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On antipodal sets on the sphere and on continuous involutions *

by

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I. Preliminaries

1. The sphere S_n. Let S_n be the n-sphere in the $(n+1)$-dimensional Euclidean space E_{n+1}, *i. e.*, the set of points $x \in E_{n+1}$ with $|x|=1$. We denote by a the *antipodal mapping* of S_n; it is defined by $a(x)=-x$, for every $x \in S_n$. The set $A \subset S_n$ is called *antipodal* if $a(A)=A$.

2. True chains. Let M be a metric space and $\varepsilon > 0$. By an *ε-simplex* of M we understand a finite subset of M with diameter $<\varepsilon$. In a known manner we introduce the notions of *ε-chains* and *ε-cycles modulo 2 of M*. Since in the sequel we shall use the homology theory modulo 2 only (with the exception of Chapter IV), the words "modulo 2" will be omitted. The *boundary* of a chain $\varkappa$ we denote by $\partial\varkappa$. By the boundary of a 0-dimensional simplex we understand the number 1 considered as a rest modulo 2. The rests 0 and 1 modulo 2 may be considered as (-1)-dimensional cycles. A p-dimensional ε-cycle γ^p is said to be *η-homologous to zero* in M if there exists in M a $(p+1)$-dimensional η-chain $\varkappa^{p+1}$ such that $\partial\varkappa^{p+1}=\gamma^p$.

A sequence of chains $\boldsymbol{\varkappa}=\{\varkappa_i\}$ is called a p-dimensional *true chain* of M if there exists a compact subset C of M and a sequence $\{\varepsilon_i\}$ of positive numbers convergent to zero and such that $\varkappa_i$ is a p-dimensional ε_i-chain of C. A true chain $\boldsymbol{\gamma}=\{\gamma_i\}$ is called a true cycle if $\partial\boldsymbol{\gamma}=\{\partial\gamma_i\}=0$. Let $\boldsymbol{\gamma}=\{\gamma_i\}$ be a p-dimensional true cycle of M. Then $\boldsymbol{\gamma}$ is said to be *homologous to zero* in M if, for every $\varepsilon>0$, there exists an i_0 such that γ_i is ε-homologous to zero in M, for $i>i_0$; it is called *convergent in M* if the true cycle $\{\gamma_i+\gamma_{i+1}\}$ is homologous to zero in M; if there exists a number $\eta>0$ such that no cycle γ_i is η-homologous to zero in M, then the true cycle $\boldsymbol{\gamma}$ is called *totally unhomologous to zero* in M.

We shall denote by $\boldsymbol{B}^p(M)$ the *p-dimensional homology group* (*modulo* 2) of M based on the convergent cycles.

* The main results of this paper were published without proof in [7] and [8].

The space M is said to be *p-acyclic* provided that every $(r-1)$-dimensional true cycle of M with $0 \leqslant r \leqslant p$ is homologous to zero in M. According to the definition of (-1)-dimensional cycles, a space is 0-acyclic if and only if it is not empty; a compact space is 1-acyclic if and only if it is a continuum; the sphere S_n is n-acyclic, but not $(n+1)$-acyclic.

The space M is said to be *acyclic* if it is p-acyclic for every p.

3. (p,φ)-system. Let φ be a continuous involution of M, *i. e.*, a continuous mapping of M into itself such that $\varphi\varphi(x)=x$, for every $x \in M$. Any sequence of true chains of M of the form

$$\Gamma_\varphi^p=(\gamma^{-1},\varkappa^0,\gamma^0,\dots,\varkappa^p,\gamma^p)$$

is called a *(p,φ)-system* of M if the following conditions are satisfied:

1° γ^{-1} is the number 1 considered as a (-1)-dimensional true cycle of M.

2° For every $r=0,1,\dots,p$, $\varkappa^r$ is an r-dimensional true chain of M such that

(1) $\partial\varkappa^r=\gamma^{r-1}$,

(2) $\gamma^r=\varkappa^r+\varphi(\varkappa^r)$.

Thus γ^r is an r-dimensional true cycle of M.

Let us observe that

(*) *If the space M is p-acyclic, then there exists a (p,φ)-system in M.*

For, given any true cycle γ^{r-1} of M, of dimension $r-1<p$, there exists an r-dimensional true chain $\varkappa^r$ of M, such that $\partial\varkappa^r=\gamma^{r-1}$. Hence the conditions 1° and 2° constitute the definition by induction of a p-system in M.

4. Chains in S_n. Antipodal system. 1-chains and 1-cycles in S_n are called briefly *chains* and *cycles in* S_n. The cycle γ in S_n which is 1-homologous to zero in S_n is called *homologous to zero* in S_n and written $\gamma\sim 0$ in S_n.

An *antipodal p-system* in S_n $(-1 \leqslant p \leqslant n)$ is assumed to be a sequence of chains in S_n,

$$\Gamma^p=(\gamma^{-1},\varkappa^0,\gamma^0,\dots,\varkappa^p,\gamma^p),$$

defined as follows:

1° $\gamma^{-1}=1$.

2° For some r $(0 \leqslant r \leqslant p)$ let an $(r-1)$-dimensional cycle γ^{r-1} in S_n such that $a(\gamma^{r-1})=\gamma^{r-1}$ be already defined. Since $r-1<n$, the cycle γ^{r-1} is homologous to zero in S_n. Let $\varkappa^r$ be a chain in S_n such that

$$\partial\varkappa^r=\gamma^{r-1}.$$

Then we put

$$\gamma^r=\varkappa^r+a(\varkappa^r).$$

Thus γ^r is an r-dimensional cycle in S_n and $a(\gamma^r)=\gamma^r$.

5. Intersection number and linking coefficient. By the *geometrical realization* of a *simplex* σ in S_n we understand the smallest convex set in S_n [1]) containing all the vertices of σ. The *geometrical realization* $|\varkappa|$ *of a chain* $\varkappa$ in S_n is assumed to be the sum of the geometrical realizations of all the simplexes belonging to $\varkappa$. By the *geometrical realization of an antipodal p-system*

$$\Gamma^p=(\gamma^{-1},\varkappa^0,\gamma^0,\dots,\varkappa^p,\gamma^p)$$

in S_n we understand the set

$$|\Gamma^p|=\sum_{s=0}^{p}(|\varkappa^s|+|a(\varkappa^s)|).$$

We denote by $\mathfrak{X}(\varkappa^p,\lambda^{n-p})$ the *intersection number* (see [2], p. 413) of any two chains $\varkappa^p$ and λ^{n-p}, which are in a general position [2]) in S_n. If γ^p and δ^{n-p-1} are two cycles in S_n such that $|\gamma^p|\cdot|\delta^{n-p-1}|=0$, then $\mathfrak{y}(\gamma^p,\delta^{n-p-1})$ denotes their *linking coefficient* (see [2], p. 416). Since only chains modulo 2 are used in this paper, the values of $\mathfrak{X}$ and $\mathfrak{y}$ are 0 and 1. In the case $p=n$, $\mathfrak{y}(\gamma^n,\delta^{-1})=1$ if and only if $\delta^{-1}=1$ and γ^n is not homologous to zero in S_n.

Let us suppose that A and B are two disjoint subsets of S_n and let $\gamma=\{\gamma_i\}$ be a p-dimensional true cycle in A and $\delta=\{\delta_j\}$ an $(n-p-1)$-dimensional true cycle in B. Then, for almost all indices i and j, $|\gamma_i|\cdot|\delta_j|=0$. If for almost all indices i and j $\mathfrak{y}(\gamma_i,\delta_j)=1$, then the true cycles γ and δ are said to be *linked*. If there exists a p-dimensional true cycle γ in A and an $(n-p-1)$-dimensional true cycle δ in B, such that the cycles γ and δ are linked, then we say that *the sets A and B are linked in the dimensions* $(p,n-p-1)$.

II. Antipodal sets

1. Introduction. S. Eilenberg proved in 1935 the following theorem on antipodal subsets of the sphere: *Any antipodal continuum on S_2 disconnects S_2 between every two antipodal points of its complement* (see [4], théorème 4, p. 269). This theorem may also be expressed by saying that

[1]) We say that a set $E\subset S_n$ of the diameter <1, is *convex* if, for every two points $a,b\in E$, the lesser of the great circle arcs passing through a and b lies in E.

[2]) *I. e.*, if $\sigma^p=(a_0,a_1,\dots,a_p)$ is a simplex of $\varkappa^p$ and $\tau^{n-p}=(b_0,b_1,\dots,b_{n-p})$ is a simplex of λ^{n-p}, then either $|\sigma^p|\cdot|\tau^{n-p}|=0$ or every system composed of $n+1$ of points $a_0,a_1,\dots,a_p,b_0,b_1,\dots,b_{n-p}$ is linearly independent.

any two antipodal continua on S_2 have common points. Obviously, this theorem is not true for spheres of higher dimension. For instance, two great circles on S_3 are antipodal continua and they may be taken to be disjoint. But, as can easily be checked, two disjoint great circles on S_3 are linked (in dimensions (1,1)). Therefore, the question arises whether any two disjoint antipodal continua on S_3 are linked. The main theorem of this paper gives a positive answer to this question. This theorem is formulated for spheres of an arbitrary dimension. The above-mentioned theorem of Eilenberg is a special case of it.

2. Fundamental results. MAIN THEOREM 1. *Let*

$$\Gamma^p_a=(\gamma^{-1},\varkappa^0,\gamma^0,\dots,\varkappa^p,\gamma^p)$$

be a (p,a)-system lying in a set $A\subset S_n$ and let

$$\Delta^{n-p-1}_a=(\delta^{-1},\lambda^0,\delta^0,\dots,\lambda^{n-p-1},\delta^{n-p-1})$$

be an $(n-p-1,a)$-system lying in a set $B\subset S_n$, with $A\cdot B=0$. Then the true cycles γ^p and δ^{n-p-1} are linked.

First we shall prove two lemmas.

LEMMA 1. *Let $\Gamma^n=(\gamma^{-1},\varkappa^0,\gamma^0,\dots,\varkappa^n,\gamma^n)$ be an antipodal n-system in S_n. Then the true cycle γ^n is not homologous to zero in S_n.*

Proof. We shall prove this lemma by induction with respect to n. Thus, Lemma 1 is evident if $n=0$. Let us suppose that Lemma 1 is proved for $n=k-1$, where $k\geqslant 1$. We shall prove it for $n=k$.

First we shall reduce the proof to the case in which all the chains of the system Γ^k are composed of simplexes of a certain triangulation of S_n.

Let $\mathfrak{T}$ be an antipodal triangulation [3]) of S_n and let $\mathfrak{T}'$ be the first barycentric subdivision of $\mathfrak{T}$. Let us consider the covering of S_n by barycentric stars of $\mathfrak{T}$. Let $\eta>0$ be the Lebesgue number of this covering. By applying, if necessary, successive barycentric subdivisions of simplexes belonging to the chains of Γ^k, we may assume that these simplexes are of diameter $<\eta$. Let us suppose that

$$\gamma^k\sim 0 \quad \text{in} \quad S_k. \tag{3}$$

We shall show that this assumption leads to a contradiction.

Let ψ be the canonical displacement assigning to every point of S_n the centre of a star which contains it. Since triangulations $\mathfrak{T}$ and $\mathfrak{T}'$ are antipodal, therefore if a star G contains x, then $a(G)$ is a star which contains $a(x)$. Consequently, we may assume that

$$\psi a=a\psi\,. \tag{4}$$

[3]) *I. e.*, if σ is a simplex of $\mathfrak{T}$, then $a(\sigma)$ is also a simplex of $\mathfrak{T}$.

The displacement ψ maps the simplexes of S_n of diameter $<\eta$ onto the simplexes of $\mathfrak{T}$. By (4), it maps the antipodal k-system Γ^k onto the antipodal k-system

$$\psi(\Gamma^k)=\big(\psi(\gamma^{-1}),\psi(\varkappa^0),\psi(\gamma^0),\dots,\psi(\varkappa^k),\psi(\gamma^k)\big)\,,$$

whose simplexes are composed of the simplexes of $\mathfrak{T}$. Since ψ is a canonical displacement, then by (3)

$$\psi(\gamma^k)\sim 0 \quad \text{in} \quad S_k\,,$$

and since $\psi(\gamma^k)$ is a chain of a triangulation of S_k, then

$$\psi(\gamma^k)=0\,. \tag{5}$$

By the definition of the antipodal system

$$\psi(\gamma^k)=\psi\big(\varkappa^k+a(\varkappa^k)\big)=\psi(\varkappa^k)+a\psi(\varkappa^k)\,,$$

and then, by (5)

$$\psi(\varkappa^k)=a\psi(\varkappa^k)\,.$$

Hence the chains $\psi(\varkappa^k)$ and $a\psi(\varkappa^k)$ are composed of the same simplexes. Two cases can occur:

(i) The chain $\psi(\varkappa^k)$ contains all the k-dimensional simplexes of $\mathfrak{T}$.

(ii) There exists a k-dimensional simplex σ_0 of $\mathfrak{T}$ which does not belong to $\psi(\varkappa^k)$.

In the first case, let U be the interior of an arbitrary k-dimensional simplex of $\mathfrak{T}$; in the second case, let U be the interior of σ_0. The set $W=S_k-U-a(U)$ is a polytope. Let $a\in U$ and let S'_{k-1} be the $(k-1)$-dimensional great sphere on S_k which is the intersection of S_k with the k-dimensional hyperplane in E_{k+1}, passing through the origin and perpendicular to the diameter $\overline{aa(a)}$. For every point $x\in W$, let $f(x)$ be the point of S'_{k-1} lying on the great circle arc $axa(a)$. Thus f is a continuous projection of W onto S'_{k-1} and satisfies the condition

$$fa=af\,. \tag{6}$$

Since W is compact, there exists a $\zeta>0$ such that for every set $E\subset W$ with diameter $<\zeta$ the set $f(E)$ is of diameter <1. If we cancel the last two terms in the antipodal k-system $\psi(\Gamma^k)$, then we obtain the antipodal $(k-1)$-system

$$\Delta^{k-1}=\big(\psi(\gamma^{-1}),\psi(\varkappa^0),\psi(\gamma^0),\dots,\psi(\varkappa^{k-1}),\psi(\gamma^{k-1})\big)$$

lying in W. Applying, if necessary, a barycentric subdivision, we may assume that the simplexes of W, and hence also the simplexes of Γ^{k-1}, are of diameters $<\zeta$. Thus f maps such simplexes onto the simplexes of S'_{k-1}. Consequently, by (6), f maps the antipodal $(k-1)$-system Δ^{k-1} onto the antipodal $(k-1)$-system

$$f(\Delta^{k-1})=\big(f\psi(\gamma^{-1}),f\psi(\varkappa^0),f\psi(\gamma^0),\ldots,f\psi(\varkappa^{k-1}),f\psi(\gamma^{k-1})\big)$$

on S'_{k-1}.

In the case (i) the chain $\psi(\varkappa^k)$ is a cycle and then

$$f\psi(\gamma^{k-1})=f\big(\partial\psi(\varkappa^k)\big)=0\,.$$

Hence the last cycle of the antipodal $(k-1)$-system $f(\Delta^{k-1})$ lying in S'_{k-1} is homologous to zero in S'_{k-1}. Therefore in this case we get a contradiction of the assumption that Lemma 1 is true for $n=k-1$.

In the case (ii) the chain $\psi(\varkappa^k)$ lies in W. It follows that f maps it onto a chain in S'_{k-1}. Moreover,

$$\partial f\psi(\varkappa^k)=f\big(\partial\psi(\varkappa^k)\big)=f\psi(\gamma^{k-1})\,.$$

Hence $f\psi(\gamma^{k-1})\sim 0$ in S'_{k-1}, and we again have a contradiction. Therefore, Lemma 1 is proved.

LEMMA 2. *Let* $-1\leqslant p\leqslant n$ *and let* $\Gamma^p=(\gamma^{-1},\varkappa^0,\gamma^0,\ldots,\varkappa^p,\gamma^p)$ *be an antipodal p-system and* $\Delta^{n-p-1}=(\delta^{-1},\lambda^0,\delta^0,\ldots,\lambda^{n-p-1},\delta^{n-p-1})$ *— an antipodal* $(n-p-1)$*-system in* S_n*, such that* $|\Gamma^p|\cdot|\Delta^{n-p-1}|=0$. *Then* $\mathfrak{y}(\gamma^p,\delta^{n-p-1})=1$.

Proof. We shall prove Lemma 2 by finite induction with respect to p. Let $p=n$. Thus, by Lemma 1, γ^n is not homologous to zero in S_n and $\delta^{n-p-1}=\delta^{-1}=1$. Hence $\mathfrak{y}(\gamma^n,\delta^{-1})=1$.

Now, let us assume that Lemma 2 is true for $p=r$. We shall prove it for $p=r-1$.

Let

$$\Gamma^{r-1}=(\gamma^{-1},\varkappa^0,\gamma^0,\ldots,\varkappa^{r-1},\gamma^{r-1})$$
$$\Delta^{n-r}=(\delta^{-1},\lambda^0,\delta^0,\ldots,\lambda^{n-r-1},\delta^{n-r-1},\lambda^{n-r},\delta^{n-r})$$

be two antipodal systems on S_n such that

(7) $$|\Gamma^{r-1}|\cdot|\Delta^{n-r}|=0\,.$$

Let us suppose that

(8) $$\mathfrak{y}(\gamma^{r-1},\delta^{n-r})=0\,.$$

By the definition of antipodal systems Γ^{r-1} and Δ^{n-}

$$\gamma^s=\varkappa^s+\alpha(\varkappa^s)\quad\text{for}\quad s=0,1,\ldots,r-1\,,$$
$$\delta^t=\lambda^t+\alpha(\lambda^t)\quad\text{for}\quad t=0,1,\ldots,n-r\,,$$

and by (7)

(9) $$|\gamma^{r-1}|\cdot\sum_{t=0}^{n-r}|\lambda^t|=0\,,\qquad |\gamma^{r-1}|\cdot\sum_{t=0}^{n-r}|\alpha(\lambda^t)|=0\,.$$

Since $r-1<n$, then $\gamma^{r-1}\sim 0$ in S_n. Hence there exists a chain $\varkappa^r$ in S_n such that

(10) $$\partial\varkappa^r=\gamma^{r-1}.$$

By (9), $|\gamma^{r-1}|\cdot|\lambda^{n-r}|=0$ and $|\gamma^{r-1}|\cdot|\alpha(\lambda^{n-r})|=0$. Hence we may choose the chain $\varkappa^r$ so that

(11) $\varkappa^r$ and λ^{n-r} are in a general position,

(12) $\varkappa^r$ and $\alpha(\lambda^{n-r})$ are in a general position.

Furthermore, by (9)

$$|\gamma^{r-1}|\cdot\sum_{t=0}^{n-r-1}|\lambda^t|=0\quad\text{and}\quad|\gamma^{r-1}|\cdot\sum_{t=0}^{n-r-1}|\alpha(\lambda^t)|=0\,.$$

Therefore, we may choose the r-dimensional chain $\varkappa^r$ so that

(13) $$|\varkappa^r|\cdot\sum_{t=0}^{n-r-1}|\lambda^t|=0\,,\qquad|\varkappa^r|\cdot\sum_{t=0}^{n-r-1}|\alpha(\lambda^t)|=0\,.$$

Since α is an isometric involution, then also by (12) and (13)

(14) $\alpha(\varkappa^r)$ and λ^{n-r} are in a general position,

(15) $$|\alpha(\varkappa^r)|\cdot\sum_{t=0}^{n-r-1}|\lambda^t|=0\,,\qquad|\alpha(\varkappa^r)|\cdot\sum_{t=0}^{n-r-1}|\alpha(\lambda^t)|=0\,.$$

Hence, by (11) and (14)

(16) $\varkappa^r+\alpha(\varkappa^r)$ and λ^{n-r} are in a general position,

and by (11) and (12)

(17) $\varkappa^r$ and $\delta^{n-r}=\lambda^{n-r}+\alpha(\lambda^{n-r})$ are in a general position.

By (8), (10) and (17)

(18) $$\mathfrak{X}(\varkappa^r,\delta^{n-r})=0\,.$$

Let

(19) $$\gamma^r=\varkappa^r+\alpha(\varkappa^r)$$

and let us compute the intersection number $\mathfrak{X}(\gamma^r,\lambda^{n-r})$, which is defined by (16). By (19)

(20) $$\mathfrak{X}(\gamma^r,\lambda^{n-r})=\mathfrak{X}(\varkappa^r,\lambda^{n-r})+\mathfrak{X}\big(\alpha(\varkappa^r),\lambda^{n-r}\big)\,.$$

Since $\lambda^{n-r}+\alpha(\lambda^{n-r})=\delta^{n-r}$, it follows that

$$X(\varkappa^r,\lambda^{n-r})=X(\varkappa^r,\delta^{n-r})+X(\varkappa^r,\alpha(\lambda^{n-r}))\,.$$

Hence by (18) and (20)

$$X(\gamma^r,\lambda^{n-r})=X(\varkappa^r,\alpha(\lambda^{n-r}))+X(\alpha(\varkappa^r),\lambda^{n-r})\,.$$

But, obviously, $X(\varkappa^r,\alpha(\lambda^{n-r}))=X(\alpha(\varkappa^r),\lambda^{n-r})$. It follows that

(21) $$X(\gamma^r,\lambda^{n-r})=0\,.$$

By (21), and since $\partial\lambda^{n-r}=\delta^{n-r-1}$, it follows that

(22) $$\mathfrak{y}(\gamma^r,\delta^{n-r-1})=0\,.$$

The cycle γ^r together with the chain $\varkappa^r$ and with the system $\boldsymbol{\Gamma}^{r-1}$ form an antipodal r-system

$$\boldsymbol{\Gamma}^r=(\gamma^{-1},\varkappa^0,\gamma^0,\ldots,\gamma^{r-1},\varkappa^r,\gamma^r)\,,$$

and the system $\boldsymbol{\Delta}^{n-r}$ after the cancelling of δ^{n-r} and λ^{n-r} forms an antipodal $(n-r-1)$-system

$$\boldsymbol{\Delta}^{n-r-1}=(\delta^{-1},\lambda^0,\delta^0,\ldots,\lambda^{n-r-1},\delta^{n-r-1})\,.$$

By (7), (13) and (15)

$$|\boldsymbol{\Gamma}^r|\cdot|\boldsymbol{\Delta}^{n-r-1}|=0\,.$$

Thus the equality (22) contradicts the assumption that Lemma 2 is true for $p=r$. Hence the supposition (8) leads to a contradiction, and therefore Lemma 2 is proved for every $p=-1,0,1,\ldots,n$.

Proof of the Main Theorem. Let

$$\boldsymbol{\gamma}^s=\{\gamma_i^s\}\,,\quad \boldsymbol{\varkappa}^s=\{\varkappa_i^s\}\,,\quad \boldsymbol{\delta}^t=\{\delta_j^t\}\,,\quad \boldsymbol{\lambda}^t=\{\lambda_j^t\}\,.$$

Thus ${}_i\boldsymbol{\Gamma}^p=(\gamma_i^{-1},\varkappa_i^0,\gamma_i^0,\ldots,\varkappa_i^p,\gamma_i^p)$ is an antipodal p-system and ${}_j\boldsymbol{\Delta}^{n-p-1}=(\delta_j^{-1},\lambda_j^0,\delta_j^0,\ldots,\lambda_j^{n-p-1},\delta_j^{n-p-1})$ — an antipodal $(n-p-1)$-system in S_n. The true chains of $\boldsymbol{\Gamma}_a^p$ lie in compact subsets of A and the true chains of $\boldsymbol{\Delta}_a^{n-p-1}$ lie in compact subsets of B. Since $A\cdot B=0$, it follows that the condition $|{}_i\boldsymbol{\Gamma}^p|\cdot|{}_j\boldsymbol{\Delta}^{n-p-1}|=0$ will be satisfied for almost all i and j. Therefore, by Lemma 2, $\mathfrak{y}(\gamma_i^p,\delta_j^{n-p-1})=1$ for almost all indices i and j, and consequently the true cycles $\boldsymbol{\gamma}^p$ and $\boldsymbol{\delta}^{n-p-1}$ are linked.

Corollary 1. *Let A and B be two disjoint subsets of S_n such that A contains an (p,a)-system and B contains an $(n-p-1,a)$-system. Then A and B are linked in the dimensions $(p,n-p-1)$.*

Theorem 2. *Let A and B be two subsets of S_n. If A contains a (p,a)-system and B contains an $(n-p,a)$-system, then $A\cdot B\neq 0$.*

Proof. Let $\boldsymbol{\Gamma}_a^p=(\boldsymbol{\gamma}^{-1},\boldsymbol{\varkappa}^0,\boldsymbol{\gamma}^0,\ldots,\boldsymbol{\varkappa}^p,\boldsymbol{\gamma}^p)$ be a (p,a)-system in A and $\boldsymbol{\Delta}_a^{n-p}=(\boldsymbol{\delta}^{-1},\boldsymbol{\lambda}^0,\boldsymbol{\delta}^0,\ldots,\boldsymbol{\delta}^{n-p-1},\boldsymbol{\lambda}^{n-p},\boldsymbol{\delta}^{n-p})$ — an $(n-p,a)$-system in B. If we suppose that $A\cdot B=0$, then, by the Main Theorem, the true cycles $\boldsymbol{\gamma}^p$ and $\boldsymbol{\delta}^{n-p-1}$ are linked. But $\boldsymbol{\delta}^{n-p-1}=\partial\boldsymbol{\lambda}^{n-p}$, and consequently the true cycle $\boldsymbol{\delta}^{n-p-1}$ is homologous to zero in the set B which is disjoint with A. Therefore true cycles $\boldsymbol{\gamma}^p$ and $\boldsymbol{\delta}^{n-p-1}$ are not linked.

3. Conclusions with regard to antipodal sets. By (*), any antipodal p-acyclic subset of S_n contains a (p,a)-system. Hence, by Corollary 1 and by Theorem 2 we obtain

Theorem 3. *If A and B are two disjoint antipodal subsets of S_n such that A is p-acyclic and B is $(n-p-1)$-acyclic, then A and B are linked in the dimensions $(p,n-p-1)$.*

Theorem 4. *If the set $A\subset S_n$ is antipodal and p-acyclic and $B\subset S_n$ is antipodal and q-acyclic with $p+q\geqslant n$, then $A\cdot B\neq 0$.*

Thus, for instance, a p-dimensional great sphere $S_p'\subset S_n$ is antipodal and p-acyclic. Two disjoint great spheres S_p' and S_{n-p-1}'' are linked; if S_p' and S_q'' are great spheres on S_n and $p+q\geqslant n$, then $S_p'\cdot S_q''\neq 0$; this can be checked immediately.

In the case of $n=3$ and $p=1$ we deduce from Theorem 3 the following

Corollary 2. *Any two disjoint antipodal continua lying in S_3 are linked in the dimensions* (1,1).

When $n=2$ and $p=1$ we obtain from Theorem 4 the following

Corollary 3. *Any two antipodal continua lying in S_2 have common points* (see [6], No 3, Lemme, p. 244, and Remarque, p. 235).

A set $A\subset S_n$ disconnects S_n between the points $a,b\in S_n-A$, if it is linked in the dimensions $(n-1,0)$ with the two-point set $(a)+(b)$. Hence, in the case $p=n-1$ we obtain from Theorem 3 the following

Corollary 4. *An $(n-1)$-acyclic subset of S_n disconnects S_n between every two antipodal points of its complement.*

In particular, if $n=2$, we have the theorem of Eilenberg (see [4], théorème 4, p. 269):

Corollary 5. *Any antipodal continuum in S_2 disconnects S_2 between every two antipodal points of its complement.*

For $p=n$, Theorem 4 reduces to the following

Corollary 6. *The only antipodal n-acyclic subset of S_n is the whole S_n.*

4. Remarks. Theorems 3 and 4 show that an antipodal p-acyclic subset of S_n is situated in S_n in some sense similarly to the great sphere of dimension $\geqslant p$. Theorem 4 can be formulated as follows:

THEOREM 4'. *If a set $A \subset S_n$ is antipodal and p-acyclic and a set $B \subset S_n$ — is antipodal and q-acyclic, with $p+q \geqslant n$, then the intersection $A \cdot B$ contains an antipodal 0-acyclic set.*

This suggests the following

Problem. *Let A and B be two antipodal subsets of S_n such that A is p-acyclic and B is q-acyclic. The question is whether the set $A \cdot B$ contains an antipodal $(p+q-n)$-acyclic subset.*

The word "contains" cannot be replaced by "is", since in that case the answer would be negative. For example, let A and B_1 be two great 2-dimensional spheres on S_3 defined by the equations $x_3=0$, and $x_4=0$, respectively. Let B_2 be a quarter of the great circle arc on S_3 defined by $x_1=0$, $x_2=0$, $x_3 \geqslant 0$, $x_4 \geqslant 0$. Then $B_1 \cdot B_2$ consists of the single point $(0,0,1,0)$. Let $B = B_1 + B_2 + a(B_2)$. Then A and B are antipodal and 2-acyclic, but $A \cdot B$ consists of the circle S, $x_1^2+x_2^2=1$, $x_3=0$, $x_4=0$, and of two points, $(0,0,0,1)$ and $(0,0,0,-1)$. Hence $A \cdot B$ is not 1-acyclic, since it is not connected. However, the set $A \cdot B$ contains an antipodal 1-acyclic subset, namely the circle S.

III. Some properties of continuous involutions

1. Involutions and mappings in spheres. The Main Theorem, concerning the antipodal mapping of S_n, which is proved in Chapter II, enables us to investigate some properties of continuous involutions of more general metric spaces. Thus, a generalization of Borsuk's theorem on antipodes (see [3], p. 178) and theorems concerning fixed points of involutions can be proved.

Now, in the case of $A = S_n$ and $p = n$, Theorem 1 can be formulated as follows:

(**) *Let $\Gamma_a^n = (\gamma^{-1}, \varkappa^0, \gamma^0, \ldots, \varkappa^n, \gamma^n)$ be an (n, a)-system in S_n. Then the true cycle γ^n is totally unhomologous to zero in S_n.*

For, if $\gamma^n = \{\gamma_i^n\}$, then γ_i^n is not 1-homologous to zero in S_n for almost all i.

LEMMA 3. *Let φ be a continuous involution of a metric space M and let us suppose that M contains an (n, φ)-system $\Delta_\varphi^n = (\delta^{-1}, \lambda^0, \delta^0, \ldots, \lambda^n, \delta^n)$. Let f be a continuous mapping of M into S_n such that $f(x) \neq f\varphi(x)$ for every $x \in M$. Then f maps the true cycle δ^n of M onto a true cycle which is totally unhomologous to zero in S_n.*

Proof. Let

$$g(x) = \frac{f(x) - f\varphi(x)}{|f(x) - f\varphi(x)|}\ ^{4)}. \tag{23}$$

Since $f(x) \neq f\varphi(x)$, the function g defined by (23) for every $x \in M$, is a continuous mapping of M into S_n and satisfied the condition

$$g\varphi = ag.$$

It follows that g maps the (n, φ)-system Δ_φ^n onto a (n, a)-system $g(\Delta_\varphi^n) = (g(\delta^{-1}), g(\lambda^0), g(\delta^0), \ldots, g(\lambda^n), g(\delta^n))$ in S_n. From (**) we conclude that the true cycle $g(\delta^n)$ is totally unhomologous to zero in S_n. Furthermore, we observe that

$$f(x) \neq ag(x) \quad \text{for every} \quad x \in M. \tag{24}$$

Indeed, if we suppose that $f(x) = ag(x)$, i. e.,

$$f(x) = -\frac{f(x) - f\varphi(x)}{|f(x) - f\varphi(x)|}$$

then we obtain

$$f\varphi(x) = f(x) \cdot (1 + |f(x) - f\varphi(x)|). \tag{25}$$

Since $|f(x)| = |f\varphi(x)| = 1$, we conclude from (25) that $|1 + |f(x) - f\varphi(x)||$ $= 1 + |f(x) - f\varphi(x)| = 1$, which is impossible since $|f(x) - f\varphi(x)| > 0$.

We conclude by (24) that f and g are homotopic (see [3], p. 179, 1)). Hence the true cycles $f(\delta^n)$ and $g(\delta^n)$ are homologous in S_n. Consequently, the true cycle $f(\delta^n)$ is totally unhomologous to zero in S_n and the proof of Lemma 3 is complete.

Every true cycle $\{\gamma_i\}$ modulo 2 of M contains a subsequence $\{\gamma_{i_k}\}$, which is a convergent cycle in M (see [1], p. 180). Therefore, under the hypotheses of Lemma 3, f maps an n-dimensional convergent cycle of M onto a convergent cycle in S_n which is totally unhomologous to zero in S_n. Since the n-dimensional homology group of S_n contains only two elements, Lemma 3 yields

THEOREM 5. *Let φ be a continuous involution of M and let us suppose that M contains a (n, φ)-system. Then every continuous mapping f of M into S_n which satisfies the condition $f(x) \neq f\varphi(x)$, for every $x \in M$, maps the n-dimensional homology group $\mathbf{B}^n(M)$ of M onto the n-dimensional homology group $\mathbf{B}^n(S_n)$ of S_n.*

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4) In the sense of operations with points in E_{n+1}.

2. Generalization of Borsuk's theorem on antipodes. By (*), every n-acyclic space M contains an (n,φ)-system, for every continuous involution φ of M. Hence Theorem 5 implies

THEOREM 6. *Let M be an n-acyclic space and φ — a continuous involution of M. Then every continuous mapping f of M into S_n satisfying the condition $f(x)\neq f\varphi(x)$ for every $x\in M$ maps the n-dimensional homology group $\boldsymbol{B}^n(M)$ onto the n-dimensional homology group $\boldsymbol{B}^n(S_n)$.*

Since the group $\boldsymbol{B}^n(S_n)$ is not trivial and since every continuous mapping of M into S_n homotopic to a constant maps the group $\boldsymbol{B}^n(M)$ into zero, we deduce

COROLLARY 7. *Let M, φ and f be as in Theorem 6. Then f is not homotopic to a constant.*

THEOREM 7. *Let M and φ be as in Theorem 6. Then, for every continuous mapping f of M into the Euclidean space E_n there exists a point $x_0\in M$ such that $f(x_0)=f\varphi(x_0)$.*

For, the mapping f of M into E_n may be considered as a mapping of M into a proper subset of S_n, and hence f is homotopic to a constant. If we suppose that $f(x)\neq f\varphi(x)$, for every $x\in M$, then we obtain a contradiction of Corollary 7.

THEOREM 8. *Let M and φ be as in Theorem 6 and let $M=M_0+M_1+ +\ldots+M_n$ be a decomposition of M into the sum of $n+1$ closed subsets of M. Then at least one of the sets M contains an involution pair $\{x,\varphi(x)\}$.*

The proof is based on the following Lemma of Borsuk (see [3], p. 188, Hilfssatz):

(**) *For any decomposition $M=M_0+M_1+\ldots+M_n$ of a metric space M into the sum of $n+1$ closed subsets of M, there exists a continuous function f mapping M into E_n such that, for every $y\in f(M)$, the set $f^{-1}(y)$ is contained in at least one of the sets M_i.*

Proof of Theorem 8. Applying Theorem 7 to the mapping f provided by the Lemma of Borsuk, we conclude that there exists an $x_0\in M$ such that $f(x_0)=f\varphi(x_0)=y_0$. Hence, for some i, $f^{-1}(y_0)\subset M_i$. Therefore, $x_0\in M_i$ and $\varphi(x_0)\in M_i$.

If $M=S_n$, and $\varphi=\alpha$, Theorems 6, 7 and 8, reduce to Borsuk's theorems I, II, and III of [3], respectively.

3. Fixed points of involutions. THEOREM 9. *Let M be a metric, separable, acyclic space, of finite dimension. Then any continuous involution φ of M has a fixed point.*

Proof. Theorem of Menger-Nöbeling (see [9], p. 235) provides a homeomorphism h of M into E_n. Since M is acyclic, and hence also n-acyclic, we conclude from Theorem 7 that there exists an $x_0\in M$ such that $h(x_0)=h\varphi(x_0)$. Hence $x_0=\varphi(x_0)$.

Theorem 9 is a special case of a theorem of P. A. Smith concerning fixed points of periodic transformations (see [10], p. 367, (13.1)-Theorem, and also [5], p. 428, Theorem I). The assumption of finite dimension of M is essential (see [5], No 8, p. 435). In particular, any continuous involution of the Euclidean space has a fixed point. However, if M is compact, the hypothesis of finite dimension of M can be omitted:

THEOREM 10. *Any continuous involution of a compact acyclic space has a fixed point.*

The proof is given in [8], p. 292.

4. Remarks. C. T. Yang proved in [11] another generalization of Borsuk's theorem on antipodes. He introduced a notion of index of a pair $\{M,\varphi\}$, where φ is a continuous involution without fixed points of a compact space M. The notion of index is related to that of an (n,φ)-system in the sense of the present paper, as follows: The index of $\{M,\varphi\}$ is the largest integer n such that M contains an (n,φ)-system. In this way Theorems 7 and 8 follow from Theorem (4.1) of C. T. Yang (see [11], p. 270).

IV. Generalization

The main result of this paper may be formulated by the use of a more general homology theory. Let $\mathfrak{R}$ be a commutative ring, containing elements which are not divisible by 2. We consider the true chains of a metric space M with coefficients belonging to $\mathfrak{R}$.

Let φ be a continuous involution of M. The notion of a (p,φ)-system may be generalized as follows: the (p,φ)-system of M is a sequence of true chains of M

$$\Gamma_\varphi^p=(\gamma^{-1},\varkappa^0,\gamma^0,\ldots,\varkappa^p,\gamma^p)$$

such that:

1° $\gamma^{-1}=\{\gamma_i^{-1}\}$, where for almost all i, γ_i^{-1} is an element of $\mathfrak{R}$ which is not divisible by 2, considered as a (−1)-dimensional cycle of M.

2° For every $r=0,1,2,\ldots,p$, $\varkappa^r$ is an r-dimensional true chain of M such that

$$\partial\varkappa^r=\gamma^{r-1},$$

$$\gamma^r=\varkappa^r-(-1)^r\alpha(\varkappa^r)\,.$$

Thus γ^r is an r-dimensional true cycle of M.

By the use of these notions, the following theorem may be proved:

THEOREM 11. *Let* $\Gamma_a^p = (\gamma^{-1}, \varkappa^0, \gamma^0, \ldots, \varkappa^p, \gamma^p)$ *be a* (p, a)*-system lying in a set* $A \subset S_n$ *and let* $\Delta_a^{n-p-1} = (\delta^{-1}, \lambda^0, \delta^0, \ldots, \lambda^{n-p-1}, \delta^{n-p-1})$ *be an* $(n-p-1)$*-system lying in a set* $B \subset S_n$, *with* $A \cdot B = 0$. *Let* $\gamma^p = \{\gamma_i^p\}$, $\delta^{n-p-1} = \{\delta_j^{n-p-1}\}$. *Then, for almost all* i *and* j, *the linking coefficient* $\mathfrak{y}(\gamma_i^p, \delta_j^{n-p-1})$ is not divisible by 2.

The proof is not essentially different from the proof of Theorem 1.

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