On the ideals' extension theorem and its equivalence to the axiom of choice

by

S. Mrówka (Warszawa)

Let us recall the following definition:

Definition 1. A set $A \in \mathfrak{R}$ is called the *maximal set* in the family $\mathfrak{R}$ if there exists no set $B \in \mathfrak{R}$ such that $A \subset B \neq A$.

Definition 2. A subset I of elements of a Boolean algebra B is called an *ideal*, if it satisfies the following conditions:

1° if $a, b \in I$ then $a+b \in I$,

2° if $a \in I$ then $a \cdot b \in I$ for arbitrary $b \in B$.

An ideal I is called *proper* if it is different from the whole algebra B. It is evident that the necessary and sufficient condition for an ideal I to be proper is $1 \notin I$.

The following "ideals' extension theorem" is known:

THEOREM 1. *If I is a proper ideal of a Boolean algebra B, then there exists a maximal proper ideal I^* which contains I* (more precisely: *in the family $\mathfrak{R}$ of all proper ideals which contain I there exists a maximal ideal I^**).

It is known that the proof of this theorem is not effective; it is based on Zermelo's theorem on well-ordering, or on Zorn's lemma equivalent to it. In both cases it is based on the axiom of choice. It is not known whether this theorem is equivalent to the axiom of choice; we only know that it is equivalent to other "non-effective" theorems, *e. g.* to the theorem stating that every free-torsion Abelian group may be ordered, to Tychonoff's theorem on the product of bicompact T_2-spaces and so on. It is remarkable that (as shown by J. L. Kelley) Tychonoff's theorem for T_1-spaces is equivalent to the axiom of choice (evidently, this equivalence is based on the axioms of the set-theory without the axiom of choice).

In this paper we shall show how it is possible to change theorem 1, so as to make it equivalent to the axiom of choice.

Let us consider the following theorem:

THEOREM 2. *Let A be a subset of elements of a Boolean algebra B and I an ideal disjoint with A. Then there exists an ideal I^* including I and disjoint with A and maximal with respect to this property (i. e. I^* is a maximal set in the family $\mathfrak{R}$ of all ideals including I and disjoint with A).*

It is evident that theorem 1 is a simple consequence of theorem 2; it is enough to put $A = \{1\}$.

We shall show that theorem 2 implies the axiom of choice.

Firstly we shall recall the following definition:

Definition 3. A set F of elements of a Boolean algebra B is called a *filter* if it satisfies the following conditions:

1° if $a, b \in F$ then $a \cdot b \in F$,

2° if $a \in F$ then $a+b \in F$ for arbitrary $b \in B$.

A filter F is called *proper* if it is different from the whole algebra B. The necessary and sufficient condition for the filter F to be proper is $0 \notin F$.

We see that if I is an ideal, the set of all a' (a' denotes the complement of a), where $a \in I$, is a filter and conversely.

Hence it follows that theorem 2 is equivalent to an analogous theorem 2′, for the filters:

THEOREM 2′. *Let A be a subset of elements of a Boolean algebra B and let F be a filter disjoint with A. Then there exists a filter F^* containing F and disjoint with A and maximal with respect to this property.*

In the sequel we shall introduce the following notation:

If B is a Boolean algebra and T an arbitrary non-empty set we shall denote by $\underset{\tau \in T}{P} B_\tau$ the product of algebras B, *i. e.* the set of all functions f defined on T with the values belonging to B and with the operations defined as follows:

$$h = f+g \quad \text{if} \quad h(\tau) = f(\tau)+g(\tau) \quad \text{for every} \quad \tau \in T,$$

$$h = f \cdot g \quad \text{if} \quad h(\tau) = f(\tau) \cdot g(\tau) \quad \text{for every} \quad \tau \in T,$$

$$h = f' \quad \text{if} \quad h(\tau) = [f(\tau)]' \quad \text{for every} \quad \tau \in T.$$

Now we shall prove the following lemmas:

LEMMA 1. *Let A be a subset of elements of a Boolean algebra B, and $\mathfrak{R}$ a family of filters $\{F_\tau\}_{\tau \in T}$ which are disjoint with the set A. Then there exists a family $\mathfrak{R}^*$ of filters $\{F^*_\tau\}_{\tau \in T}$ such that every F^*_τ is disjoint with A and contains F_τ and is maximal with respect to this property.*

Proof. Let us set $\overline{B} = \underset{\tau \in T}{P} B_\tau$. Let $\overline{A}$ be the set of all those $f \in \overline{B}$, for which $f(\tau) \in A$ for at least one $\tau \in T$ and $\overline{F}$ be the set of all those $f \in \overline{B}$

for which $f(\tau)\in F_\tau$ for every $\tau\in T$. Evidently, the set $\bar{F}$ is a filter and $\bar{F}\cdot\bar{A}=0$. By theorem 2′ there exists a filter $\bar{F}^*$ which contains $\bar{F}$ and is disjoint with $\bar{A}$ and maximal with respect to this property. Let $F^*_{\tau_0}$ be the set of all those $a\in B$ for which the element $f\in\bar{B}$ where $f(\tau_0)=a$ and $f(\tau)=1$ for $\tau\neq\tau_0$, belongs to $\bar{F}^*$. It is evident that $F^*_{\tau_0}$ is a filter including F_{τ_0} and disjoint with A. We shall show that $F^*_{\tau_0}$ is the maximal filter disjoint with A. In fact, if it is not maximal there should exist a filter $F\subset B$ such that $F^*_{\tau_0}\subset F\neq F^*_{\tau_0}$ and $F\cdot A=0$. If we denote by $\bar{F}_1$ the set of all those $f\in\bar{B}$ for which $f(\tau_0)\in F$ and $f(\tau)\in F^*_\tau$ for $\tau\neq\tau_0$, we see that $\bar{F}_1$ is a filter disjoint with $\bar{A}$ and $\bar{F}^*\subset\bar{F}_1\neq\bar{F}^*$; but this leads to a contradiction, because the filter $\bar{F}^*$ is the maximal filter disjoint with $\bar{A}$.

LEMMA 2. *Let S be a sublattice of a Boolean algebra B and let $0\in S$. Let $\mathfrak{R}$ be a family of proper filters $\{F_\tau\}_{\tau\in T}$ of the lattice S (i. e. $F_\tau\subset S$). Then there exists a family $\mathfrak{R}^*$ of maximal proper filters $\{F^*_\tau\}_{\tau\in T}$ such that $F_\tau\subset F^*_\tau\subset S$.*

This lemma is a simple consequence of lemma 1. It is enough to set $A=B-S+\{0\}$ and apply lemma 1 to the family $\mathfrak{R}$.

Now we shall show that lemma 2 implies the axiom of choice.

Let $\mathfrak{K}=\{C_\tau\}_{\tau\in T}$ be a family of non-empty and mutually disjoint sets. Let us set $P=\sum_{\tau\in T} C_\tau$. We shall introduce the topology in the set P in the following manner: the family S of all closed subsets of the space P is a family consisting of

(α) the empty set 0,

(β) all one-point sets,

(γ) all the sets C_τ $(\tau\in T)$,

(δ) all finite unions of the sets of (β) and (γ).

It is evident that the space P thus defined is a bicompact T_1-space. The lattice S of all closed subsets of the space P is contained in the Boolean algebra B of all subsets of the space P and $0\in S$. Let us denote by F_τ a filter consisting of closed sets containing C_τ. Since C_τ is a non-empty set, F_τ is a proper filter and evidently $F_\tau\subset S$. Let us consider the family $\mathfrak{R}$ of all these filters. By lemma 2 there exists a family $\mathfrak{R}^*$ of maximal proper filters $\{F^*_\tau\}_{\tau\in T}$ contained in S and satisfying the condition $F_\tau\subset F^*_\tau$. Let us notice that if $D_1,\ldots,D_k\in F^*_\tau$ then $D_1\cdot\ldots\cdot D_k\in F^*_\tau$ and since $0\notin F^*_\tau$ (F^*_τ being a proper filter) we have $D_1\cdot\ldots\cdot D_k\neq 0$ for every $D_1,\ldots,D_k\in F^*_\tau$. But P being a bicompact space implies that $\prod_{D\in F^*_\tau} D\neq 0$. We shall show that $\prod_{D\in F^*_\tau} D$ is a one-point set. In fact, in the opposite case the class F of closed subsets of P which contain the element $a\in\prod_{D\in F^*_\tau} D$ would be a proper filter contained in S and $F^*_\tau\subset F\neq F^*_\tau$; but this leads to contradiction because F^*_τ is the maximal filter. Since $C_\tau\in F^*_\tau$, $\prod_{D\in F^*_\tau} D\subset C_\tau$. In this manner we may choose one element from each of the sets C_τ.

References

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Reçu par la Rédaction le 12.3.1955