A constructivist theory of plane curves

by

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Introduction. This paper develops a theory of p-curves, which are finite matrices with binary fraction elements. Roughly speaking, a p-curve is a finite assemblage of points in serial order on a grid, the jump from one point to the next being of fixed amount in one or other of two "directions". The concept of a plane curve is then introduced in terms of sequences of p-curves. The emphasis throughout the paper is on the strictly finitist character of the proof processes.

The present work on *analysis situs* is a preliminary to a study of curvilinear integrals.

Definitions. We denote integers by $i, j, k, l, m, n, \mu, \nu, p, q, r, s, t, \varrho, \sigma, \tau$ with or without suffixes, and binary fractions $m/2^p$ by $a, b, c, d, x, y, \xi, \eta$ with or without suffixes or affixes; more specifically, for a given p we write x^p, etc., for $m/2^p$. The ordered pair (x,y) is called a *point*, and the ordered pair $\langle x_1, y_2\rangle$ an *interval*; the ordered pair of intervals $\langle x_1, x_2\rangle\langle y_1, y_2\rangle$ (where $x_1 < x_2$, $y_1 < y_2$) is called a *rectangle* with *vertices* (x_r, y_s), $r=1,2$ and $s=1,2$. If

$$2^p a_r^p\,, \quad 0 \leqslant r \leqslant \mu_p, \quad 2^p b_s^p, \quad 0 \leqslant s \leqslant \nu_p$$

are the integers from $2^p x_1^p$ to $2^p x_2^p$ and from $2^p y_1^p$ to $2^p y_2^p$, respectively, then the points

$$(a_r^p, b_s^p), \quad 0 \leqslant r \leqslant \mu_p, \quad 0 \leqslant s \leqslant \nu_p,$$

are called the *lattice points of the network*

$$F_p\begin{pmatrix} x_1^p & x_2^p \\ y_1^p & y_2^p \end{pmatrix}$$

in the rectangle $\langle x_1^p, x_2^p\rangle\langle y_1^p, y_2^p\rangle$; the rectangles $\langle a_r^p, a_{r+1}^p\rangle\langle b_s^p, b_{s+1}^p\rangle$, $0 \leqslant r \leqslant \mu_p$, $0 \leqslant s \leqslant \nu_p$ are called the *p-cells* of the rectangle $\langle x_1^p, x_2^p\rangle\langle y_1^p, y_2^p\rangle$ or of the network

$$F_p\begin{pmatrix} x_1^p & x_2^p \\ y_1^p & y_2^p \end{pmatrix}.$$

For a given $k \geq 1$ the integers i_r, j_r satisfy the equation

(0.1) $$|i_{r+1}-i_r|+|j_{r+1}-j_r|=1$$

for all r, $0 \leq r \leq k-1$, and $x_r^p = i_r/2^p$, $y_r^p = j_r/2^p$; then the *ordered* set of points

(0.2) $$(x_r^p, y_r^p), \quad 0 \leq r \leq k,$$

is called a *plane curve*, or specifically a *plane p-curve*, joining the points (x_0^p, y_0^p) and (x_k^p, y_k^p). If

(0.3) $$|i_r - i_s| + |j_r - j_s| > 0$$

for all r, s satisfying $0 \leq r < s \leq k$, then the curve (0.2) is said to be *simple* and *open*.

(0.31) If i_r, j_r satisfy the condition (0.3) for all r, s such that $0 \leq r \leq s \leq k-1$ or $1 \leq r < s \leq k$, and if in addition, $i_k = i_0$, $j_k = j_0$, $k \geq 4$, then the curve (0.2) is said to be *simple* and *closed*.

If x_r, y_r are periodic, with period k, and if (x_r, y_r), $0 \leq r \leq k$, is a simple closed curve, then *eo ipso* the curve (x_r, y_r), $m \leq r \leq m+k$, is closed and simple. The curves (x_r, y_r), $0 \leq r \leq k$, and (x_r, y_r), $m \leq r \leq m+k$ are said to be *equivalent* and mutually interchangeable.

If
$$i_{2r}^{p+1} = 2i_r^p, \quad i_{2r+1}^{p+1} = i_r^p + i_{r+1}^p,$$
and
$$j_{2r}^{p+1} = 2j_r^p, \quad j_{2r+1}^{p+1} = j_r^p + j_{r+1}^p,$$
where i_r^p, j_r^p satisfy (0.1), then obviously i_r^{p+1}, j_r^{p+1} satisfy (0.1). The p-curve (x_r^p, y_r^p), $0 \leq r \leq k$, and the $(p+1)$-curve (x_r^{p+1}, y_r^{p+1}), $0 \leq r \leq 2k$, (where $x_r^q = i_r^q/2^q$, $y_r^q = j_r^q/2^q$ for $q = p, p+1$) are said to be *equivalent* and mutually interchangeable.

We shall denote by ξ (with one or more suffixes) a value taken by one or more of the numbers x_r, $0 \leq r \leq k$, and by η a value taken by one or more of y_r, $0 \leq r \leq k$. In a network F_p, $\xi + 2^{-p}$ will be called the *successor* of ξ and denoted by ξ', and $\xi - 2^{-p}$ will be called the *predecessor* of ξ and denoted by ξ^*. Similarly we define η', η^*. The successor of x_r of course is x_{r+1}, $r < k$, and the predecessor of x_{r+1} is x_r, $r \geq 0$; we shall also call x_1 the successor of $x_k (= x_0)$ in a *closed* curve, and x_{k-1} the predecessor of x_0. Similarly the successor of y_k and the predecessor of y_0 are y_1 and y_{k-1} respectively, when the curve is closed.

Such a pair as ξ, ξ' will be called a *vertical strip*, and such as η, η' a *horizontal strip*.

If r_n, $0 \leq n \leq \mu$, are all the suffixes r such that, either
$$x_r = \xi, \quad x_{r+1} = \xi' \quad \text{or} \quad x_r = \xi', \quad x_{r+1} = \xi$$

in a simple closed curve (x_r, y_r), $0 \leq r \leq k$, then the values η_n of y_{r_n}, $0 \leq n \leq \mu$, are called the *boundary levels* in the vertical strip ξ, ξ' (note that $y_r = y_{r+1}$ since $x_r \neq x_{r+1}$). The η's are all different since the points (x_n, η_n), (x_{n+1}, η_n) cannot *both* occur twice in the set (x_r, y_r), $0 \leq r \leq k$. Similarly we define the boundary levels in the horizontal strip η, η'.

The foregoing definitions, and the proofs which follow, may be regarded as definition and proof *schemata* formalisable by replacing the unspecified numbers and functions introduced by definite numbers and functions. The definitions, however, are also susceptible of formalisation in a free variable calculus.

1. Theorem 1. *If (x_r, y_r), $0 \leq r \leq k$, is a simple curve, and if for some m, n (where $0 \leq m < n \leq k$ if the curve is open, and $0 \leq m < n < k$ or $0 < m \leq n < k$ if the curve is closed), and for all s, t satisfying $m \leq s < t \leq n$, we have $y_s = y_t$, then the sequence x_r, $m \leq r \leq n$, is strictly monotonic.*

We may suppose $n \geq m+2$, else there is nothing to prove. Let $x_s = \xi$ so that, since $y_{s+1} = y_s$, x_{s+1} must be either ξ' or ξ^*; suppose the former, then since x_{s+2} differs from both x_s and x_{s+1}, and the *values* of x_{s+1}, x_{s+2} are consecutive in F_p, therefore $x_{s+2} = \xi''$. Similarly, if $x_{s+1} = \xi^*$ then $x_{s+2} = \xi^{**}$, and so, since $x_m \gtrless x_{m+1}$, the sequence x_r, $m \leq r \leq n$, is strictly monotonic.

2. Theorem 2. *If the integers i_r satisfy the equation (0.1), and if $0 \leq m < n \leq k$, then i_r takes every integral value between i_m and i_n for a value of r between m and n.*

For if $i_m < v < i_n$ and if μ is the *smallest* integer, greater than m, such that $i_\mu \geq v$, then $i_{\mu-1} \leq v-1$; but
$$0 \leq (i_\mu - v) + (v - 1 - i_{\mu-1}) = (i_\mu - i_{\mu-1}) - 1 \leq 0,$$
and so $i_\mu = v$.

It follows that if (x_r, y_r), $0 \leq r \leq k$, is a p-curve then x_r attains every value $l/2^p$ between x_m, x_n for an r between m, n and y_r attains every value $l/2^p$ between y_m, y_n for an r between m, n.

3. Theorem 3. *If the integers i_r, $0 \leq r \leq k$, satisfy (0.1), and $i_k = i_0$, and if for some m, v, $i_0 < v$, $i_m = v$, then if $\lambda < k$ is the greatest suffix such that $i_\lambda = v$, we have $i_{\lambda+1} = v-1$; for otherwise $i_{\lambda+1} = v+1$, and by Theorem 2 we should have $i_r = v$ for some r, where $\lambda+1 < r < k$. Similarly, if $i_0 > v$ and μ is the least suffix for which $i_\mu = v$, then $i_{\mu-1} = v+1$.*

4. Theorem 4. *If (x_r, y_r), $0 \leq r \leq k$, is a simple closed curve, and if for some m, n*
$$x_m = \xi, \quad x_{m+1} = \xi' \quad \text{and} \quad x_n = \xi', \quad x_{n+1} = \xi,$$
then $m \neq n+1$ and $n \neq m+1$.

For by 0.31, $x_{m+2} \geq \xi' > x_{n+1}$ and $x_{n+2} \leq \xi < x_{m+1}$.

5. THEOREM 5. *If* (x_r, y_r), $0 \leqslant r \leqslant k$, *is a simple closed curve, then each pair of values* ξ, ξ' *is taken by consecutive* x*'s an even number of times.*

Proof. If there is no value of r, $0 \leqslant r \leqslant k$, such that for a given pair ξ, ξ', $x_r = \xi$ and $x_{r+1} = \xi'$ or $x_r = \xi'$ and $x_{r+1} = \xi$, then the theorem is proved.

If there is a *unique* value of r, $0 \leqslant r \leqslant k$, such that $x_r = \xi$, $x_{r+1} = \xi'$ then there is an s, $0 \leqslant s \leqslant k$, such that $x_s = \xi'$, $x_{s+1} = \xi$. For if $x_0 < \xi'$ and s is the greatest suffix such that $x_s = \xi'$ then, by Theorem 3, $x_{s+1} = \xi$. Moreover, by Theorem 4, $s > r+1$.

If there is more than one value of r for which $x_r = \xi$, $x_{r+1} = \xi'$ then between any consecutive two such values, m and n say, there is a ϱ such that

$$x_\varrho = \xi', \quad x_{\varrho+1} = \xi, \quad \text{and} \quad \varrho + 1 < n.$$

For if $\varrho+1$ is the *least* suffix between $m+1$ and n such that $x_{\varrho+1} = \xi$ then, by Theorem 3, $x_\varrho = \xi'$ (and $m+1 < \varrho < n-1$, by Theorem 4). Similarly, between consecutive values of r for which $x_r = \xi'$, $x_{r+1} = \xi$ there is a value of r for which $x_r = \xi$, $x_{r+1} = \xi'$.

Let m_s, $0 \leqslant s \leqslant \sigma$, where $m_s < m_{s+1}$, be *all* the values of r, $0 \leqslant r \leqslant k$, for which $x_r = \xi$, $x_{r+1} = \xi'$, and let μ_t, $0 \leqslant t \leqslant \tau$, where $\mu_t < \mu_{t+1}$ be all the values of r, $0 \leqslant r \leqslant k$, for which $x_r = \xi'$, $x_{r+1} = \xi$.

We may without loss of generality suppose that $m_0 < \mu_0$. Since there is a μ between m_0 and m_1 therefore μ_0 lies between m_0 and m_1, and since there is an m between μ_0 and μ_1, therefore m_1 lies between μ_0 and μ_1, and so μ_0 is the *only* μ between m_0 and m_1.

Similarly μ_s is the only μ between m_s and m_{s+1}, $0 \leqslant s \leqslant \sigma - 1$. By considering the *equivalent* closed curve (x_r, y_r), $m_\sigma \leqslant r \leqslant m_\sigma + k$, it follows that there is a *unique* τ between m_σ and $m_0 + k$ such that $x_r = \xi'$, $x_{r+1} = \xi$ and so there is just one μ greater than m_σ (for the least μ, μ_0, exceeds m_0). Thus $\tau = \sigma$, which completes the proof.

In the same way we can show that each pair of values η, η' is taken an even number of times by consecutive y's.

It follows from Theorem 5 that in a simple closed p-curve there are an *even* number of boundary levels in each horizontal strip, and an even number in each vertical strip. If x_l, x_g are the least and greatest values of x_r, $0 \leqslant r \leqslant k$, and y_l, y_g the least and greatest values of y_r, $0 \leqslant r \leqslant k$, and if $2^p X_s$, $0 \leqslant s \leqslant \sigma$ are the integers from $2^p x_l$ to $2^p x_g$ inclusive, and $2^p Y_t$, $0 \leqslant t \leqslant \tau$, the integers from $2^p y_l$ to $2^p y_g$, and if, finally, η_r^s, $1 \leqslant r \leqslant 2\mu_s$, are the boundary levels in the strip X_s, X_{s+1} and ξ_r^t, $1 \leqslant r \leqslant 2\nu_t$, the boundary levels in the strip Y_t, Y_{t+1} then the cells of the network F_p in all the rectangles $\langle X_s, X_{s+1}\rangle\langle \eta_{2r-1}^s, \eta_{2r}^s\rangle$, $1 \leqslant r \leqslant \mu_s$, $0 \leqslant s \leqslant \sigma-1$, are called the *interior* p_x*-cells* of the curve, and the cells of the network F_p in all the rectangles $\langle \xi_{2r-1}^t, \xi_{2r}^t\rangle$, $\langle Y_t, Y_{t+1}\rangle$, $1 \leqslant r \leqslant \nu_t$, $0 \leqslant \tau \leqslant r-1$ are called the *interior* p_y*-cells* of the curve. A cell of the network F_p, in a vertical strip, which is not an interior p_x-cell is called an *exterior* p_x*-cell*, and a cell in a horizontal strip which is not an interior p_y-cell is called an *exterior* p_y*-cell.*

6. Linked boundary levels. (x_r, y_r), $0 \leqslant r \leqslant k$, is a simple closed curve on a network F_p, where x_r, y_r are periodic with period k.

A level α (of the closed curve) in the strip ξ^*, ξ is said to be *linked along* ξ to the level β in the strip ξ, ξ' if either $\beta = \alpha$ or $\beta \gtrless \alpha$ and for some μ, $\nu > 1$

$$x_\mu = \xi^*, \quad x_{\mu+\nu+1} = \xi', \quad x_r = \xi, \quad \mu+1 \leqslant r \leqslant \mu+\nu,$$

and $y_\mu = \alpha$, $y_{\mu+\nu+1} = \beta$ and $2^p y_r$, $\mu+1 \leqslant r \leqslant \mu+\nu$, are the integers from $2^p\alpha$ to $2^p\beta$ inclusive.

Two levels α, β of the *same* strip ξ^*, ξ or ξ, ξ' are said to be *linked along* ξ if $y_\mu = \alpha$, $y_{\mu+\nu+1} = \beta$ and $2^p y_r$, $\mu+1 \leqslant r \leqslant \mu+\nu$, are the integers from $2^p\alpha$ to $2^p\beta$ inclusive, and $x_r = \xi$, $\mu+1 \leqslant r \leqslant \mu+\nu$, and either $x_\mu = x_{\mu+\nu+1} = \xi^*$ or $x_\mu = x_{\mu+\nu+1} = \xi'$.

6.1. *If a level* α *is linked to a level* β *along* ξ *then* α *is not linked to another level along* ξ; for, in a simple curve, there cannot be *two* values of r, $0 \leqslant r \leqslant k$, for which $y_r = y_{r+1} = \alpha$ and either $x_r = \xi^*$, $x_{r+1} = \xi$ or $x_r = \xi'$, $x_{r+1} = \xi$.

6.2. *If* α, β *are consecutive boundary levels in either of the strips* ξ^*, ξ; ξ, ξ' *and if* $\alpha \leqslant \eta < \eta' \leqslant \beta$, *then if for some* μ, $x_\mu = x_{\mu+1} = \xi$ *and* $y_\mu = \eta$, $y_{\mu+1} = \eta'$ *(or* $y_\mu = \eta'$, $y_{\mu+1} = \eta$*) the levels* α, β *are linked along* ξ.

Let $2^p\varrho_s$, $0 \leqslant s \leqslant \sigma$, be the integers from $2^p\alpha$ to $2^p\beta$ inclusive, where $\varrho_t = \eta$, $\varrho_{t+1} = \eta'$, say, $0 \leqslant t < \sigma$. If $\alpha = \eta$, $\beta = \eta'$ there is nothing to prove; hence we may suppose $\mu \geqslant 2$ so that for $1 \leqslant s \leqslant \sigma - 1$, ϱ_s not a boundary level in either of the strips ξ^*, ξ; ξ, ξ'.

Accordingly $x_{\mu+2} = \xi$, $y_{\mu+2} = \varrho_{t+2}$ and, by induction, $x_n = \xi$, $y_n = \varrho_{n+t-\mu}$ for $\mu \leqslant n \leqslant \mu + \sigma - t$. Similarly $x_m = \xi$, $y_m = \varrho_{m+t-\mu}$ for $\mu - t \leqslant m \leqslant \mu - 1$, which completes the proof.

6.21. It follows from 6.2 that *if the consecutive boundary levels* α, β *are not linked along* ξ *then there is no value of* μ, $0 \leqslant \mu \leqslant k$, *for which* $x_\mu = x_{\mu+1} = \xi$ *and* $2^p y_\mu$, $2^p y_{\mu+1}$ *are consecutive integers between* $2^p\alpha$ *and* $2^p\beta$ *inclusive.*

6.3. *If* α, β, γ $(\alpha < \beta < \gamma)$ *are consecutive boundary levels in either of the strips* ξ^*, ξ *or* ξ, ξ' *then either* β *is a level in both strips, or* β *is linked along* ξ *either to* α *or to* γ.

For if β is not a level in both strips and if $y_\mu = y_{\mu+1} = \beta$ and one of x_μ, $x_{\mu+1}$ is ξ, then two cases arise:

(a) $x_{\mu+1} = \xi$, then $x_{\mu+2} = x_{\mu+1}$;

(b) $x_\mu = \xi$, then $x_{\mu-1} = x_\mu$.

In case (a) either $y_{\mu+2} = \beta'$ or $y_{\mu+2} = \beta^*$; if the former then β is linked along ξ to γ, and if the latter, β is linked to α, by 6.2. The result follows in the same way in case (b). We observe that if γ is the *greatest* boundary level in either strip, and if γ is a level in only one of the two strips, and there is no greater boundary level in the other strip, then the foregoing considerations show that γ is necessarily linked to β along ξ. If γ is a level in both strips then, by definition, γ is linked along ξ.

6.4. We take for granted the definitions and theorems on linked levels in *horizontal* strips corresponding to 6-6.3 above.

6.5. *If $\langle \xi^*, \xi\rangle\langle \eta, \eta'\rangle$ and $\langle \xi, \xi'\rangle\langle \eta, \eta'\rangle$ are interior p_x cells of a simple closed contour (x_r, y_r), $0 \leqslant r \leqslant k$, then ξ is not a boundary level in the strip η, η'.*

There are an *odd* number of levels in the strip ξ^*, ξ which are not less than η', and an odd number in the strip ξ, ξ'. Hence there are an *even* number of levels which lie in one or other of the two strips, and which are not less than η' (counting twice a value of y_r which is a level in both strips). Let these levels, in decreasing order of magnitude be h_r, $1 \leqslant r \leqslant 2n$, (where for some values of r, h_r may equal h_{r+1}). By 6.3, h_1 is linked to h_2, h_3 to h_4 and hence by induction, h_{2r-1} is linked to h_{2r} for $1 \leqslant r \leqslant n$. If h_{2n+1} is the first level in either strip which is less than η' (and so not greater than η) then h_{2n}, being linked to h_{2n-1}, is not linked to h_{2n+1}, and so by 6.21 there is no μ, $0 \leqslant \mu < k$, such that $x_\mu = x_{\mu+1} = \xi$ and $y_\mu = \eta$, $y_{\mu+1} = \eta'$ (or $y_{\mu+1} = \eta$, $y_\mu = \eta'$) which proves that ξ is not a boundary level in the horizontal strip η, η'.

6.51. *If one of the p_x-cells $\langle \xi^*, \xi\rangle\langle \eta, \eta'\rangle$, $\langle \xi, \xi'\rangle\langle \eta, \eta'\rangle$ is interior, and one exterior, then ξ is a boundary level in the strip η, η', and conversely.*

Proof similar to 6.5.

7. THEOREM 7. *In a simple closed curve (x_r, y_r), $0 \leqslant r \leqslant k$, the interior p_x-cells are interior p_y-cells, and vice-versa.*

Let the boundary levels in the strip η, η' be ξ_s, $1 \leqslant s \leqslant 2\mu$, where $\xi_{s+1} > \xi_s$, $1 \leqslant s \leqslant 2\mu - 1$.

If σ is less than x_r, $0 \leqslant r \leqslant k$, then, by 6.51, there are no interior p_x-cells in the rectangle $\langle \sigma, \xi_1\rangle\langle \eta, \eta'\rangle$, and therefore, by 6.51, all the interior p_x-cells which lie in the strip η, η' lie in the rectangles $\langle \xi_{2r-1}, \xi_{2r}\rangle \langle \eta, \eta'\rangle$, and are therefore p_y-cells. Similarly the interior p_y-cells are p_x-cells.

In view of Theorem 7 we now drop the suffix and refer to interior p_x- or p_y-cells as the interior p-cells, and the cells of the network F_p which are not interior p-cells, as exterior p-cells.

8. For any s, $0 \leqslant s \leqslant k-1$, the simple curves (x_r, y_r), $s \leqslant r \leqslant s+1$, joining the points (x_s, y_s), (x_{s+1}, y_{s+1}), are called the *boundary lines* of the simple closed curve (x_r, y_r), $0 \leqslant r \leqslant k$; for any assigned value of r between 0 and k inclusive, the point (x_r, y_r) is called a *boundary p-point* or *vertex* of the simple closed curve.

9. Let $2^p a_r$, $0 \leqslant r \leqslant \mu$, be the integers from $2^p a$ to $2^p b$ inclusive, and $2^p \gamma_r$, $0 \leqslant r \leqslant \nu$, the integers from $2^p c$ to $2^p d$ inclusive, then if

$$\begin{aligned}
&x_r = a_r, && y_r = c, && 0 \leqslant r \leqslant \mu,\\
&x_r = b, && y_r = \gamma_{r-\mu}, && \mu \leqslant r \leqslant \mu+\nu,\\
&x_r = a_{2\mu+\nu-r}, && y_r = d, && \mu+\nu \leqslant r \leqslant 2\mu+\nu,\\
&x_r = a, && y_r = \gamma_{2\mu+2\nu-r}, && 2\mu+\nu \leqslant r \leqslant 2(\mu+\nu),
\end{aligned}$$

and

$$x_r^* = x_{2\mu+2\nu-r}, \quad y_r^* = x_{2\mu+2\nu-r}, \qquad 0 \leqslant r \leqslant 2(\mu+\nu),$$

then the simple closed curve (x_r^*, y_r^*), $0 \leqslant r \leqslant 2(\mu+\nu)$, (or an equivalent p-curve) is called the *clockwise p-path* round the rectangle $\langle a, b\rangle\langle c, d\rangle$, and (x_r, y_r), $0 \leqslant r \leqslant 2(\mu+\nu)$, is called the *anticlockwise p-path*.

The sides of a p-path round a p-cell are called the *sides of the cell*. An interior cell of a simple closed curve, which has a side in common with the curve, is called an *interior boundary cell*.

9.1. For any lattice point (ξ, η) of a network F_p we define:

$$\left.\begin{aligned}
x_r^s &= \xi, & r &= 0, 3, 4\\
x_r^s &= \xi', & r &= 1, 2\\
y_0^s = y_1^s &= y_4^s, & y_2^s &= y_3^s,
\end{aligned}\right\} \; 1 \leqslant s \leqslant 4,$$

and

$$\begin{aligned}
&y_0^s = \eta, && s = 1, 2, && y_2^s = \eta', && s = 1,\\
&y_0^s = \eta', && s = 3, && y_2^s = \eta, && s = 3, 4,\\
&y_0^s = \eta^*, && s = 4, && y_2^s = \eta^*, && s = 2.
\end{aligned}$$

It is readily verified that the simple closed p-curves (x_r^s, y_r^s), $0 \leqslant r \leqslant 4$, are clockwise for $s = 2, 3$ and anticlockwise for $s = 1, 4$. In the curves given by $s = 1, 2$ the point (ξ, η) precedes the point (ξ', η), and in the curves $s = 3, 4$ the point (ξ', η) precedes (ξ, η).

Thus of the four curves there is one clockwise and one anticlockwise curve in which (ξ, η) precedes (ξ', η), and one clockwise and one anticlockwise curve in which (ξ', η) precedes (ξ, η).

9.11. Let η be a boundary level, in the strip ξ, ξ', of a simple closed curve Γ; then of the four curves (x_r^s, y_r^s), $0 \leqslant r \leqslant 4$, either those with $s = 1, 3$ or those with $s = 2, 4$ are paths round the interior boundary cell with vertices (ξ,η), (ξ',η). In either case there is only *one* curve in which the order of the points (ξ,η), (ξ',η) is the *same* as in Γ. Similarly there is only one path round an interior boundary cell with vertices (ξ,η), (ξ',η) in which the order of these points is the same as in Γ. Thus with each boundary line of Γ we have associated a unique path, round an interior boundary cell, and this path is said to be described in *the same sense* as Γ. We shall show that all the paths described round interior boundary cells of Γ, and described in the same sense as Γ, are either all clockwise, or all anticlockwise. In the former case the curve Γ is said to be clockwise, and in the latter, anticlockwise.

9.12. Let the path round the interior boundary cell of Γ, with vertices (ξ,η), (ξ',η), be the anticlockwise curve (x_r^1,y_r^1), $0\leqslant r\leqslant 4$, and consider the successor, in Γ, of the points (ξ,η), (ξ',η).

If the successor is (ξ'',η) then ξ' is not a boundary level in the strip η,η' and so $\langle\xi',\xi''\rangle\langle\eta,\eta'\rangle$ is an *interior* boundary cell; thus the path associated with the side of Γ joining the points (ξ',η), (ξ'',η) is the *anticlockwise* curve

$$(\xi',\eta),\quad (\xi'',\eta),\quad (\xi'',\eta'),\quad (\xi',\eta'),\quad (\xi',\eta),$$

(*i. e.* the curve (x_r^1,y_r^1), $0\leqslant r\leqslant 4$, with ξ replaced by ξ' and so ξ' replaced by ξ'').

If (ξ',η') is the successor, then ξ' is a boundary level in the strip η,η' and so $\langle\xi,\xi'\rangle\langle\eta,\eta'\rangle$ is the interior cell associated with the side of Γ joining the points (ξ',η), (ξ',η'), and the path associated with this side is the *anticlockwise* curve (x_r^1,y_r^1), $0\leqslant r\leqslant 4$.

If (ξ',η^*) is the successor, then, since η is a boundary level in the strip (ξ,ξ'), $\langle\xi,\xi'\rangle\langle\eta^*,\eta\rangle$ is an exterior cell and so $\langle\xi',\xi''\rangle\langle\eta^*,\eta\rangle$ is the interior cell associated with the side joining the points (ξ',η), (ξ',η^*) and the path associated with this side is the *anticlockwise* curve (x_r^4,y_r^4), $0\leqslant r\leqslant 4$, with ξ' replacing ξ and ξ'' replacing ξ'.

The same analysis may be applied to a consideration of the successor of the points (ξ,η), (ξ,η'). Thus if the path round one interior boundary cell, described in the same sense as the curve Γ, is anticlockwise, then so too is the path, described in the same sense as Γ, round any other interior boundary cell. And if one is clockwise, then all are clockwise.

10. If Γ_{p+1} is the closed curve on a network F_{p+1} which is equivalent to a curve Γ_p on a network F_p, then the interior $(p+1)$-cells of Γ_{p+1} are the $(p+1)$-cells of the interior p-cells of Γ_p. For if y_r^p is a boundary level in the strip x_r^p, x_{r+1}^p then $y_{2r}^{p+1}=y_r^p$ is a boundary level in each of the strips $x_{2r}^{p+1},x_{2r+1}^{p+1}$; $x_{2r+1}^{p+1},x_{2r+2}^{p+1}$. Similarly for boundary levels in horizontal strips.

11. Γ and γ are simple closed curves on a network F_p. If all the interior cells of γ are interior cells of Γ, and all the interior cells of Γ which have a vertex in common with Γ, *i. e.* the *interior boundary cells* of Γ, are exterior to γ, then γ is said to be *completely contained* in Γ.

γ and Γ are said to be *completely exterior* to each other if no interior cell of one is interior to the other and no boundary p-point of one is a boundary p-point of the other.

11.1. *If γ is completely contained in Γ, all the cells exterior to γ, with a vertex in common with γ, are interior to Γ.*

Let $\langle\xi,\xi'\rangle\langle\eta,\eta'\rangle$ be a cell exterior to γ, and (ξ,η) a boundary p-point of γ. If ξ is a boundary level of γ in the strip η,η' then the cell $\langle\xi^*,\xi\rangle\langle\eta,\eta'\rangle$ is an interior cell of γ, and so of Γ; hence neither (ξ,η) nor (ξ,η') are boundary points of Γ and so ξ is not a boundary level of Γ in η,η'. Therefore $\langle\xi,\xi'\rangle\langle\eta,\eta'\rangle$ is interior to Γ. The corresponding result holds if η is a boundary level of γ in the strip ξ,ξ'.

If ξ is not a boundary level of γ in η,η' and η is not a boundary level in ξ,ξ', then $\langle\xi,\xi'\rangle\langle\eta^*,\eta\rangle$ is exterior to γ; since (ξ,η) is necessarily contained between (ξ^*,η) and (ξ,η^*), in γ, therefore $\langle\xi^*,\xi\rangle\langle\eta^*,\eta\rangle$ is interior to γ, and so interior to Γ. Hence (ξ,η) is not a boundary point of Γ, and so ξ is not a boundary level of Γ in the strip (η^*,η) and η is not a boundary level of Γ in ξ,ξ'; accordingly both $\langle\xi,\xi'\rangle\langle\eta^*,\eta\rangle$ and $\langle\xi,\xi'\rangle\langle\eta,\eta'\rangle$ are interior cells of Γ.

11.2. *If $\gamma_1,\gamma_2,\gamma_3$ are simple closed curves, and if γ_1 is completely contained in γ_2 and γ_2 is completely contained in γ_3, then γ_1 is completely contained in γ_3.*

For the interior cells of γ_1 are interior cells of γ_2, and the interior cells of γ_2 are interior cells of γ_3, so that the interior cells of γ_1 are interior to γ_3. Moreover, any cell which has a vertex in common with γ_3, is exterior to γ_2 and so exterior to γ_1.

11.3. A vertex of an interior p-cell of a simple closed curve which is not also a vertex of the curve, is called an *interior p-point* of the curve. A vertex of an exterior p-cell which is not a vertex of the curve, is called an *exterior p-point.*

If L is an interior and M an exterior p-point of a simple closed curve γ, then any simple p-curve joining L to M has a vertex in common with γ.

Let k be a simple p-curve from L to M. There are boundary p-points of k which are exterior to γ (*e. g.* M); let (a_{r+1},b_{r+1}) be the first point of k which is *not* an interior p-point of γ, so that (a_r,b_r) is an interior p-point. Of the two cells with vertices (a_r,b_r), (a_{r+1},b_{r+1}) one at least is an interior p-cell, and so (a_{r+1},b_{r+1}) is a boundary p-point of γ.

12. *Γ is the simple closed curve (x_r^p, y_r^p), $0 \leqslant r \leqslant k$, on a network F_p, and γ is a simple closed p-curve (a_r^p, b_r^p), $0 \leqslant r \leqslant \lambda$, such that each point (a_r^p, b_r^p), $0 \leqslant r \leqslant \lambda$, is an interior point of Γ. Then γ is completely contained in Γ.*

Let b_ν^p be a boundary level of γ in the strip $(a_\nu^p, a_{\nu+1}^p)$ with $b_\nu^p = b_{\nu+1}^p$; we may without loss of generality suppose that $a_{\nu+1}^p > a_\nu^p$. Since (a_ν^p, b_ν^p) and $(a_{\nu+1}^p, b_{\nu+1}^p)$ are interior points of Γ, b_ν^p is contained between consecutive boundary levels of Γ in the strip $(a_\nu^p, a_{\nu+1}^p)$, say η_1, η_2 where $\eta_1 < \eta_2$, and let

$$x_\varrho = a_\nu^p, \quad x_{\varrho+1} = a_{\nu+1}^p, \quad x_\sigma = a_{\nu+1}^p, \quad x_{\sigma+1} = a_\nu^p,$$

$$y_\varrho = \eta_1, \quad y_{\varrho+1} = \eta_1, \quad y_\sigma = \eta_2, \quad y_{\sigma+1} = \eta_2.$$

We denote by $2^{p+1}e_r$, $0 \leqslant r \leqslant h$, the integers from $2^p\eta_2$ to $2^p\eta_2$ respectively, and by D the simple closed $(p+1)$-curve (X_r, Y_r) where

$$X_r = x_{r+2\varrho+1}^{p+1}, \quad Y_r = y_{r+2\varrho+1}^{p+1}, \qquad 0 \leqslant r \leqslant 2(\sigma-\varrho),$$

$$X_r = x_{2\varrho+1}^{p+1}, \quad Y_r = e_{2(\sigma-\varrho)+h-r}, \qquad 2(\sigma-\varrho)+1 \leqslant r \leqslant 2(\sigma-\varrho)+h,$$

and

$$x_{2r}^{p+1} = x_r^p, \quad x_{2r+1}^{p+1} = \tfrac{1}{2}(x_r^p + x_{r+1}^p),$$

$$y_{2r}^{p+1} = y_r^p, \quad y_{2r+1}^{p+1} = \tfrac{1}{2}(y_r^p + y_{r+1}^p),$$

(so that (x_r^{p+1}, y_r^{p+1}), $0 \leqslant r \leqslant 2k$, is equivalent to Γ).

Further, L is the simple $(p+1)$-curve (a_r^{p+1}, b_r^{p+1}), $2(\nu+1) \leqslant r \leqslant 2(\nu+\lambda)$, joining the points (a_ν^p, b_ν^p), $(a_{\nu+1}^p, b_{\nu+1}^p)$, where (a_r^{p+1}, b_r^{p+1}), $0 \leqslant r \leqslant 2\lambda$, is equivalent to (a_r^p, b_r^p), $0 \leqslant r \leqslant \lambda$, and both a_r^{p+1}, b_r^{p+1} are periodic with period 2λ.

The only boundary level of D, between a_ν^p and $a_{\nu+1}^p$, in any of the strips e_r, e_{r+1}, $0 \leqslant r \leqslant h-1$, is X_0, and so one of the points (a_ν^p, b_ν^p), $(a_{\nu+1}^p, b_{\nu+1}^p)$ is interior to D, and the other exterior to D. Hence by 11.3, L and D have a boundary $(p+1)$-point in common. It may be shown that Γ and γ have no point in common, and so the common points of L and D are (X_0, e_r) for some values of r. Let the common points be (a_r^{p+1}, b_r^{p+1}), $r = 2(\nu+1)+r_m$, $1 \leqslant m \leqslant n$, where $r_{m+1} > r_m$ and $r_n < 2\lambda$. The relation of (a_r^{p+1}, b_r^{p+1}) to (a_r^p, b_r^p) shows that, for $r = 2(\nu+1)+r_m$, $1 \leqslant m \leqslant n$, b_r^{p+1} is a boundary level of γ in the strip $a_\nu^p, b_{\nu+1}^p$, and a boundary level of L in each of the strips $a_{2\nu}^{p+1}, a_{2\nu+1}^{p+1}$ and $a_{2\nu+1}^{p+1}, a_{2\nu+2}^{p+1}$. It follows that for $r = 2(\nu+1)+r_m+(-1)^m$, $1 \leqslant m \leqslant n$, the points (a_r^{p+1}, b_r^{p+1}) lie on the same side of D as $a_{\nu+1}^p$, and those for which $r = 2(\nu+1)+r_m-(-1)^m$, $1 \leqslant m \leqslant n$, lie on the same side as a_ν^p; but $2(\nu+1)+r_n$ is the greatest value of r (below $2(\nu+\lambda)$) for which (a_r^{p+1}, b_r^{p+1}) is a common point of L and D, and so $r_n+(-1)^n < r_n-(-1)^n$, which proves that n is odd. Since b_ν^p itself is also a boundary level of γ in the strip $a_\nu^p, a_{\nu+1}^p$, it follows that there are an *even* number of boundary levels of γ in the strip $a_\nu^p, a_{\nu+1}^p$ which lie between the consecutive boundary levels of Γ, η_1 and η_2.

Thus between any two consecutive boundary levels of Γ, in a strip ξ, ξ' lie an even number of boundary levels of γ, (and no boundary level of γ lies outside Γ since the vertices of γ are interior points of Γ) so that, if f_r, $1 \leqslant r \leqslant 2i$, are the boundary levels of Γ in ξ, ξ' (in increasing order of magnitude) then the interior cells of γ in this strip lie between f_{2r-1} and f_{2r}, $1 \leqslant r \leqslant i$, and are therefore all interior cells of Γ, and the boundary cells of Γ are exterior cells of γ. Thus γ is completely contained inside Γ.

13. The p-curve of a relatively continuous function.

13.1. A rational recursive function (see [1]) $f(n,x)$ is convergent in n, and continuous in x, relative to n, (op. cit., p. 174) in the interval $\langle a, b\rangle$, if there are recursive functions $N(k,x)$, $a^k(r), \beta(k), \sigma(k,r)$ and $C(x,y,k)$ such that, for all positive integers k,

$$|f(n,x) - f(N(k,x),x)| < 1/2^k$$

for all integers n not less than $N(k,x)$, and all rational x in $\langle a, b\rangle$, and, for $0 \leqslant r \leqslant \beta(k)$,

$$|f(n,x) - f(n, a^k(r))| < 1/2^k$$

for all x satisfying $a^k(r) \leqslant x \leqslant a^k(r+1)$, and $n \geqslant C(x, a^k(r), k)$, where $a^k(0) = a$, $a^k(\beta(k)+1) = b$, and $a^k(r) < a^k(r+1)$, $0 \leqslant r \leqslant \beta(k)$, and $a^{k+1}(r) = a^k(\sigma(k,r))$.

13.2. $f(n,x)$ is convergent in n, and continuous relative to n, in $\langle a, b\rangle$; then each of the differences

$$|f(n, a^{k+2}(r)) - f(N(k+2, a^{k+2}(r)), a^{k+2}(r))|,$$

$$|f(n, a^{k+2}(r+1)) - f(N(k+2, a^{k+2}(r+1)), a^{k+2}(r+1))|,$$

$$|f(n, a^{k+2}(r+1)) - f(n, a^{k+2}(r))|$$

is less than $1/2^{k+2}$, whence

$$|f(N(k+2, a^{k+2}(r+1)), a^{k+2}(r+1)) - f(N(k+2, a^{k+2}(r)), a^{k+2}(r))| < 3/2^{k+2}$$

and so, if

$$f_k(r) = [2^k f(N(k+2, a^{k+2}(r)), a^{k+2}(r))] / 2^k$$

(where $[x]$ denotes the greatest integer not exceeding x, if x is non-negative, and $[x] = -[-x]$ if x is negative) then

$$|f_k(r+1) - f_k(r)| \leqslant 1/2^k$$

and so *the integers* $2^k f_k(r)$ *are equal or consecutive for consecutive values of* r. $f_k(r)$ is called the *lacing* of the function $f(n,x)$; the lacing depends, of course, upon the *subdivision* $a^k(r)$.

13.3. $f(n,x)$ and $g(n,x)$ are both convergent in n, and continuous relative to n, in $\langle a,b\rangle$. By combining the subdivisions of (a,b) associated with $f(n,x)$ and $g(n,x)$ respectively we may form lacings of these functions, $f_k(r)$ and $g_k(r)$, on a *common* subdivision $a^k(r)$, $0\leqslant r\leqslant\beta(k)+1$, say. Let

$$\begin{aligned}\Theta(0)&=0, &\quad \Theta(r+1)&=2^p\{f_p(r+1)-f_p(r)\}\,,\\ \varphi(0)&=0, &\quad \varphi(r+1)&=2^p\{g_p(r+1)-g_p(r)\}\,,\end{aligned}\qquad 0\leqslant r\leqslant\beta(p)\,,$$

so that $\Theta(r)$ and $\varphi(r)$ take only the values 0, ±1.

Further, let $r_0=0$, and let r_{r+1} be the least integer greater than r_n, if any, such that $|\Theta(r_{n+1})|+|\varphi(r_{n+1})|>0$; otherwise $r_{n+1}=r_n$. μ_p is the greatest positive integer, if any, such that $\mu_p\leqslant\beta(p)+1$ and $r_{\mu_p}>r_{\mu_p}-1$; otherwise $\mu_p=0$.

Hence if $f_p^*(i)=f_p(r_i)$ and $g_p^*(i)=g_p(r_i)$, $0\leqslant i\leqslant\mu_p$, and

$$\begin{aligned}\Theta^*(0)&=0, &\quad \Theta^*(i+1)&=2^p\{f_p^*(i+1)-f_p^*(i)\}\\ \varphi^*(0)&=0, &\quad \varphi^*(i+1)&=2^p\{g_p^*(i+1)-g_p^*(i)\}\end{aligned}$$

then $\Theta^*(i)$, $\varphi^*(i)$ take only the values 0, ±1, and are not simultaneously zero for $i>0$.

Next, let $k_1+1, k_2+1, \ldots, k_\nu+1$ be the values of i, (if any) in increasing order of magnitude, where $|\Theta^*(i)\varphi^*(i)|>0$, then we define:

$$\left.\begin{aligned}f^p(k_r+r)&=f_p^*(k_r+1), &\quad f^p(j)&=f_p^*(j-s)\\ g^p(k_r+r)&=g_p^*(k_r), &\quad g^p(j)&=g_p^*(j-s)\end{aligned}\right\}\begin{aligned}&1\leqslant r\leqslant\nu,\\ &k_s+s+1\leqslant j\leqslant k_{s+1}+s,\\ &1\leqslant s\leqslant\nu-1\end{aligned}$$

and

$$\begin{aligned}f^p(j)&=f_p^*(j), &\quad g^p(j)&=g_p^*(j), &\quad &0\leqslant j\leqslant k_1,\\ f^p(j)&=f_p^*(j-\nu), &\quad g^p(j)&=g_p^*(j-\nu), &\quad &k_\nu+\nu+1\leqslant j\leqslant\mu_p+\nu.\end{aligned}$$

If $\Theta^*(i)\varphi^*(i)=0$ for all i, $0\leqslant i\leqslant\mu_p$, then we define

$$f^p(i)=f_p^*(i),\qquad g^p(i)=g_p^*(i),\qquad 0\leqslant i\leqslant\mu_p.$$

In either case

$$2^p\{|f^p(i+1)-f^p(i)|+|g^p(i+1)-g^p(i)|\}=1$$

and therefore

$$\big(f^p(r),g^p(r)\big),\qquad 0\leqslant r\leqslant\mu_p+\nu,$$

(where ν is the number of values of i for which $|\Theta^*(i)\varphi^*(i)|>0$) is a plane p-curve, which we shall call *the p-curve derived from* $f(n,x), g(n,x)$.

Thus a pair of functions $f(n,x), g(n,x)$, each convergent in n, and continuous in x, relative to n, determine a sequence of curves, the p-curves derived from the pair, for all positive integral values of p.

14. If $f(r)$ is the lacing of a function $f(n,x)$ on a subdivision $a(r)$, $0\leqslant r\leqslant\beta+1$, then $\sum_{r=0}^{\beta}|f(r+1)-f(r)|$ is called the *relative variation* of $f(n,x)$ on the subdivision $a(r)$.

14.1. If V^1, V^2 are the relative variations of $f(n,x)$ on the subdivisions $a_1(r)$, $0\leqslant r\leqslant\beta_1+1$; $a_2(r)$, $0\leqslant r\leqslant\beta_2+1$, and if there is a positive integer function v_p such that, for $k\geqslant1$,

$$|V^1-V^2|<1/k$$

for any subdivisions $a_1(r)$, $a_2(r)$ satisfying

$$\max_{\substack{0\leqslant r\leqslant\beta_1\\ 0\leqslant s\leqslant\beta_2}}\left\{\big(a_1(r+1)-a_1(r)\big),\big(a_2(s+1)-a_2(s)\big)\right\}<1/v_k,$$

then (anticipating the following theorem) $f(n,x)$ is said to be of *convergent variation relative to* n.

14.2. *If $f(n,x)$ is of convergent variation relative to n, and if V_p is the relative variation on a subdivision $a^p(r)$, $0\leqslant r\leqslant\beta(p)+1$, such that* $\max_{0\leqslant r\leqslant\beta(p)}\{a^p(r+1)-a^p(r)\}\to0$, *then V_p is convergent.*

For we can determine p_k such that $\max_{0\leqslant r\leqslant\beta(p)}\{a^p(r+1)-a^p(r)\}<1/v_k$ for $p\geqslant p_k$, and so if q is any positive integer, then by 14.1,

$$|V_{p+q}-V_p|<1/k$$

which proves that V_p converges.

14.3. If $f(n,x), g(n,x)$ are of convergent variation relative to n, then the sequence of p-curves derived from the pair of functions is said to be *rectifiable*.

Reference

[1] R. L. Goodstein, *Recursive function theory*, Acta Math. 92 (1954), p. 171-190.

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Reçu par la Rédaction le 10.10.1954