On cyclic groups

by

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To the memory of my dearly beloved,
prematurely dead Master: professor T. Szele.

We treat in this paper the following group-theoretical problem for a certain characterization of cyclic groups among all groups. Let G be an arbitrary (and thus not necessarily abelian) group. *The kth power of the group G* is a subgroup G^k generated by the set of the kth powers of all group-elements. An arbitrary group G is called *a group with property P* if there exists to every subgroup H a rational integer k for which $H=G^k$ holds. *E. g.* any cyclic group has the property P.

We shall prove in this paper that the property P postulated for groups and the cyclic behaviour of the groups are equivalent, namely that *a group with property P is always cyclic.*

As we do not assume the group-operation to be commutative we write it multiplicatively. We call the neutral element of the group the *unity* and denote it by 1. A group is called *torsion free* if it has no element of finite order other than the unity. Moreover a group is called *a torsion group* if every element of the group has finite order. A torsion group is a *p-group* if the orders of group-elements are the powers of a fixed prime number p. A torsion group is called an *elementary group* if the order of any group-element is a square free number. We refer here to the ground theorem of finite abelian groups, according to which every abelian group with finitely many elements is the direct product of cyclic groups.

Every subgroup of an abelian group is evidently a *normal subgroup* of the group. A non-commutative group is termed *hamiltonian* if every subgroup is a normal subgroup of the group. *E. g.* the *quaternion group Q* generated by the elements a and b for which the relations $a^{-1}ba=b^{-1}$ and $a^2=b^2$ are true is a hamiltonian group. The fundamental theorem of R. Baer supplies an explicit description of all hamiltonian groups: *every hamiltonian group is the direct product of the quaternion group and of an abelian group whose 2-component* (*i. e.* the set of all group-elements of 2-power order) *is an elementary p-group.* It is obvious that the factor group G/Z of a hamiltonian group G by the centre Z of the group G is always commutative [1].

We now prove the following theorem:

THEOREM. *An arbitrary group G is cyclic if and only if the group G has the property P.*

First of all we prove some Lemmas.

LEMMA 1. *Every homomorphic image G' of a group G with property P has likewise the property P.*

Proof. Let G be an arbitrary group with property P. If φ is a homomorphic mapping of G onto the group G', then there exists in the group G a uniquely defined complete inverse image $H=\varphi^{-1}(H')$ of an arbitrary subgroup H' of the group G'. There exists a rational integer k for which $H=G^k$ holds. Let $h\in H$ and $h'\in H'$ be given so that $\varphi(h)=h'$. Then we can write that $h=g_1^k\dots g_s^k\in H=G^k$ where $g_1,\dots,g_s$ are suitable elements of G. For this reason evidently $h'\in(G')^k$, therefore $\varphi(G^k)\subseteq(G')^k$ holds. On the other hand, if $g\in G$, then $g^k\in H$, therefore by $(G')^k\subseteq\varphi(G^k)$ obviously $H'=(G')^k$ is true.

LEMMA 2. *A group with property P is finite or torsion free.*

Proof. If $g\in G$ is an element of finite order and $O(g)=n>1$, then with a suitable rational integer k the connection $\{g\}=G^k$ holds. Therefore the order of any group-element is no greater than kn, namely G is at most kn-bounded. In this case G has only finitely many different powers, whence only finitely many different subgroups, namely G is itself finite.

LEMMA 3. *If an abelian group has the property P, then it is cyclic.*

Proof. Let G be an abelian torsion free group with property P. Then there exists to the element $1\neq g\in G$ a rational integer $n\neq0$ for which $\{g\}=G^n$ holds. Then the mapping $x\to x^n$ is evidently an isomorphism of G into itself, therefore the group G is in this case cyclic.

Now let G be a finite abelian group with property P, which is by the above-mentioned ground theorem the direct product of cyclic groups. It is evidently sufficient to prove our Lemma for p-groups, because every p-component, as a direct factor, has by Lemma 1 likewise the property P, and therefore from this proof of our Lemma already follows the cyclic behaviour of G itself. There exists a direct factor $A_p\neq\{1\}$ of the p-component G_p of the group G for which with a suitable subgroup B_p obviously $G_p=A_p\times B_p$ holds. Then $A_p=G^s$, and we can assume that $s=p^m$ $(m\geqslant0)$. In this case $A_p=A_p^s\times B_p^s$, and thus $A_p=A_p^s$, namely $s=1$. Therefore the subgroup G_p and the group G are cyclic.

The proof of the theorem. Let G be an arbitrary group with property P. Then every subgroup of the group G is by the definition of the kth power of the group G a normal subgroup of G. Let Z be the centre of G, then the factor group G/Z is by the above-mentioned ground theorem of R. Baer on the hamiltonian groups, necessarily abelian, and by Lemma 1 has the property P. Therefore G/Z is by Lemma 3 cyclic, which is evidently equivalent to the commutativity of the group G. But then G is by Lemma 3 likewise cyclic.

On the other hand every cyclic group has the property P, and therefore the proof is complete.

References

[1] R. Baer, *Situation der Untergruppen und Struktur der Gruppe*, S. B. Heidelberg. Akad. Wiss. 2 (1933), p. 12-17.

[2] А. Г. Курош, *Теория групп.* Москва 1953.

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On antipodal sets on the sphere and on continuous involutions *

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I. Preliminaries

1. The sphere S_n. Let S_n be the n-sphere in the $(n+1)$-dimensional Euclidean space E_{n+1}, *i. e.*, the set of points $x\in E_{n+1}$ with $|x|=1$. We denote by α the *antipodal mapping* of S_n; it is defined by $\alpha(x)=-x$, for every $x\in S_n$. The set $A\subset S_n$ is called *antipodal* if $\alpha(A)=A$.

2. True chains. Let M be a metric space and $\varepsilon>0$. By an *ε-simplex* of M we understand a finite subset of M with diameter $<\varepsilon$. In a known manner we introduce the notions of *ε-chains* and *ε-cycles modulo 2 of M*. Since in the sequel we shall use the homology theory modulo 2 only (with the exception of Chapter IV), the words "modulo 2" will be omitted. The *boundary* of a chain $\varkappa$ we denote by $\partial\varkappa$. By the boundary of a 0-dimensional simplex we understand the number 1 considered as a rest modulo 2. The rests 0 and 1 modulo 2 may be considered as (-1)-dimensional cycles. A p-dimensional ε-cycle γ^p is said to be *η-homologous to zero* in M if there exists in M a $(p+1)$-dimensional η-chain $\varkappa^{p+1}$ such that $\partial\varkappa^{p+1}=\gamma^p$.

A sequence of chains $\boldsymbol{\varkappa}=\{\varkappa_i\}$ is called a p-dimensional *true chain* of M if there exists a compact subset C of M and a sequence $\{\varepsilon_i\}$ of positive numbers convergent to zero and such that $\varkappa_i$ is a p-dimensional ε_i-chain of C. A true chain $\gamma=\{\gamma_i\}$ is called a true cycle if $\partial\gamma=\{\partial\gamma_i\}=0$. Let $\gamma=\{\gamma_i\}$ be a p-dimensional true cycle of M. Then γ is said to be *homologous to zero* in M if, for every $\varepsilon>0$, there exists an i_0 such that γ_i is ε-homologous to zero in M, for $i>i_0$; it is called *convergent in M* if the true cycle $\{\gamma_i+\gamma_{i+1}\}$ is homologous to zero in M; if there exists a number $\eta>0$ such that no cycle γ_i is η-homologous to zero in M, then the true cycle γ is called *totally unhomologous to zero* in M.

We shall denote by $\boldsymbol{B}^p(M)$ the *p-dimensional homology group* (*modulo* 2) of M based on the convergent cycles.

* The main results of this paper were published without proof in [7] and [8].