Undecidability of first order sentences in the theory of free groupoids

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1. J. Pepis has proved some theorems about reducibility of the decision problem for predicate calculus. One of those theorems (Pepis [2], theorem 35, p. 325) states that the satisfiability problem for sentences of the predicate calculus is reducible to the satisfiability problem for sentences of the form

$$(1)\qquad \prod_y \prod_z \sum_x P(x,y,z)\cdot \prod_{x_1}\cdots\prod_{x_s}\pi(x_1,\dots,x_s,P,E)$$

such that, $\psi(x,y)$ being a function from natural numbers to natural number and

$$(2)\qquad \begin{gathered}\prod_x\prod_y\big(x<\psi(x,y)\cdot y<\psi(x,y)\big)\,,\\ \prod_x\prod_y\prod_z\prod_t\big(\psi(x,z)=\psi(y,t)\to x=y\cdot z=t\big)\end{gathered}$$

being fulfilled, the sentence (1) is satisfiable if and only if (1) possesses a model in which individuals are natural numbers and $P(x,y,z)$ is interpreted as $x=\psi(y,z)$ (Pepis [2], definition 4, p. 290). In (1), E is a unary predicate and no quantifier occurs in π.

The present paper contains a more general result concerning algebraic sentences and a short proof of Pepis theorem. Some consequences of our theorem 7 were presented in an earlier paper (Jaśkowski [1]), where theorem 7 was given without proof.

2. i being a positive integer, x_i and y_i are individual variables, q_i and r_i are positive integers, f_i is the symbol of a q_i-ary function from individuals to individual, R_i is the symbol of a r_i-ary relation. Beyond that, x, y and z are individual variables, the dot $\bullet$ a binary function symbol, P and E — relation symbols in some particular cases. Individual variables are atomic terms. If $\vartheta_1,\dots,\vartheta_{q_i}$ are terms, $f_i(\vartheta_1,\dots,\vartheta_{q_i})$ is a term. If $\vartheta_1,\dots,\vartheta_{r_i}$ are terms, $R_i(\vartheta_1,\dots,\vartheta_{r_i})$ is an atomic formula. Identity symbol does not occur in formulae. α and β being formulae,

$$\sim\alpha\,,\quad \alpha\vee\beta\,,\quad \alpha\cdot\beta\,,\quad \alpha\to\beta\,,\quad \alpha\equiv\beta\,,\quad \prod_{x_i}\alpha\,,\quad \sum_{x_i}\alpha\,,\quad \prod_{R_i}\alpha\,,\quad \sum_{R_i}\alpha$$

are formulae. Formulae without quantifiers are *matrices*, formulae without bound relation symbols are *elementary*. Elementary formulae without free individual variables are *elementary sentences*. Formulae without free individual variables and without free relation symbols are *first order sentences*.

By Gödel's method of arithmetisation, the set of terms and formulae may be ordered in a sequence. This holds for any finite or infinite sequences $\{q_i\}$, $\{r_i\}$. Let $\mathrm{Ar}(\alpha)$ denote Gödel's number for α and Γ_j denote the term or formula with number j. An operation Φ from formula to formula is called *recursive* if and only if the function

$$\varphi(j)=\mathrm{Ar}\big(\Phi(\Gamma_j)\big)$$

is primitive recursive.

$\alpha=\alpha(x_1,\dots,x_n,f_1,\dots,f_l,R_1,\dots,R_m)$ denotes that no free variable, different from those written in parentheses, occurs in α.

$\mathcal{M}=\langle \mathrm{M},\mathrm{f}_1^0,\dots,\mathrm{f}_l^0,\mathrm{R}_1^0,\dots,\mathrm{R}_m^0\rangle$ being an algebra with q_i-ary functions f_i^0 and r_i-ary relations R_i^0 over the set of individuals M, the expression obtained from α by replacing f_i by f_i^0 and R_i by R_i^0, is called realization of α in $\mathcal{M}$ and denoted by

$$\alpha[\mathcal{M}](x_1,\dots,x_n)=\alpha(x_1,\dots,x_n,\mathrm{f}_1^0,\dots,\mathrm{f}_l^0,\mathrm{R}_1^0,\dots,\mathrm{R}_m^0)\,.$$

In the realization of α in $\mathcal{M}$, bound individual variables run over the set M, bound variables R_i run over the set of r_i-ary relations on M. If α is an elementary or first order sentence and $\alpha[\mathcal{M}]$ is true, we say that α is satisfied in $\mathcal{M}$ or that $\mathcal{M}$ is a model for α. Hence a first order sentence $\beta=\beta(f_1,\dots,f_l)$, may have a model $\mathcal{M}=\langle \mathrm{M},\mathrm{f}_1^0,\dots,\mathrm{f}_l^0\rangle$ without relations. An elementary or first order sentence α is *satisfiable* if and only if α possesses a model: $\sum_{\mathcal{M}}\alpha[\mathcal{M}]$. The sentence α is *tautological* if and only if $\sim\alpha$ is not satisfiable: $\sim\sum_{\mathcal{M}}(\sim\alpha)[\mathcal{M}]$.

$=$ is the symbol of the identity relation over the set of elements of an algebra; it is distinct from the metalogical identity symbol $=$, to avoid ambiguity.

We do not apply here Gödel's completeness theorem, according to which satisfiability of elementary sentences is equivalent to consistency.

$\boldsymbol{K}$ being a class of sentences, we distinguish three kinds of the decision problem for $\boldsymbol{K}$. We may ask for any $\alpha\in\boldsymbol{K}$: (i) whether α is satisfiable, or (ii) whether α is tautological, or (iii) whether α is satisfied in a given algebra $\mathcal{M}$ (decision problem in $\mathcal{M}$). The decision problem of one kind A for the class $\boldsymbol{K}$ is reducible to the decision problem of the same or another kind B for a class of sentences $\boldsymbol{L}$ if and only if there exists a recursive mapping Φ of $\boldsymbol{K}$ in $\boldsymbol{L}$ such that the decision of kind A for any $\alpha\in\boldsymbol{K}$ is equivalent to the decision of kind B for $\Phi(\alpha)$.

3. A sentence ε is called *normal* if and only if quantifiers occur only in the prefix of ε. A normal elementary sentence without existential quantifiers $\sum$ is called an *open sentence.* Thus an open sentence has the form

$$\prod_{x_1}\cdots\prod_{x_n}\mu \tag{3}$$

where $\mu=\mu(x_1,\ldots,x_n,f_1,\ldots,f_l,R_1,\ldots,R_m)$ is a matrix. It is well known that:

THEOREM 1. *Every satisfiable open sentence possesses a model with a single generator.*

Indeed, let $\mathcal{M}=\langle \mathrm{M},\mathrm{f}_1^1,\ldots,\mathrm{f}_l^1,\mathrm{R}_1^1,\ldots,\mathrm{R}_m^1\rangle$ be a model for the open sentence α. m_0 being an element of M, there exists a minimal subset M^0 of M over m_0, closed with respect to $\mathrm{f}_1^1,\ldots,\mathrm{f}_l^1$, and there exist functions $\mathrm{f}_1^0,\ldots,\mathrm{f}_l^0$ and relations $\mathrm{R}_1^0,\ldots,\mathrm{R}_m^0$, defined over M^0 and equal to $\mathrm{f}_1^1,\ldots,\mathrm{f}_l^1$ and $\mathrm{R}_1^1,\ldots,\mathrm{R}_m^1$ on M^0. Now, $\mathcal{M}^0=\langle \mathrm{M}^0,\mathrm{f}_1^0,\ldots,\mathrm{f}_l^0,\mathrm{R}_1^0,\ldots,\mathrm{R}_m^0\rangle$ is the model for α, with the single generator m_0.

The reduction of the satisfiability problem for elementary sentences to the satisfiability problem for open sentences is an analogon for the well known theorem on the reduction of the satisfiability problem for the predicate calculus to sentences with the Skolem prefix. ε being an elementary sentence, there exists a sentence $\mathbf{N}(\varepsilon)$ (*normal form* of ε) with properties: (i) $\varepsilon\equiv\mathbf{N}(\varepsilon)$ is a tautology, (ii) $\mathbf{N}(\varepsilon)$ is a normal sentence with an initial general quantifier. $\mathbf{N}(\varepsilon)$ has the form

$$\prod_{x_1}\cdots\prod_{x_{t_1}}\sum_{y_1}\prod_{x_{t_1+1}}\cdots\prod_{x_{t_2}}\sum_{y_2}\prod_{x_{t_2+1}}\cdots\prod_{x_{t_p}}\sum_{y_p}\prod_{x_{t_p+1}}\cdots\prod_{x_n}\mu \tag{4}$$

where $\mu=\mu(x_1,\ldots,x_n,y_1,\ldots,y_p,f_1,\ldots,f_l,R_1,\ldots,R_m)$ is a matrix, $1\leqslant t_1\leqslant t_2\leqslant\ldots\leqslant t_p\leqslant t_{p+1}=n$ and $\prod_{x_{t_i+1}}\cdots\prod_{x_{t_{i+1}}}$ is to be cancelled if $t_i=t_{i+1}$.

$\mathbf{O}(\varepsilon)$ — the *open form* of ε — is the open sentence:

$$\prod_{x_1}\cdots\prod_{x_n}\varrho \tag{5}$$

with matrix

$$\begin{aligned}\varrho&=\varrho(x_1,\ldots,x_n,f_1,\ldots,f_{l+p},R_1,\ldots,R_m)\\&=\mu(x_1,\ldots,x_n,f_{l+1}(x_1,\ldots,x_{t_1})\ldots f_{l+p}(x_1,\ldots,x_{t_p}),f_1,\ldots,f_l,R_1,\ldots,R_m)\,.\end{aligned}$$

$\mathbf{O}(\varepsilon)$ is a recursive operation: it is the known operation of elimination of existential quantifiers from the normal form by introducing choice functions $f_{l+1},\ldots,f_{l+p}$ with $q_{l+i}=t_i$. For open ε, we have $\mathbf{O}(\varepsilon)=\varepsilon$. By aid of the axiom of choice, one proves in a known manner

THEOREM 2. *The satisfiability problem for the class of elementary sentences ε is reducible to the satisfiability problem of the class of open sentences, by operation $\mathbf{O}$:*

$$\sum_{\mathcal{M}}\varepsilon[\mathcal{M}]\equiv\sum_{\mathcal{M}}\mathbf{O}(\varepsilon)[\mathcal{M}]\,.$$

4. The dot $\bullet$ is a binary function symbol, E — a set symbol, *i. e.* symbol of a unary relation written in the form $x\in E$ instead of $E(x)$. $\{q_i\}$, $\{r_i\}$ being sequences of positive integers and k any positive integer, we put

$$\prod_{i=1}^{1}{}^{\bullet}x_i=x_1,\qquad \prod_{i=1}^{k+1}{}^{\bullet}x_i=\Big(\prod_{i=1}^{k}{}^{\bullet}x_i\Big)\bullet x_{k+1},\qquad x^{\bullet k}=\prod_{i=1}^{k}{}^{\bullet}x\,, \tag{6}$$

$$\mathrm{f}_k^{\bullet}(x_1,\ldots,x_{q_k})=\Big(\prod_{i=1}^{q_k}{}^{\bullet}x_i\Big)^{\bullet k+1}, \tag{7}$$

$$\mathrm{R}_k^{\bullet}\langle E\rangle(x_1,\ldots,x_{r_k})\equiv\Big(\prod_{i=1}^{r_k}{}^{\bullet}x_i\Big)^{\bullet k+1}\in E\,. \tag{8}$$

Hence $x^{\bullet k}$ is the kth power of x in some sense: $x^{\bullet 1}=x$, $x^{\bullet k+1}=x^{\bullet k}\bullet x$.

If $\alpha=\alpha(x_1,\ldots,x_n,f_1,\ldots,f_l,R_1,\ldots,R_m)$ is a formula then

$$\alpha^{\bullet}\langle E\rangle=\alpha(x_1,\ldots,x_n,\mathrm{f}_1^{\bullet},\ldots,\mathrm{f}_l^{\bullet},\mathrm{R}_1^{\bullet}\langle E\rangle,\ldots,\mathrm{R}_m^{\bullet}\langle E\rangle)=\alpha^{\bullet}\langle E\rangle(x_1,\ldots,x_n) \tag{9}$$

denotes the formula obtained from α by the successive operations:

(i) formal substitution of $\mathrm{f}_i^{\bullet}$ for f_i and of $\mathrm{R}_i^{\bullet}\langle E\rangle$ for R_i,

(ii) substitution of the right sides of (6)-(8) for $\mathrm{f}_i^{\bullet}$ and $\mathrm{R}_i^{\bullet}\langle E\rangle$.

Finally, only $\bullet$ and E may occur in $\alpha^{\bullet}\langle E\rangle$ as function and relation symbols. ε being an elementary sentence and the open form $\mathbf{O}(\varepsilon)$ being represented by (3), the first order sentence

$$\sum_E\prod_{x_1}\cdots\prod_{x_n}\mu^{\bullet}\langle E\rangle \tag{10}$$

is called the *satisfiability reduct* of ε and denoted by $\mathbf{SR}(\varepsilon)$. The negation of the satisfiability reduct of non-ε may be expressed in the form

$$\prod_E\sum_{x_1}\cdots\sum_{x_n}\varrho \tag{11}$$

with matrix $\varrho=\varrho(x_1,\ldots,x_n,\bullet,E)$. (11) is called the *tautology reduct* of ε and denoted by $\mathbf{TR}(\varepsilon)$. Evidently

$$\mathbf{TR}(\varepsilon)\equiv\sim\mathbf{SR}(\sim\varepsilon) \tag{12}$$

is a tautology. **SR** and **TR** are recursive operations.

5. $\mathcal{F}=\langle \mathrm{F},*\rangle$ is a *free groupoid* if and only if there exists a finite or infinite subset $G\subset \mathrm{F}$ such that every element $a\in \mathrm{F}$ may be represented in a single manner in the form:

$$a=\vartheta^*(g_1,\ldots,g_q) \tag{13}$$

where the right side of (13) is a term in symbols $*$ and $g_1,\dots,g_q$ are elements of G. G is called the *set of free generators*.

Applying definitions (6) to $*$, we easily prove

LEMMA 3. *$\mathcal{F}=\langle \mathrm{F},*\rangle$ being a free groupoid, $a,a_1,a_2,\dots,b,b_1,b_2,\dots$ being elements of F; moreover k, j being positive integers and $\{s_i\}$ a sequence of positive integers,*

a. $\prod_{i=1}^{k}{}^{*} a_i=\prod_{i=1}^{k}{}^{*} b_i$ *if and only if* $a_1=b_1,\dots,a_k=b_k$,

b. $a^{*j+1}=b^{*k+1}$ *if and only if* $j=k$ *and* $a=b$,

c. $(\prod_{i=1}^{s_j}{}^{*} a_i)^{*j+1}=(\prod_{i=1}^{s_k}{}^{*} b_i)^{*k+1}$ *if and only if* $j=k$ *and* $a_1=b_1,\dots,a_{s_j}=b_{s_k}$.

Putting $*$ for $\bullet$ in definitions (7), (8), we get functions f_k^* from F to F and relations R_k^* with r_k individual arguments and one argument E of the type of a set.

LEMMA 4 *(on a homomorphism). $\mathcal{F}=\langle \mathrm{F},*\rangle$ being a free groupoid, $\mathcal{M}=\langle \mathrm{M},\mathrm{f}_1^0,\dots,\mathrm{f}_l^0,\mathrm{R}_1^0,\dots,\mathrm{R}_m^0\rangle$ an algebra with one generator m_0, with q_i-ary functions f_i^0 and r_i-ary relations R_i^0, there exists a single function h from F to M which fulfils the following conditions:*

(a) *For any $k=1,\dots,l$ and any $x,x_1,\dots,x_{q_k}\in \mathrm{F}$, if*

$$x=\mathrm{f}_k^*(x_1,\dots,x_{q_k}) \tag{14}$$

then

$$hx=\mathrm{f}_k^0(hx_1,\dots,hx_{q_k}) . \tag{15}$$

(b) *If $x\in\mathrm{F}$ and if (14) is false for any $k=1,\dots,l$ and for any $x_1,\dots,x_{q_k}\in\mathrm{F}$, then*

$$hx=m_0 . \tag{16}$$

Proof. For any $a\in\mathrm{F}$, the number of symbols $*$ which occur in the unique representation of a by generators (13) is called the *degree* of a. The proof is inductive with respect to the degree of x. x being an element of F of degree 0, x is a generator and $hx=m_0$ is the unique value of hx determined by conditions (a), (b). Suppose now that the value of the function h is uniquely determined for all elements of lower degree than n and that x is an element of degree n. By lemma 3c, x may be represented in the form (14) in one manner at most. If that representation is possible, $x_1,\dots,x_{q_k}$ possess lower degrees than n and $hx_1,\dots,hx_{q_k}$ are uniquely determined. Hence the conditions (a), (b) determine exactly one value for hx.

LEMMA 5 *(on a homomorphism, continued). Suppositions of lemma 4 remaining valid, h being the function from F to M, determined by conditions (a), (b), there exist a single subset $E\subset\mathrm{F}$, satisfying following conditions:*

(c) *For any $k=1,\dots,m$ and any $x,x_1,\dots,x_{r_k}\in\mathrm{F}$, if*

$$x=\Big(\prod_{i=1}^{r_k}{}^{*} x_i\Big)^{*k+1} \tag{17}$$

then

$$x\in E\equiv \mathrm{R}_k^0(hx_1,\dots,hx_{r_k}) . \tag{18}$$

(d) *If $x\in\mathrm{F}$ and if (17) is false for any $k=1,\dots,m$ and any $x_1,\dots,x_{r_k}\in\mathrm{F}$, then*

$$\sim(x\in E) . \tag{19}$$

The proof is analogous to that of lemma 4, by lemma 3c.

6. THEOREM 6. *$\mathcal{F}$ being a free groupoid, the satisfiability problem for the class of elementary sentences ε is reducible to the decision problem for the class of first order sentences with prefix $\sum_E\prod_{x_1}\dots\prod_{x_n}$, in $\mathcal{F}$. The reduction is made by the operation* **SR**:

$$\sum\nolimits_{\mathcal{M}}\varepsilon[\mathcal{M}]=\mathbf{SR}(\varepsilon)[\mathcal{F}] .$$

By theorem 2, we limit the proof to the case of an open sentence ε having the form (3). By (10), $\mathbf{SR}(\varepsilon)[\mathcal{F}]$ has the form

$$\sum\nolimits_{E\subset\mathrm{F}}\prod\nolimits_{x_1\in\mathrm{F}}\dots\prod\nolimits_{x_n\in\mathrm{F}}\mu^*\langle E\rangle . \tag{20}$$

Now, if (20) is true, there exists a subset E of F such that

$$\mathcal{M}=\langle \mathrm{F},\mathrm{f}_1^*,\dots,\mathrm{f}_l^*,\mathrm{R}_1^*\langle E\rangle,\dots,\mathrm{R}_m^*\langle E\rangle\rangle$$

is a model for (3). Hence (20) implies the satisfiability of (3). It remains to prove the converse implication. (3) being satisfiable, (3) possesses a model

$$\mathcal{M}=\langle \mathrm{M},\mathrm{f}_1^0,\dots,\mathrm{f}_l^0,\mathrm{R}_1^0,\dots,\mathrm{R}_m^0\rangle$$

with single generator m_0, by theorem 1. By lemmas 4 and 5, there exist a homomorphism h and a subset $E\subset\mathrm{F}$ which satisfy conditions (a), (b), (c), (d). Consider an arbitrary term

$$\vartheta(x_1,\dots,x_t)=\vartheta(x_1,\dots,x_t,f_1,\dots,f_l) .$$

By condition (a), the substitution of f_i^* and f_i^0 for f_i gives functions: ϑ^* on F and ϑ^0 on M, which satisfy, for any $x_1,\dots,x_t\in\mathrm{F}$

$$h\vartheta^*(x_1,\dots,x_t)=\vartheta^0(hx_1,\dots,hx_t) . \tag{21}$$

By (c), (9), (21), an inductive proof exists for

$$\alpha^*\langle E\rangle(x_1,\dots,x_k)\equiv\alpha[\mathcal{M}](hx_1,\dots,hx_k) \tag{22}$$

where a is an arbitrary matrix with symbols: $a=a(x_1,\ldots,x_k,f_1,\ldots,f_l,R_1,\ldots,R_m)$. (3) being true in $\mathcal{M}$, the right side of (22) is true for any $x_1,\ldots,x_n\in \mathrm{F}$, if $a=\mu$. Hence (20).

THEOREM 7. *$\mathcal{F}$ being a free groupoid, the tautology problem for the class of elementary sentences ε is reducible to the decision problem for the class of first order sentences with prefix $\prod_E \sum_{x_1}\ldots\sum_{x_n}$, in $\mathcal{F}$. The reduction is made by operation* **TR**:

$$\sim\sum_{\mathcal{M}}(\sim\varepsilon)[\mathcal{M}]\equiv \mathbf{TR}(\varepsilon)[\mathcal{F}]\,.$$

Proof. By theorem 6 and by (12).

In the satisfiability or tautology reduct, the matrix may be expressed in the normal form. Denote by **NTR**(ε) the tautology reduct (11) in which the matrix has the normal form of a logical sum with h summands, each summand being product of k factors:

$$\varrho=\bigcup_{i=1}^{h}\bigcap_{j=1}^{k}(\vartheta_j\in E^{e_{ij}})$$

where ϑ_j are terms, $\{e_{ij}\}$ is a finite double sequence with $e_{ij}=\pm 1$, $i=1,\ldots,h$, $j=1,\ldots,k$ and $E^{e_{ij}}$ denotes E or its complement E':

$$E^1=E\,,\qquad E^{-1}=E'\,.$$

Let N be the set of non-negative integers and

$$\psi(x,y)=\tfrac{1}{2}(x+y)(x+y+1)+y+1\,.$$

Hence $\mathcal{N}=\langle \mathrm{N},\psi\rangle$ is a free groupoid. By theorem 7, the tautology problem of any elementary sentence ε is reducible to the decision problem of **NTR**$(\varepsilon)[\mathcal{N}]$. In Jaśkowski [1] the last result was quoted, without proof and the tautology reduct in $\mathcal{N}$ was called the Pepis problem.

The operation **SR** may be replaced by a simpler one, if we have to reduce the satisfiability problem for open sentences ω with one binary function and one unary relation: $\omega=\omega(\bullet,E)$. In that case, $\sum_E\omega$ is a first order sentence.

THEOREM 8. *$\mathcal{F}$ being a free groupoid, the satisfiability problem for the class of open sentences $\omega=\omega(\bullet,E)$ is reducible to the same decision problem as in theorem 6 by operation $\sum_E$*:

$$\sum_{\mathcal{M}}\omega[\mathcal{M}]\equiv\Big(\sum_E\omega\Big)[\mathcal{F}]\,.$$

The proof is like that of theorem 6, it is based on the following lemmas analogous to lemmas 4, 5:

$\mathcal{F}=\langle \mathrm{F},*\rangle$ being a free groupoid, $\mathcal{M}=\langle \mathrm{M},\circ,\mathrm{E}^0\rangle$ an algebra with one generator m_0, one binary function $\circ$ and one constant set symbol E^0,

(i) *there exists a single function h from F to M which fulfils:*

(a) *for any $x,x_1,x_2\in\mathrm{F}$, if $x=x_1*x_2$, then $hx=hx_1\circ hx_2$,*

(b) *for every $x\in\mathrm{F}$, if x non$=x_1*x_2$ for any $x_1,x_2\in\mathrm{F}$, then $hx=m_0$,*

(ii) *there exists one subset $E\subset\mathrm{F}$ which fulfils:*

$$x\in E\equiv hx\in\mathrm{E}^0\,.$$

SOR(ε) *(open reduct of satisfiability)* denotes now the open sentence obtained from the satisfiability reduct **SR**(ε) by cancelling the initial $\sum_E$.

THEOREM 9. *The satisfiability problem for the class of elementary sentences ε is reducible* (a) *to the satisfiability problem for the class of open sentences with matrix $\omega=\omega(\bullet,E)$ and* (b) *to the satisfiability problem for the class of first order sentences with prefix $\sum_E\prod_{x_1}\ldots\prod_{x_n}$ and with matrix $\omega=\omega(\bullet,E)$. The reductions are made*

(a) *by operation* **SOR**

$$\sum_{\mathcal{M}}\varepsilon[\mathcal{M}]\equiv\sum_{\mathcal{M}}\mathbf{SOR}(\varepsilon)[\mathcal{M}]\,,$$

(b) *by operation* **SR**

$$\sum_{\mathcal{M}}\varepsilon[\mathcal{M}]=\sum_{\mathcal{M}}\mathbf{SR}(\varepsilon)[\mathcal{M}]\,.$$

Proof. By **SR**$(\varepsilon)=\sum_E$**SOR**(ε), (a) is an immediate consequence of theorems 6 and 8. But the satisfiability of $\omega(\bullet,E)$ is equivalent to the satisfiability $\sum_E\omega(\bullet,E)$. Hence (b).

7. By a known method, we eliminate the function symbol $\bullet$ from the open reduct of satisfiability and we introduce a new ternary relation symbol. Let $\vartheta_1,\ldots,\vartheta_s$ be all terms which occur in an open sentence $\omega=\omega(\bullet,E)$. If ϑ_i is not a single variable, it has the form $\vartheta_i=\vartheta_j\bullet\vartheta_k$ for some $j,k=1,\ldots,s$. By a suitable choice of indices, there exist finite sequences $\{a_i\}$, $\{b_i\}$ with integer a_i,b_i, satisfying $1\leqslant a_i<i$ and $1\leqslant b_i<i$ for $i=n+1,\ldots,s$ such that

$$\vartheta_i=\begin{cases}x_i & \text{for}\quad i=1,\ldots,n\,,\\ \vartheta_{a_i}\bullet\vartheta_{b_i} & \text{for}\quad i=n+1,\ldots,s\,.\end{cases}\tag{23}$$

Let $\vartheta_{j_1},\ldots,\vartheta_{j_k}$ be those terms which occur in ω as arguments in atomic formulae $\vartheta_{j_i}\in E$. ω may be written as

$$\prod_{x_1}\ldots\prod_{x_n}\varkappa(\vartheta_{j_1},\ldots,\vartheta_{j_k},E)\,.\tag{24}$$

Let P be a ternary relation symbol. Write

$$\pi=\pi(x_1,\ldots,x_s,P,E)=\Big(\bigcap_{i=n+1}^{s}P(x_i,x_{a_i},x_{b_i})\Big)\rightarrow\varkappa(x_{j_1},\ldots,x_{j_k},E)\,. \tag{25}$$

This notation for π being accepted, the Pepis sentence (1) is called the *predicative form* of ω and denoted by $\mathbf{Pr}(\omega)$. $\mathbf{Pr}$ is a recursive operation.

THEOREM 10. *Every open sentence $\omega=\omega(\bullet,E)$ with a satisfiable predicative form, is satisfiable*:

$$\sum_{\mathcal{M}}\mathbf{Pr}(\omega)[\mathcal{M}]\rightarrow\sum_{\mathcal{M}}\omega[\mathcal{M}]\,.$$

Proof. Let (24) be the sentence ω and (1) the sentence $\mathbf{Pr}(\omega)$. The symbol $\bullet$ does not occur in (1), it will be used as a symbol of the choice function by constructing $\mathbf{O}\big(\mathbf{Pr}(\omega)\big)$:

$$\prod_y\prod_z\prod_{x_1}\cdots\prod_{x_s}\big(P(y\bullet z,y,z)\cdot\pi(x_1,\ldots,x_s,P,E)\big)\,. \tag{26}$$

(26) is satisfiable by theorem 2 and implies (24) by (23), (25). Hence (24) is satisfiable.

THEOREM 11. *$\mathcal{F}=\langle\mathrm{F},*\rangle$ being a free groupoid, $\omega=\omega(\bullet,E)$ a satisfiable open sentence, $\mathbf{Pr}(\omega)$ is satisfiable in a model $\mathcal{P}=\langle\mathrm{F},\mathrm{P}^1,\mathrm{E}^1\rangle$, where*

$$\mathrm{P}^1(x,y,z)\equiv x=y*z\,. \tag{27}$$

Proof. Let (24) be the sentence ω. By theorem 8, $\sum_E\omega$ is satisfied in $\mathcal{F}$ and there exists a subset $\mathrm{E}^1\subset\mathrm{F}$ such that ω is satisfied in a model $\langle\mathrm{F},*,\mathrm{E}^1\rangle$. Now, ω and (27) imply (26) by (23), (24) and (25). Hence (26) is satisfied in the model $\langle\mathrm{F},*,\mathrm{P}^1,\mathrm{E}^1\rangle$ in which P^1 is defined by (27). By (27), the sentence (26) implies (1) which is $\mathbf{Pr}(\omega)$ and does not contain the symbol $*$. Thus $\mathbf{Pr}(\omega)$ is satisfied in $\mathcal{P}$.

THEOREM 12. (a) *The satisfiability problem for the class of elementary sentences ε is reducible to the satisfiability problem for the class of elementary sentences of the Pepis form (1) with matrix π represented by (25). The reduction is made by the operation $\mathbf{Pr}\big(\mathbf{SOR}(\varepsilon)\big)$*:

$$\sum_{\mathcal{M}}\varepsilon[\mathcal{M}]\equiv\sum_{\mathcal{M}}\mathbf{Pr}\big(\mathbf{SOR}(\varepsilon)\big)[\mathcal{M}]\,.$$

(b) *If $\mathcal{F}=\langle\mathrm{F},*\rangle$ is a free groupoid, P^1 satisfies (27) and $\mathbf{Pr}\big(\mathbf{SOR}(\varepsilon)\big)$ is satisfiable, then $\mathbf{Pr}\big(\mathbf{SOR}(\varepsilon)\big)$ has a model $\mathcal{P}=\langle\mathrm{F},\mathrm{P}^1,\mathrm{E}^1\rangle$ for some $\mathrm{E}^1\subset\mathrm{F}$*:

$$\sum_{\mathcal{M}}\mathbf{Pr}\big(\mathbf{SOR}(\varepsilon)\big)[\mathcal{M}]\rightarrow\sum_{E^1\subset F}\mathbf{Pr}\big(\mathbf{SOR}(\varepsilon)\big)[\langle\mathrm{F},\mathrm{P}^1,\mathrm{E}^1\rangle]\,.$$

Proof. (a) is an immediate consequence of theorems 9(a), 10, 11; (b) is a consequence of theorem 11.

Pepis conditions (2) imply that natural numbers are a free groupoid with respect to ψ. Thus the Pepis theorem is a particular case of theorem 12.

References

[1] S. Jaśkowski, *Example of a class of systems of ordinary differential equations having no decision method for existence problems*, Bull. Acad. Pol. Sci., Cl. III, 2 (1954), p. 155-157.

[2] J. Pepis, *Untersuchungen über das Entscheidungsproblem der mathematischen Logic*, Fund. Math. 30 (1938), p. 257-348.

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