Note on quasi-Lie rings

by

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1. Introduction. Let X be a simply-connected space and let $[\alpha,\beta]\in\pi_{l+m-1}(X)$ be the Whitehead product of $\alpha\in\pi_l(X)$, $\beta\in\pi_m(X)$. Then the homotopy group $\pi(X)=\sum_n \pi_n(X)$ may be turned into a (non-associative) ring by defining $\alpha\beta=[\alpha,\beta]$. For an arbitrary path-connected space we will understand by its homotopy-ring the group $\pi(X)=\sum_{n>1}\pi_n(X)$, furnished with the Whitehead product. This ring satisfies precisely the identities:

(i) $\alpha\beta=(-1)^{lm}\beta\alpha$,

(ii) $(-1)^{l(n+1)}\alpha(\beta\gamma)+(-1)^{m(l+1)}\beta(\gamma\alpha)+(-1)^{n(m+1)}\gamma(\alpha\beta)=0,\quad \gamma\in\pi_n(X)$.

In the light of (i), (ii) is equivalent to

(ii') $(-1)^{ln}(\alpha\beta)\gamma+(-1)^{ml}(\beta\gamma)\alpha+(-1)^{nm}(\gamma\alpha)\beta=0$.

Let T be a wedge of spheres; that is, T is the union of spheres $S_1,\ldots,S_t$ with a single common point. Let S_i be of dimension n_i+1, $i=1,\ldots,t$, $n_i\geqslant 1$. Then it was shown in [4] that

$$\pi_r(T)\cong\sum\pi_r(S^{q_i+1}),$$

where the summand $\pi_r(S^{q_i+1})$ is embedded in $\pi_r(T)$ by composition with the basic product (bp) p_i. Let ι_i be a chosen generator of $\pi_{n_i+1}(S_i)$. Then the p_i belong to the subring, $L(T)$, of $\pi(T)$ generated by $\iota_1,\ldots,\iota_t$.

Our object in this note is to define a class of rings, namely quasi-Lie rings, of which homotopy rings are examples. We will give some elementary properties of these rings, and then discuss the role of basic products in these rings.

2. Graded Lie rings and quasi-Lie rings. A *graded Lie ring* [1]) is a graded abelian group $L=\sum_{n\geqslant 0}L_n$, together with a multiplication satisfying $L_lL_m\subseteq L_{l+m}$ and verifying the axioms

[1]) P. Cartier has considered the notion of a graded Lie algebra in [2]. The torsion inherent in the definition of a graded Lie ring does not, of course, appear in the algebra. Note that Cartier's identities agree with (G1) and (G2) if Cartier's product of a and b is identified with $(-1)^l ab$.

(G1) $ab=(-1)^{(l+1)(m+1)}ba,\quad a\in L_l,\quad b\in L_m$;

(G2) $(-1)^{(l+1)n}a(bc)+(-1)^{(m+1)l}b(ca)+(-1)^{(n+1)m}c(ab)=0,\quad c\in L_n$.

Notice that a *Lie ring* is a graded Lie ring with $L=L_0$, $a^2=0$. The homotopy ring of a path-connected space X is a graded Lie ring if we define $L_0=0$, $L_n=\pi_{n+1}(X)$, $n>0$, since $[\pi_{l+1}(X),\pi_{m+1}(X)]\subseteq\pi_{l+m+1}(X)$; here we have $L_0=0$. We will write $\overline{\pi}(X)$ for the graded Lie ring associated with the homotopy ring $\pi(X)$; it is given by $\overline{\pi}_0(X)=0$, $\overline{\pi}_n(X)=\pi_{n+1}(X)$, $n>0$. Let $a_1,\ldots,a_t$ be elements in a graded Lie ring L. We define a monomial in $a_1,\ldots,a_t$ inductively by saying that the *monomials of degree* 1 are $a_1,\ldots,a_t$ themselves and that a *monomial of degree* $d>1$ is a product m_1m_2 where m_j is a monomial of degree d_j, $j=1,2$, and $d_1+d_2=d$. Then the additive subgroup of L generated by the monomials in $a_1,\ldots,a_t$ is clearly a subring of L, which we call the subring generated by $a_1,\ldots,a_t$. In particular if $L=\overline{\pi}(T)$ and $a_i=\iota_i$, we write $L(T)$ for the subring generated by $\iota_1,\ldots,\iota_t$. Since the ι_i are homogeneous elements, *i. e.* $\iota_i\in L_{n_i}$ for some n_i, so are the monomials in the ι_i. In particular, basic products [2]), whose definition we now recall, are monomials in the ι_i and hence are homogeneous elements of the subring $L(T)$ of $\overline{\pi}(T)$.

We now define basic products in the elements $a_1,\ldots,a_t$ of L. For clarity we adopt a device used by Barratt and Whitehead [1] and first define *symbolic basic products* (sbp). These are defined and ordered as follows: The sbp of weight 1 are the symbols $(1),\ldots,(t)$ in increasing order. Suppose sbp of weight $<w$ defined and ordered. Then the sbp of weight $w\geqslant 2$ are symbols $(\varkappa,\lambda)$ where $\varkappa$ is an sbp of weight u, λ is an sbp of weight v, $u+v=w$, $\varkappa<\lambda$, and if $\lambda=(\lambda_1,\lambda_2)$ then $\varkappa\geqslant\lambda_1$. The sbp of weight w are greater than those of smaller weight and are ordered arbitrarily among themselves. We may define symbolic products generally in the symbols $(1),\ldots,(t)$ (by induction on weight, as above) and these form a multiplicative system P; we map P into L by the unique homomorphism φ with $\varphi(i)=a_i$, and the *basic products* in the a_i are then the φ-images of the symbolic basic products. We define the weight of a basic product (bp) by the requirement that φ be weight-preserving. Then Theorem A of [4] gives a precise relation between the ring $\overline{\pi}(T)$, its subring $L(T)$, and homotopy groups of spheres, expressed by means of the bp's in $\iota_1,\ldots,\iota_t$, which are homogeneous elements in $L(T)$.

It will be observed that axioms (G1) and (G2) are concerned only with the parity of the degree of homogeneous elements of L. We are thus led to a generalization of graded Lie rings, namely, quasi-Lie rings.

[2]) The bibliography in [4] should have contained a reference to [3], where the author introduced the notion of basic commutators independently of the work of P. Hall.

A *quasi-Lie ring* L is a pair (M, A) where

(α) A is a ring satisfying

(α1) $aa' = -a'a$,

(α2) $a(a'a'') + a'(a''a) + a''(aa') = 0, \quad a, a', a'' \in A$;

(β) M is a left A-module[3]), such that $a(a'm) = a'(am) + (aa')m$, $a, a' \in A$, $m \in M$;

(γ) there is a symmetrical bilinear pairing $\varphi\colon M \times M \to A$ satisfying

(γ1) $\varphi(m, m')m'' + \varphi(m', m'')m + \varphi(m'', m)m' = 0, \; m, m', m'' \in M$,

(γ2) $a\varphi(m, m') = \varphi(m, am') + \varphi(am, m'), \quad a \in A$.

Then addition and multiplication are defined in L by

$$(m_1, a_1) + (m_2, a_2) = (m_1 + m_2, a_1 + a_2),$$
$$(m_1, a_1)(m_2, a_2) = \big(a_1 m_2 + a_2 m_1, \varphi(m_1, m_2) + a_1 a_2\big).$$

If we turn M into a two-sided module by defining $ma = am$ and if we write mm' for $\varphi(m, m')$, $m + a$ for (m, a), then L is (additively) the direct sum of M and A and rules (α2), (β), (γ1), and (γ2) may be compressed into

(λ1) $a(bc) + b(ca) + c(ab) = 0$, where $a, b, c \in A$ or $a, b, c \in M$;

(λ2) $a(bc) = b(ca) + c(ab)$, where $a \in A$, $b \in L$, $c \in M$.

Clearly a graded Lie ring is a quasi-Lie ring with $A = \sum L_{2n}$, $M = \sum L_{2n+1}$. Thus, in particular, the homotopy ring $\pi(X)$ is quasi-Lie ring if we put $A = \sum_{n>0} \pi_{2n+1}(X)$, $M = \sum_{n>0} \pi_{2n}(X)$.

We now prove

THEOREM 1. (i) *If $a \in A$, then $2a^2 = 0$, $ba^2 = 0$, $b \in L$.*

(ii) *If $m \in M$, then $m^2 b = -2m(bm)$, $3m^3 = 0$, $bm^3 = 0$, $(m^2)^2 = 0$, $b \in L$.*

Proof. If $a \in A$, then $a \cdot a = -a \cdot a$. If $b \in A$, then, by (λ1), $a(ba) + ba^2 + a(ab) = 0$; but $ab = -ba$, so that $ba^2 = 0$. If $b \in M$, then, by (λ2), $a(ab) = a(ba) + ba^2$; but $ab = ba$, so that $ba^2 = 0$. Thus $ba^2 = 0$ for arbitrary $b \in L$, by distributivity.

If $m \in M$ and $b \in M$, then, by (λ1), $m(bm) + bm^2 + m($ ... 0; but $mb = bm$, $m^2 b = bm^2$, so that $m^2 b = -2m(bm)$. If $m \in M$, $b \in A$, t... by (λ2), $bm^2 = m(mb) + m(bm)$; but $mb = bm$, $m^2 b = -bm^2$, so that $m^2 b = -2m(bm)$. Thus $m^2 b = -2m(bm)$ for arbitrary $b \in L$.

Now $m^2 m = m \cdot m^2$, so that m^3 is unambiguous; moreover $m(m^2) + m(m^2) + m(m^2) = 0$, so that $3m^3 = 0$. By (λ2), $m^2(bm) = b(m \cdot m^2) + m(m^2 b)$; by the argument above, $m^2(bm) = -2m\big((bm)m\big)$ and $m(m^2 b) = -2m\big(m(bm)\big)$. Since $(bm)m = m(bm)$ (since $mm' = m'm$, $ma = am$), it follows that $bm^3 = 0$.

Finally, we have $m^2 \cdot m^2 = -2m(m^2 \cdot m)$, so that $3(m^2)^2 = 0$. On the other hand, we have $2(m^2)^2 = 0$ since $m^2 \in A$, so that $(m^2)^2 = 0$.

COROLLARY 1. (i) *Let X be an arbitrary connected space and let $\alpha \in \pi_{2n+1}(X)$, $\beta \in \pi_r(X)$, $n \geq 1$, $r > 1$. Then $2[\alpha, \alpha] = 0$, $\big[\beta, [\alpha, \alpha]\big] = 0$.*

(ii) Let $\alpha \in \pi_{2n}(X)$, $\beta \in \pi_r(x)$, $n \geq 1$, $r > 1$. *Then* $\big[[\alpha, \alpha], \beta\big] = -2\big[\alpha, [\beta, \alpha]\big]$, $3\big[\alpha, [\alpha, \alpha]\big] = 0$, $\Big[\beta, \big[\alpha, [\alpha, \alpha]\big]\Big] = 0$, $\big[[\alpha, \alpha], [\alpha, \alpha]\big] = 0$.

This follows immediately by applying Theorem 1 to $\overline{\pi}(X)$.

It should be noted that higher powers of arbitrary elements of L are, in general, non-zero. Thus, for example, if $x = m + a$, then

$$x \cdot x^2 = 2a(am) + 2am^2 + m^3,$$
$$x^2 x = 2a(am) + m^3,$$
$$x^2 x^2 = 4am(am + m^2).$$

3. Basic products. Let L be a quasi-Lie ring. We describe $x \in L$ as homogeneous if $x \in M$ or $x \in A$. Choose elements $\alpha_1, \ldots, \alpha_t \in L$ so that $\alpha_i \in M$, $1 \leq i \leq s$, $\alpha_i \in A$, $s + 1 \leq i \leq t$, and form basic products in the α_i. Then the basic products, being monomials in the α_i, are homogeneous elements of L. Let L_α be the subring of L generated by $\alpha_1, \ldots, \alpha_t$. We prove

THEOREM 2. *L_α is additively generated by the basic products together with their squares and cubes.*

Write L^* for the additive subgroup of L generated by the bp, their squares and cubes. Clearly $L^* \subseteq L_\alpha$; we wish to prove $L_\alpha \subseteq L^*$. The most important step in the proof is that showing that if p, p' are bp, then $pp' \in L^*$. To achieve this step, we prove the following sharper result:

LEMMA 1. *Let p be a* bp *of weight u, p' a* bp *of weight u'. Then*

$$pp' = \sum n_i p_i + \sum n_j p_j^2 + \sum n_k p_k^3$$

where the n_i, n_j, n_k are integers, the p_i run over the bp *of weight $u + u'$, the p_j run over the* bp *of weight $(u + u')/2$, and the p_k run over the* bp *of*[4]) *weight $(u + u')/3$.*

Proof. Write $w(p)$ for the weight of p. We may suppose $p \leq p'$ since $pp' = \pm p'p$. The lemma is true if $w(p') = 1$ since then $w(p) = 1$ and pp' is a bp of weight 2 unless $p = p'$. Thus we may suppose the lemma proved for all p, p' with $p \leq p'$ and $w(p') < u'$, $u' \geq 2$, and prove it for p, p'

[3]) That is, M is an additive abelian group such that, for all $a \in A$, $m \in M$, an element $am \in M$ is defined such that $(a + a')m = am + a'm$, $a(m + m') = am + am'$. We do not suppose that $(aa')m = a(a'm)$, $a, a' \in A$, $m \in M$.

[4]) Of course, the terms in p_j^2, p_k^3 can only occur if the appropriate weights are integers.

with $p \leqslant p'$, $w(p') = u'$. The lemma is true if $p = p'$. We suppose it proved for all p_1, p_1' with $p < p_1 \leqslant p_1', w(p_1') = u'$ and prove it [5]) for p, p'.

Let $w(p) = u$ and let p' appear as qq'. Then pp' is basic if $p \geqslant q$ so we suppose $p < q < q' < p'$, and $u \leqslant v = w(q) \leqslant v' = w(q') < u'$, $v + v' = u'$; moreover,

$$pp' = p(qq') = \pm q(pq') \pm q'(pq).$$

Consider first $q(pq')$; applying the inductive hypothesis,

$$pq' = \sum n_i p_i + \sum n_j p_j^2 + \sum n_k p_k^3, \quad w(p_i) = u + v', \quad w(p_j) = (u + v')/2,$$

and, by Theorem 1, $q(pq') = \sum n_i q p_i + \sum n_j q p_j^2$, the second term (or first residue) being present only if $(u+v')/2$ is an integer. The terms $n_i q p_i$ may be disposed of by the inductive hypothesis since $w(p_i) \leqslant u'$, $q > p$ and $v + u + v' = u + u'$, so that it remains to consider qp_j^2. Now $qp_j^2 = 0$ if $p_j \in A$ and $qp_j^2 = \pm 2 p_j(qp_j)$ if $p_j \in M$. If $(u+v')/2 = v$, then either $p_j(qp_j)$ is basic or $p_j(p_j q)$ is basic or $p_j = q$. Thus we may assume $(u+v')/2 \neq v$. Put $v = u_0$, $(u+v')/2 = u_1$, and define u_r inductively as the arithmetic mean of u_{r-2}, u_{r-1}, $r \geqslant 2$. Then if 2^m is the highest power of 2 dividing $u_1 - u_0$, it is easy to see that u_r is an integer if and only if $r \leqslant m+1$. Moreover, $u < u_r$ and $u_{r-1} + 2u_r = u + u'$, $r \geqslant 1$, whence $2u_r < u'$, $r \geqslant 2$.

By the inductive hypothesis (since $u_0 < u'$, $u_1 < u'$),

$$qp_{j_1} = \sum n_{j_1 l} p_{j_1 l} + \sum n_{j_1 j} p_{j_1 j}^2 + \sum n_{j_1 k} p_{j_1 k}^3,$$

where $w(p_{j_1 l}) = 2u_2$, $w(p_{j_1 j}) = u_2$. Thus

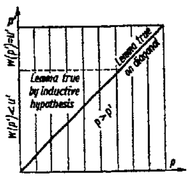
$$p_{j_1}(qp_{j_1}) = \sum n_{j_1 i} p_{j_1} p_{j_1 i} + \sum n_{j_1 j} p_{j_1} p_{j_1 j}^2,$$

the second term (or second residue) being present only if u_2 is an integer. The terms $n_{j_1 i} p_{j_1} p_{j_1 i}$ may be disposed of by the inductive hypothesis since $u_1 < u'$, $2u_2 < u'$, $u_1 + 2u_2 = u + u'$, and we treat the terms $n_{j_1 j} p_{j_1} p_{j_1 j}^2$ as above (in the case of qp_j^2). We obtain

$$p_{j_1 j_2}(p_{j_1} p_{j_1 j_2}) = \sum n_{j_1 j_2 i} p_{j_1 j_2} p_{j_1 j_2 i} + \sum n_{j_1 j_2 j} p_{j_1 j_2} p_{j_1 j_2}^2,$$

$w(p_{j_1 j_2 i}) = 2u_3$, $w(p_{j_1 j_2 j}) = u_3$, the second term (or third r[illegible]e) being present only if u_3 is an integer. Proceeding in this way it is clear that,

[5]) The method of the induction may be illustrated by a diagram. Our argument enables us to deduce the truth of the lemma successively for points on the vertical lines starting from the right.



at worst, the $(m+2)$nd residue vanishes, so that $q(pq')$ will by then have been expressed in the desired form.

We treat $q'(pq)$ in the same way. The only point to observe is that if $u = v$ then pq is basic so that $q'(pq)$ may be expressed in the desired form by the inductive hypothesis. We may therefore assume that $v > u$; with this assumption, the changes in the argument are just those resulting from the interchange [6]) of q, q' and v, v'. Thus the lemma is completely proved.

We now show that L^* is closed to multiplication. To do so it remains only to prove that (i) $pp'^2 \in L^*$, (ii) $p^2 p'^2 \in L^*$.

We prove (i) by a simplified version of the argument in Lemma 1, proceeding by induction on the exponent of the highest power of 2 dividing $w(p') - w(p)$. For $pp'^2 = 0$ or $\pm 2p'(pp')$ and $pp' = \sum n_i p_i + \sum n_j p_j^2 + \sum n_k p_k^3$, $w(p_i) = u + u'$, $w(p_j) = (u+u')/2$, $w(p) = u$, $w(p') = u'$. Thus if $u' -$ [illegible] odd, $pp' = \sum n_i p_i + \sum n_k p_k^3$ and $p'(pp') = \sum n_i p' p_i \in L^*$ by the lemm[illegible] f we suppose $2^m | u' - u$, $2^{m+1} \nmid u' - u$ and that (i) holds if $(m-1)$ is the exponent of the highest power of 2 dividing $w(p') - w(p)$, then we have $p'(pp') = \sum n_i p' p_i + \sum n_j p' p_j^2$ and $w(p_j) - w(p') = (u+u')/2 - u' = (u-u')/2$, so that $p' p_j^2 \in L^*$ by the inductive hypothesis, whence $pp'^2 \in L^*$.

To prove (ii), we observe that $p^2 p'^2 = 0$ or $\pm 2p'(p^2 p')$; but $p^2 p' = \sum n_i p_i + \sum n_j p_j^2 + \sum n_k p_k^3$, so that $p'(p^2 p') = \sum n_i p' p_i + \sum n_j p' p_j^2 \in L^*$. Thus L^* is a ring and since $a_i \in L^*$, $i = 1, \ldots, t$, we have $L_a \subset L^*$ as required.

We will say that the quasi-Lie ring L is *free* if it contains a set S of homogeneous generators $a_1, \ldots, a_t$, $a_i \in M$, $1 \leqslant i \leqslant s$, $a_i \in A$, $s+1 \leqslant i \leqslant t$, such that for every quasi-Lie ring $\bar{L} = \bar{M} + \bar{A}$ and mapping $f_0 \colon S \to \bar{L}$ with $f_0 a_i \in \bar{M}$, $1 \leqslant i \leqslant s$, $f_0 a_i \in \bar{A}$, $s+1 \leqslant i \leqslant t$, there is an extension of f_0 to a ring-homomorphism $f \colon L \to \bar{L}$.

We prove

THEOREM 3. *If L is free, then it is, additively, the direct sum of*
(a) *a free abelian group, basis* $\{p_i\} + \{p_j^2\}$, $p_j \in M$,
(b) *a free* mod 2 *module, basis* $\{p_k^2\}$, $p_k \in A$, *and*
(c) *a free* mod 3 *module, basis* $\{p_l^3\}$, $p_l \in M$.

(Here the p_i are, of course, bp in the generators $a_1, \ldots, a_t$).

Proof. We know that the given elements generate L additively; it remains to show that they are independent. Suppose then that $\sum a_i p_i + \sum b_j p_j^2 + \sum c_k p_k^2 + \sum d_l p_l^3 = 0$, $0 \leqslant c_k \leqslant 1$, $0 \leqslant d_l \leqslant 2$. Let T be a wedge of spheres of which the first s are even-dimensional, the remaining $(t-s)$ odd-dimensional, let $\iota_1, \ldots, \iota_t$ be the generators of $L(T)$ as in § 2 and let f:

[6]) The possibility $(u+v)/2 = v'$ does not, in fact, arise.

$L \to L(T)$ be the homomorphism characterised by $fa_i = \iota_i$, $i = 1, \dots, t$. Embed $L(T)$ in $\overline{\pi}(T)$ and let $\varrho\colon \overline{\pi}(T) \to H(\Omega T)$ be defined as in [4], where ΩT is the space of loops on T. The basic products p_i are mapped by f to bp in the ι_i and ϱ maps bp to basic quasi-commutators (bqc) in the elements $e_i = \varrho\iota_i$. Moreover if $\varrho f p_i = q_i$, then

$$\varrho f p_j^2 = -2q_j^2, \quad p_j \in M, \quad \varrho f p_k^2 = 0, \quad p_k \in A, \quad \varrho f p_l^3 = 0, \quad p_l \in M.$$

Thus

$$\sum a_i q_i - 2\sum b_j q_j^2 = 0.$$

However, since the q_i, q_j^2 are certainly monomials in the q_i of "zero disorder", it follows (see 2.1 and 3.2 of [4]) that $a_i = 0$, all i. $b_j = 0$, all j. Thus

$$\sum c_k p_k^2 + \sum d_l p_l^3 = 0.$$

Now let p_{k_0} be an arbitrary bp in A and let it involve a_i n_i times, $i = 1, \dots, t$. Define the M-weight of p_{k_0} to be $w_1 = \sum_{i=1}^{s} n_i$ and the A-weight to be $w_2 = \sum_{i=s+1}^{t} n_i$. Then not both of w_1, w_2 are zero.

Let the even-dimensional spheres in T all have dimension $2m$ and let the odd-dimensional spheres all have dimension $2n+1$. Then p_{k_0} is mapped by f to a bp $\overline{p}_{k_0}$ in $\overline{\pi}(T)$ of degree $r = w_1(2m-1) + 2w_2 n$. Choose m, so that [7]) r is not of the form $2^q - 2$. In general, let the degree of $\overline{p}_k = fp_k$ be r_k, $r = r_{k_0}$. Then

$$\sum c_k[\overline{p}_k, \overline{p}_k] + \sum d_l[\overline{p}_l, [\overline{p}_l, \overline{p}_l]] = 0.$$

But $[\overline{p}_k, \overline{p}_k] = \overline{p}_k \circ [\iota_{r_k+1}, \iota_{r_k+1}]$ and

$$[\overline{p}_l, [\overline{p}_l, \overline{p}_l]] = p_l \circ [\iota_{r_l+1}, [\iota_{r_l+1}, \iota_{r_l+1}]].$$

Thus

$$\sum \overline{p}_k \circ c_k[\iota_{r_k+1}, \iota_{r_k+1}] + \sum \overline{p}_l \circ d_l[\iota_{r_l+1}, [\iota_{r_l+1}, \iota_{r_l+1}]] = 0.$$

By Theorem A of [4], it follows that $c_k[\iota_{r_k+1}, \iota_{r_k+1}] = 0$; but $[\iota_{r+1}, \iota_{r+1}] \neq 0$; so that $c_{k_0} = 0$. Since p_{k_0} was chosen arbitrarily it follows that $c_k = 0$, all k.

By a similar argument, choosing the dimensions of the spheres in T to avoid accidental zeros of $[\iota, [\iota, \iota]]$ (it was proved in [5] that $[\iota, [\iota, \iota]] \in \pi_{6n-2}(S^{2n})$ can only be zero if n is a power of 3), we prove that each $d_l = 0$, and so complete the proof of the theorem.

[7]) This is obviously possible; for example, observe that no three distinct integers in arithmetic progression can be powers of 2.

It would be most satisfactory to replace Theorem A of [4] by an invariant description of the ring $\overline{\pi}(T)$ in terms of the subring $L(T)$ generated by $\iota_1, \dots, \iota_t$, and homotopy groups of spheres. We indicate a possible first step in this direction.

Now an element of $L_q(T)$ is represented by a map $S^{q+1} \to T$. Thus we may define, for $\varrho \in L_q(T)$, $\alpha \in \pi_n(S^{q+1})$, the composition element $\varrho \circ \alpha \in \pi_n(T)$. Let $L_q(T) \circ \pi_n(S^{q+1})$ be the subgroup of $\pi_n(T)$ generated by all such elements $\varrho \circ \alpha$, and let V_m, $m \geqslant 1$, be the subgroup-union of the groups $L_q(T) \circ \pi_n(S^{q+1})$ for all $q \geqslant m$. Then certainly $V_m \supseteq V_{m+1}$. Now by Theorem A of [4], any element $\beta \in \pi_n(T)$ may be represented as $\sum p_i \circ \beta_i$ (finite sum), where the p_i are bp's. Then $\beta_i \in \pi_n(S^{q_i+1})$, $p_i \in L_{q_i}(T)$, for some q_i, so that $\beta \in V_m$, where m is the smallest integer q_i such that $\beta_i \neq 0$. Thus $\pi_n(T) = \bigcup_{m \geqslant 1} V_m$; in other words, $\pi_n(T)$ is (decreasingly) filtered by its subgroups V_m.

Consider the mapping $\psi\colon L_m(T) \times \pi_n(S^{m+1}) \to V_m/V_{m+1}$, given by

$$\psi(\varrho, \alpha) = \varrho \circ \alpha, \bmod V_{m+1}.$$

Then ψ is bilinear since $\varrho \circ (\alpha_1 + \alpha_2) = \varrho \circ \alpha_1 + \varrho \circ \alpha_2$ and [8])

$$(\varrho_1 + \varrho_2) \circ \alpha = \varrho_1 \circ \alpha + \varrho_2 \circ \alpha + [\varrho_1, \varrho_2] \circ H_0\alpha + \dots \equiv \varrho_1 \circ \alpha + \varrho_2 \circ \alpha, \bmod V_{2m}.$$

Thus ψ induces a homomorphism $\overline{\psi}\colon L_m(T) \otimes \pi_n(S^{m+1}) \to V_m/V_{m+1}$ which is obviously onto V_m/V_{m+1}. However it is not, in general, (1-1), for if $l \in L_m(T)$, then

$$\overline{\psi}(l \otimes [\iota, \iota]) = l \circ [\iota, \iota] = [l, l] \equiv 0 \bmod V_{m+1}.$$

Thus to express the graded group $\sum V_m/V_{m+1}$, associated with the group $\pi_n(T)$, in invariant terms, it would be necessary to examine the kernel of $\overline{\psi}$.

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[8]) See proof of (6.5) in [4].

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