 2, \ldots, s ,
\end{align*}
\]

Since the assignment \( x \to z \) is one-to-one, we easily see that axioms (5.7.3) can be written in the form

\[
(5.7.4) \quad \text{sub}_x \left( x_1, \ldots, x_s \right) = \text{sub}_x \left( z_1, \ldots, z_s \right)_{(z_1^0, \ldots, z_s^0)} , \quad i = 1, 2, \ldots, s .
\]

We have noted above that the elements (5.7.1) need not be distinct; let us assume that they form a set with \( n \) elements

\[(5.7.5) \quad X_n = \{ x_1, \ldots, x_n \} = \{ z_1, \ldots, z_n \}
\]

where \( x_i \neq x_j \) for \( i \neq j \). Each of the sets

\[X_i = \{ g(x_1), \ldots, g(x_n) \} \quad i = 1, 2, \ldots, s
\]

has exactly \( n \) elements and is ordered similarly to \( X_n \). The set \( X^* \) is the union of the sets \( X_1, X_2, \ldots, X_s \):

\[X^* = X_1 \cup X_2 \cup \ldots \cup X_s = \{ z_1', \ldots, z_s' \}
\]

Let \( M \) be a model of \( S \) over an infinite set \( Y \). We can assume that the set \( Y \) is ordered and denote by \( < \) the ordering relation.

Let \( U \) be an element of \( Y^* \), i.e., a subset of \( Y \) with exactly \( n \) elements. A sequence \( (u_1, \ldots, u_n) \) with \( p \) (not necessarily distinct) terms \( u \in U \) will be called a distinguished ordering of \( U \) if \( u_i < u_j = x_i < x_j \) for \( h, j < p \). It is evident that for each \( U \in Y^* \) there exists exactly one such distinguished ordering.

We now define a partition of \( Y^* \) into \( 2^s \) sets \( C_{\alpha_n} \), where \( \alpha_n = \pm 1 \) for \( i = 1, 2, \ldots, s \) by including a set \( U \in Y^* \) to \( C_{\alpha_n} \) if the distinguished ordering \( (u_1, \ldots, u_n) \) of \( U \) satisfies the condition

\[
(5.7.6) \quad \text{sub}_x \left( z_1, \ldots, z_s \right)_{(u_1, \ldots, u_n)} \quad \text{for} \quad i = 1, 2, \ldots, s
\]

It is evident that the union of all sets \( C_{\alpha_n} \) is \( Y^* \) and that two different sets \( C_{\alpha_n} \) and \( C_{\alpha'_{n'}} \) are disjoint. By theorem 5.1 there is a fixed system \( \epsilon_1, \ldots, \epsilon_s \) of indices \( \pm 1 \) and an infinite set \( Y \subseteq Y^* \) such that (5.7.6) holds for each \( U \in Y^* \). We select from \( Y \) a set \( \{ y_1, \ldots, y_m \} \) with \( m \) elements ordered (by the relation \( < \)) similarly to \( X^* \):

\[
(5.7.7) \quad y_i < y_j = z_i' < z_j' \quad i, j < m .
\]

We can now define a model \( M^* \) of \( S^* \) over \( Y \) by taking

\[M^*_\alpha = M^*_\alpha \quad \text{for} \quad \alpha \in \mathcal{P} \{ S \}, \quad M^*_\alpha = y_j \quad \text{for} \quad j = 1, 2, \ldots, m.
\]

It is evident that axioms of \( S \) and axioms (5.7.2) are valid in \( M^* \).

It remains therefore to prove that axioms (5.7.4) are valid in \( M^* \). We first prove the following auxiliary statements.

(5.7.8) The sequence \( (M^*_{\alpha_1}, \ldots, M^*_{\alpha_s}) \) is a distinguished ordering of the set \( (M^*_\alpha, \ldots, M^*_\alpha) \).

(5.7.9) The sequence \( (M^*_\alpha, \ldots, M^*_\alpha) \) is a distinguished ordering of the set \( (M^*_\alpha, \ldots, M^*_\alpha) \).

(Note that both sets \( (M^*_\alpha, \ldots, M^*_\alpha) \) and \( (M^*_\alpha, \ldots, M^*_\alpha) \), have exactly \( n \) elements).

**Proof of (5.7.8).** Each \( a_\alpha (h < p) \) is identical with \( z_\alpha \) where \( m < \alpha \) (cf. (5.7.5)). Assume that \( h, j < p \) and \( a_\alpha = z_i', j = z_j' \). Hence we have the equivalence

\[
M^*_\alpha < M^*_\alpha < M^*_\alpha, \quad M^*_\alpha = y_\alpha < y_\alpha,
\]

which together with (5.7.7) yields

\[
M^*_\alpha < M^*_\alpha = z_i' < z_i' = z_j' < z_j', \quad \text{e. o. d.}
\]

**Proof of (5.7.9).** Each \( g(x_\alpha) (h < p) \) is an element of \( X_\alpha \) and hence identical with an element of the form \( g(z_\alpha) \) where \( m < \alpha < n \). Assume that \( h, j < p \) and \( g(x_\alpha) = g(z_i') \), \( g(y_\alpha) = g(z_j') \). Since \( g(z_i') \) and \( g(z_j') \) belong to \( X_\alpha \), they are identical with elements \( z_i' \) of \( z_\alpha \) for \( i < m \). Hence, on account of (5.7.7), we obtain

\[
M^*_\alpha < M^*_\alpha = M^*_\alpha < M^*_\alpha = M^*_\alpha < M^*_\alpha
\]

\[
= y_\alpha < y_\alpha = z_i' < z_i' = g(z_i') < g(z_i') = g(x_\alpha) < g(z_j').
\]

Since \( e \in O(X_\alpha, <) \), it preserves the ordering relation \( < \) and hence the last part of the above formula is equivalent to \( x_\alpha < z_\alpha \), e. o. d.

We can now prove that axioms (5.7.4) are valid in \( M^* \). From (5.7.8), (5.7.9), and the remark that \( M^*_\alpha, \ldots, M^*_\alpha \) are elements of \( Y \), we obtain the formulas

\[
\text{sub}_x \left( z_1', \ldots, z_s' \right)_{(u_1, \ldots, u_n)} , \quad i = 1, 2, \ldots, s
\]
whence, on account of theorem 3.8, we further obtain

$$(5.7.10) \quad \text{sub}_M \psi \bigg|_{\begin{array}{l}
E_i, \ldots, E_i \\
E_i, \ldots, E_i
\end{array}} = \psi_{i=1,2,\ldots,s}.$$  

From (5.7.9) we obtain in the same manner

$$\text{sub}_M \psi \bigg|_{\begin{array}{l}
E_i, \ldots, E_i \\
E_i, \ldots, E_i
\end{array}} = \psi_{i=1,2,\ldots,s}.$$  

Since $\psi_i$ contains only the variables $E_{i-1}$, $E_i$, the last formula can be written in the form

$$\text{sub}_M \psi \bigg|_{\begin{array}{l}
E_i, \ldots, E_i \\
E_i, \ldots, E_i
\end{array}} = \psi_{i=1,2,\ldots,s}.$$  

We now remark that $\psi_i$ results from $\psi_i$ by a substitution of variables $g((x_{i-1},y_{i-1}))$, $g((x_i))$ for the variables $E_{i-1}$, $E_i$. Hence we can write the last formula in the form

$$\text{sub}_M \psi \bigg|_{\begin{array}{l}
E_i, \ldots, E_i \\
E_i, \ldots, E_i
\end{array}} = \psi_{i=1,2,\ldots,s}.$$  

We simplify this formula by inserting the "fictitious" variables $g((x_{i+k}))$ $(j \neq i, k = 1,2,\ldots,t)$ in the upper row. The validity of the formula is unaffected since these variables do not occur in $\psi_i$. We thus obtain

$$\text{sub}_M \psi \bigg|_{\begin{array}{l}
E_i, \ldots, E_i \\
E_i, \ldots, E_i
\end{array}} = \psi_{i=1,2,\ldots,s}.$$  

or, what amounts to the same,

$$\text{sub}_M \psi \bigg|_{\begin{array}{l}
E_i, \ldots, E_i \\
E_i, \ldots, E_i
\end{array}} = \psi_{i=1,2,\ldots,s}.$$  

Using theorem 3.8 we finally obtain the formula

$$\text{sub}_M \psi \bigg|_{\begin{array}{l}
E_i, \ldots, E_i \\
E_i, \ldots, E_i
\end{array}} = \psi_{i=1,2,\ldots,s},$$  

which together with (5.7.10) proves that the matrix (5.7.4) is valid in $M^*$.  

Theorem 5.7 is thus proved.

6. We shall conclude by proving one more theorem, which is not directly connected with the subject-matter of the present paper but which will be needed in one of the subsequent papers mentioned at the end of the introduction. It seems appropriate to include the proof here because the method of proof is very close to that used in the proof of theorem 5.7.

**Theorem 6.1.** Let $G$ be a set ordered by a relation $\supset$ and $S$ an open theory which possesses a model $M$ over an infinite set $Y$ ordered by a relation $\supset$. Further let $Y^*$ be an infinite set contained in $Y$ and $\eta$ an open matrix of $S$ with the free variables $\xi_1, \ldots, \xi_s$ such that

$$\eta \bigg|_{\begin{array}{l}
\xi_1, \ldots, \xi_s \\
\eta_1, \ldots, \eta_s
\end{array}}$$

for each sequence $(\eta_1, \ldots, \eta_s)$ of elements of $Y^*$ satisfying the conditions $\eta_1 < \eta_2 < \cdots < \eta_s$. Under these assumptions there exists a model $N$ of $S$ over a set $X$, $\exists X$ such that

$$(6.1.1) \quad N \supset X, \exists X$$

$$(6.1.2) \quad \text{sub}_N \eta \bigg|_{\begin{array}{l}
\xi_1, \ldots, \xi_s \\
\eta_1, \ldots, \eta_s
\end{array}}$$

for each sequence $(\eta_1, \ldots, \eta_s)$ of elements of $X^*$ satisfying the conditions $\eta_1 < \eta_2 < \cdots < \eta_s$.  

**Proof.** We first show that $S^*(X)$ remains consistent if we add to its axioms the axioms which are arbitrary elements of $X$, $\exists X$ all matrices

$$(6.1.3) \quad \text{sub}_N \eta \bigg|_{\begin{array}{l}
\xi_1, \ldots, \xi_s \\
\eta_1, \ldots, \eta_s
\end{array}}$$

where $\eta_1, \ldots, \eta_s \in \exists X$ and $\eta_1 < \eta_1 < \cdots < \eta_1$. As before it is sufficient to exhibit for each finite set $X^*$ of $X$ a model $N^*$ of a theory $S^*$ such that $N^* = N | X^*$. $S^*$ consists of $S$ and of those axioms (5.7.2), (5.7.3), and (6.1.3) which contain no $x$ from the outside of $X^*$.

To achieve this result we repeat word for word the construction carried out in the proof of theorem 5.7 with the only change that we construct the partition not of the whole set $Y$ but of its part $Y^*$. In this way we obtain a pseudo-model $M^*$ of $S^*$ over $X^*$ in which $M^*| X^*$ for $x \in X$ and in which axioms belonging $A(Y)$ as well as the axioms (5.7.2) and (5.7.3) are valid. If $x_1, \ldots, x_s \in X$ and $x_1 < \cdots < x_s$, then $M^*_{x_1} < M^*_{x_2} < \cdots < M^*_{x_s}$ (cf. (5.7.7)) and, since $M^*_{x_1}, \ldots, M^*_{x_s}$ belong to $Y^*$, the assumptions of the theorem yield

$$\text{sub}_M \psi \bigg|_{\begin{array}{l}
\xi_1, \ldots, \xi_s \\
\eta_1, \ldots, \eta_s
\end{array}}.$$  

The consistency of $S^*(X)$ extended as indicated above is thus proved.

We now select a complete set $I$ which contains $A(Y^*)$ as well as matrices (5.4.1), and (6.1.3), and consider the model $M^*(X, I)$ of $S^*$.
over \( \mathcal{S}_X \). Each function \( h \in \mathcal{G}(X, \mathcal{S}_X) \) determines an automorphism \( f_h \) of \( \mathcal{M}(X, I) \) (cf. lemma 5.4), and the formula

\[
\text{st}_{\mathcal{M}(X, I)}(\mathcal{S}_X) \eta \begin{pmatrix} \xi_1 & \ldots & \xi_k \end{pmatrix}_{\mathcal{S}_X^k}
\]

holds for each sequence \((\xi_1, \ldots, \xi_k)\) such that \( x \prec x_1 \prec \ldots \prec x_k \) (cf. lemma 3.6). Owing to the fact that \([x'] \neq [x'']\) for \( x \neq x'' \), we can identify the classes \([x]\) where \( x \in X \) with the elements \( x \), and obtain thus a model \( \mathcal{M}_a \) satisfying (6.1.1) and (6.1.2).

References


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On the extending of models (III)

Extensions in equational definable classes of algebras

by J. Słomiński (Toruń)

In his paper [4] V. Pták deals with the problem of extending semigroups to groups. This problem was first discussed and solved by Malcev ([2, 3]), who constructed a set of general sentences such that their validity in a semigroup \( S \) forms a necessary and sufficient condition for the existence of a group \( G \) which is an extension of \( S \).

Another solution of this problem is given by V. Pták in the above mentioned paper [4]. Let \( S \) be a semigroup generated by the set \( A_0 \), \( \gamma(A_0) \) and \( \mathcal{G}(A_0) \) the respective semigroup and group freely generated by the set \( A_0 \). Obviously \( \gamma(A_0) \) is a subsemigroup of \( \mathcal{G}(A_0) \). Let \( h(\gamma(A_0)) = S \) be a homomorphism with \( h(a) = a \) for \( a \in A \) and \( X \) the least normal subgroup of \( \mathcal{G}(A_0) \) such that \( h(x) = h(x) \) implies \( x = x \) for \( x \in \gamma(A_0) \). Pták has shown that for the existence of a group \( G \) which is an extension of \( S \) it is necessary and sufficient that for \( s_1, s_2 \) in \( \gamma(A_0) \), \( s_1 = s_2 \) implies \( h(s_1) = h(s_2) \).

It may seem that the solution of Malcev is of a "logical" character whereas that of Pták is more "algebraical" and more closely connected with the normal methods of group-theoretical researches. The purpose of this paper is to show that this is not true. The construction of Pták may be generalized to the case of equational definable classes of algebras which fulfill some additional conditions (see main theorem on p. 72) without introducing any new idea and therefore the construction itself is not connected with groups.

On the other hand, although it is true that the solution of Malcev is of a logical character, it follows from the results obtained by Los (see [1], theorem 1, p. 45) that this is the correct manner of solving the problem in question. Moreover, it is easy to see that this solution is closely connected with groups: the sentences found by Malcev express the specific properties of those semigroups which may be extended.

§ 1. Terms and notations. By a \( k \)-ary operation on the set \( A \) we understand a function \( e(x_1, x_2, \ldots, x_k) \) defined on \( A \) and with values in \( A \). A system \( \langle A, e_1, e_2, \ldots, e_n \rangle \), where \( A \) is a non-empty set and \( e \) are...