On a theorem of Borsuk

by

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Recently Borsuk [1] showed that if a compact space of dimension at most k has a positive k-dimensional Betti number, then for $m \geqslant k$ the function space of maps of the original space into the m-sphere has a positive $(m-k)$th Betti number. The case $m=k$ is covered by the Hopf Classification Theorem [9], and the object of this paper is to show that the case $m>k$ may be viewed as a generalization of this theorem, and that there is in fact a sort of duality between the homology of the function space and the cohomology of the original space.

1. The critical dimension

Notation and conventions. In this paper space will always mean Hausdorff space. For any space X, let $H^q(X)$ denote the q-dimensional Čech cohomology group of X with integer coefficients [3], and if A is a subspace of X let $H^q(X,A)$ be the q-dimensional Čech cohomology group of the pair (X,A). Similarly let $H_q(X)$ and $H_q(X,A)$ be the q-dimensional singular homology groups with integer coefficients of the space X and of the pair (X,A).

If X is a space and $x \in X$, denote the q-dimensional homotopy group of X based at the point x by $\pi_q(X,x)$. If X is a function space containing a unique constant map, this map will be taken as the base point, and the preceding notation will be abbreviated to $\pi_q(X)$. Finally, if X has a multiplication with an identity up to homotopy, and an inverse up to homotopy, let $\pi_0(X)$ denote the group of path components of X.

For any non-negative integer k, let S^k denote the k-dimensional sphere, and let y^k be a point of S^k. Let I denote the closed interval of real numbers from -1 to $+1$.

Definition. If X is a space, define $s(X)$ to be the identification space of $X \times I$ obtained by the following identifications: $(x,1)$ is identified with $(x',1)$, and $(x,-1)$ is identified with $(x',-1)$ for $x,x' \in X$. Define $s_+(X)$ to be the image of those pairs (x,t) such that $t \geqslant 0$, define $s_-(X)$

to be the image of those pairs (x,t) such that $t \leqslant 0$, and identify $x \in X$ with $(x,0) \in s(X)$. The space $s(X)$ is called the *suspension* of the space X.

Assumption. A positive integer m has been chosen once and for all.

Definition. If (X,A) is a pair, let $E(X,A)$ denote the function space of maps $f\colon (X,A) \to (S^m, y^m)$ topologized with the compact-open topology, and let $E'(X,A)$ denote the component of the constant map. $E(X)$ will denote $E(X,\emptyset)$ where $\emptyset$ is the empty set.

Definition. Let X be a space with base point $x \in X$. Then the space of loops in X based at x is the space of maps $f\colon I \to X$ such that $f(-1)=f(1)=x$. Denote this space by $\Omega(X,x)$ and take as base point the map $e^1\colon I \to X$ defined by $e^1(t)=x$. Now let $\Omega^1(X,x)=\Omega(X,x)$ and proceed to define $\Omega^{k+1}(X,x)$ assuming we have defined $\Omega^k(X,x)$ with base point e^k. In this case $\Omega^{k+1}(X,x)=\Omega\big(\Omega^k(X,x),e^k\big)$ and $e^{k+1}\colon I \to \Omega^{k+1}(X,x)$ is the map such that $e^{k+1}(t)=e^k$. The space $\Omega^k(X,x)$ is called the *k-fold loop space in X based at the point x.*

THEOREM 1. *If (X,A) is a finite dimensional compact pair, and $H^q(X,A)=0$ for $q>k$, where $k \leqslant m$ is an integer, then $\pi_q\big(E(X,A)\big)=0$ for $q<m-k$, and $\pi_{m-k}\big(E(X,A)\big) \approx H^k(X,A)$.*

Proof. Proceed by induction on $m-k$. If $m=k$, the preceding is just the Hopf theorem ([9] and [2]). Suppose now that $m>k$. Define $p\colon E\big(s_+(X),s_+(A)\big) \to E(X,A)$ by $p(f)=f|X$. Then p is a fibre mapping in the sense of Serre ([5], p. 443), and the total space is contractible. Therefore, since the fibre of p is $E\big(s(X),s(A)\big)$, we have $\pi_q\Big(E\big(s(X),s(A)\big)\Big) \approx \pi_{q+1}\big(E(X,A)\big)$. Further [1] $H^{q+1}\big(s(X),s(A)\big) \approx H^q(X,A)$. Consequently,

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[1]) We have an exact sequence for the triad $\big(s(X),s_+(X),s(A)\big)$:

$$\ldots \to H^q\big(s_+(X),s_+(X)\cap s(A)\big) \to H^{q+1}\big(s(X),s_+(X)\cup s(A)\big)$$
$$\to H^{q+1}\big(s(X),s(A)\big) \to H^{q+1}\big(s_+(X),s_+(X)\cap s(A)\big) \to \ldots$$

Since $s_+(X)$ and $s_+(A)$ are contractible we have $H^{q+1}\big((s(X),s_+(X)\cup s(A)\big) \approx H^{q+1}\big(s(X),s(A)\big)$.

However, $H^{q+1}\big(s(X),s_+(X)\cup s(A)\big) \approx H^{q+1}\big(s_-(X),X\cup s_-(A)\big)$ by excision, and also $H^q\big(X\cup s_-(A),s_-(A)\big) \approx H^q(X,A)$.

Now we have an exact sequence for the triple $\big(s_-(X),X\cup s_-(A),s_-(A)\big)$:

$$\ldots \to H^q\big(s_-(X),s_-(A)\big) \to H^q\big(X\cup s_-(A),s_-(A)\big)$$
$$\to H_{q+1}\big(s_-(X),s_-(A)\cup(X)\big) \to H^{q+1}\big(s_-(X),s_-(A)\big) \to \ldots$$

and since $s_-(X)$, and $s_-(A)$ are contractible, we have

$$H^q\big(X\cup s_-(A),s_-(A)\big) \approx H^{q+1}\big(s_-(X),X\cup s_-(A)\big)$$

which implies the desired result.

the truth of the theorem for all compact finite dimensional pairs such that $m-k \leqslant n$ implies the theorem for all finite dimensional compact pairs such that $m-k \leqslant n+1$, and the result follows.

COROLLARY. *Under the conditions of the preceding theorem, if $m>k$, then $H_q\big(E(X,A)\big)=0$ for $0<q<m-k$, and $H_{m-k}\big(E(X,A)\big) \approx H^k(X,A)$.*

Proof. This follows immediately from the Hurewicz theorem, and the preceding theorem.

COROLLARY. *If X is a k-dimensional compact space and $k<m$ then $H_q\big(E(X)\big) \approx H_q\big(E(X,x)\big)+H_q(S^m)$ for $q \leqslant 2(m-1)-k$, where $x \in X$.*

Proof. The space $E(X)$ is a fibre space over S^m with fibre $E(X,x)$ and this fibre space has a cross-section. Since $H_i(S^m)=0$ for $0<i<m$, $H_i\big(E(X,x)\big)=0$ for $0<i<m-k$ by preceding corollary, we have the exact sequence ([5], Proposition 5, p. 468):

$$H_{2m-k-1}\big(E(X,x)\big) \xrightarrow{i_*} H_{2m-k-1}\big(E(X)\big) \xrightarrow{p_*} H_{2m-k-1}(S^m) \to H_{2m-k-2}\big(E(X,x)\big) \to \ldots$$

Now, let $f\colon S^m \to E(X)$ be the cross-section. Then $p_*f_*\colon H_q(S^m) \to H_q(S^m)$ is identity and it follows that $H_q\big(E(X)\big)=f_*H_q(S^m)+p_*^{-1}(0)=f_*H_q(S^m)+ i_*H_q\big(E(X,x)\big)$ for $q \leqslant 2m-k-1$, where f_* is isomorphism. Thus $p_*\colon H_q\big(E(X)\big) \to H_q(S^m)$ is onto for $q \leqslant 2m-k-1$ and the exactness implies that i_* maps $H_q\big(E(X,x)\big)$ isomorphically for $q \leqslant 2m-k-2$.

In order to study further the relationship between the homology of $E(X,A)$ and the cohomology of (X,A) it would be convenient to have a natural mapping $\varphi\colon H_r\big(E(X,A)\big) \to H^{m-r}(X,A)$ for $r \leqslant m$ such that if $r=m-k$, φ is an isomorphism. The next two sections will be devoted to setting up such a mapping and studying a few of its properties. In the process of doing this the preceding corollary will be reproved in another manner.

2. The homomorphism φ

If X is a space, denote by $C(X)$ the singular chain complex of X ([3], p. 187). Now if X and Y are spaces it has been proved that the homology of $C(X\times Y)$ is naturally isomorphic with the homology of the tensor product complex $C(X)\otimes C(Y)$ [4], where $\big(C(X)\otimes C(Y)\big)_n$ is

$$\sum_{r+s=n} C_r(X)\otimes C_s(Y),$$

and if $x \in C_r(X)$, $y \in C_s(Y)$, then $\partial(x\otimes y)=\partial x\otimes y+(-1)^r x\otimes \partial y$. Further the singular cohomology of Y is the cohomology of the chain complex $\operatorname{Hom}\big(C(Y),Z\big)$ ([3], p. 152) where Z denotes the ring of integers, and the singular cohomology of the space $X\times Y$ is naturally isomorphic with the cohomology of the chain complex $\operatorname{Hom}\big(C(X)\otimes C(Y),Z\big)$ [4]. For this last complex the cochains of dimension n are the elements of the group

$$\operatorname{Hom}\Big(\sum_{r+s=n}C_r(X)\otimes C_s(Y),Z\Big),$$

and if f is an n-cochain, then δf is an $(n+1)$-cochain such that $\delta f(x\otimes y)=f\big(\partial(x\otimes y)\big)$.

Definition. If X and Y are spaces, and $f\in\operatorname{Hom}\Big(\big(C(X)\otimes C(Y)\big)_n,Z\Big)$, define φ_f: $C_r(X)\to\operatorname{Hom}\big(C_{n-r}(Y),Z\big)$ for $0\leqslant r\leqslant n$ by $\varphi_f(x)(y)=f(x\otimes y)$.

LEMMA 1. *If* $f\in\operatorname{Hom}\Big(\big(C(X)\otimes C(Y)\big)_n,Z\Big)$, $\delta f=0$, *and* $x\in C_r(X)$, *then* $\delta\varphi_f(x)=(-1)^{r+1}\varphi_f(\partial x)$.

LEMMA 2. *If* $f,g\in\operatorname{Hom}\Big(\big(C(X)\otimes C(Y)\big)_n,Z\Big)$, $\delta f=\delta g=0$,

$$h\in\operatorname{Hom}\Big(\big(C(X)\otimes C(Y)\big)_{n-1},Z\Big),$$

$f-g=\delta h$, *and* $x\in C_r(X)$, *then* $\varphi_f(x)-\varphi_g(x)=\varphi_h(\partial x)+(-1)^r\delta\varphi_h(x)$.

The proofs of the preceding two lemmas are straight-forward, and will be omitted.

Definitions. If X is a space, denote by $\widetilde{H}^q(X)$ the q-dimensional singular cohomology group of X. If X and Y are spaces, and $f\in\widetilde{H}^n(X\times Y)$ define φ_f: $H_r(X)\to\widetilde{H}^{n-r}(Y)$ for $0\leqslant r\leqslant n$ as follows: Let g be a cocycle representing f, and let φ_f be the homomorphism induced by φ_g. This procedure is permissable and independent of the choice of g by Lemmas 1 and 2. Similarly if X is a space, (Y,B) is a pair, and $f\in\widetilde{H}^n\big(X\times(Y,B)\big)$ define φ_f: $H_r(X)\to\widetilde{H}^{n-r}(Y,B)$ for $0\leqslant r\leqslant n$ by the same procedure.

Definition. Let $a\in\widetilde{H}^m(S^m)$ be a generator, and for any compact pair (X,A) let ξ: $E(X,A)\times(X,A)\to(S^m,y^m)$ be the map defined by $\xi(f,x)=f(x)$. Define $\overline{\varphi}$: $H_r\big(E(X,A)\big)\to\widetilde{H}^{m-r}(X,A)$ for $0\leqslant r\leqslant m$ by $\overline{\varphi}=\varphi_{\xi^*(a)}$, where ξ^*: $\widetilde{H}^m(S^m,y^m)\to\widetilde{H}^m\big(E(X,A)\times(X,A)\big)$ is the homomorphism induced by ξ.

PROPOSITION 1. *If* (X,A) *and* (Y,B) *are compact pairs,* f: $(X,A)\to(Y,B)$ *is a map, and* $\widetilde{f}$: $E(Y,B)\to E(X,A)$ *is the induced map, then* $\overline{\varphi}\widetilde{f}_*=f^*\overline{\varphi}$, *where* $\widetilde{f}_*$: $H_r\big(E(Y,B)\big)\to H_r\big(E(X,A)\big)$ *and* f^*: $\widetilde{H}^{m-r}(Y,B)\to\widetilde{H}^{m-r}(X,A)$ *are the homomorphisms induced by* $\widetilde{f}$ *and* f *respectively.*

Definition. Let (X,A) be a compact pair, let $\{(X_j,A_j),\pi^{j_1}_{j_2}\}$ be an inverse system of finite simplicial complexes whose inverse limit is (X,A). Now $H^q(X,A)=\lim\widetilde{H}^q(X_j,A_j)$, the direct limit group of the direct system $\{\widetilde{H}^q(X_j,A_j),\pi^{j_1}_{j_2}\}$, and $H_q\big(E(X,A)\big)=\lim H_q\big(E(X_j,A_j)\big)$ (see [3], Chapters VIII-X). Define φ: $H_r\big(E(X,A)\big)\to H^{m-r}(X,A)$ to be the homomorphism induced by $\{\varphi$: $H_r\big(E(X_j,A_j)\big)\to\widetilde{H}^{m-r}(X_j,A_j)\}$ for $0\leqslant r\leqslant m$.

3. Duality

THEOREM 2. *If* (X,A,B) *is a triple of finite dimensional compact spaces,* $H^r(A,B)=0$ *for* $r>p$, $H^r(X,A)=0$ *for* $r>q$, $m>\max\{p,q\}$, $s=2m-(p+q+1)$, *and* $n=\min\{s,m\}$, *then there is an exact sequence*

$$H_s\big(E(X,A)\big)\to\ldots\to H_r\big(E(X,A)\big)\to H_r\big(E(X,B)\big)\to H_r\big(E(A,B)\big)$$
$$\to H_{r-1}\big(E(X,A)\big)\to\ldots\to H_1\big(E(A,B)\big)\to 0\,,$$

and further there is a diagram

$$\begin{array}{ccccccccc}
H_n\big(E(X,A)\big)\to\ldots\to & H_r\big(E(X,A)\big) & \to & H_r\big(E(X,B)\big) & \to & H_r\big(E(A,B)\big) & \to & H_{r-1}\big(E(X,A)\big) & \to\ldots\\
\downarrow\varphi & \downarrow\varphi & & \downarrow\varphi & & \downarrow\varphi & & \downarrow\varphi & \\
H^{m-n}(X,A)\to\ldots\to & H^{m-r}(X,A) & \to & H^{m-r}(X,B) & \to & H^{m-r}(A,B) & \to & H^{m-r+1}(X,A) & \to\ldots
\end{array}$$

such that the horizontal lines are exact, and the squares are commutative up to sign.

Proof. The natural map of $E(X,B)$ into $E(A,B)$ is a fibre map with fibre $E(X,A)$. Further, by Theorem 1, the base space is $m-p-1$ connected, and the fibre is $m-q-1$ connected. Consequently the first part of the theorem follows from [5], Proposition 5, p. 468. The proof may now be completed in the standard manner using the exactness of the cohomology sequence of the triple (X,A,B). Those squares where the index r does not change are actually commutative. In the remaining type of square there is commutativity up to sign, the sign being that determined by Lemma 1.

LEMMA 3. *If* $k<m$, *and* $n=\min\{2(m-k)-1,m\}$ *then*

$$\varphi\colon\ H_r\big(E(S^k,y^k)\big)\to H^{m-r}(S^k,y^k)$$

is an isomorphism for $0<r\leqslant n$, *and* $H_r\big(E(S^k,y^k)\big)=0$ *for* $n<r<2(m-k)$.

Proof. The space $E(S^k,y^k)$ is naturally homeomorphic with the k-fold loop space in S^m, $\Omega^k(S^m,y^m)$. Applying the suspension theorem ([5], Proposition 10, p. 483) we have that $H_{q-1}\big(\Omega(S^m,y^m)\big)\approx H_q(S^m)$ for $1<q<2m-1$, $H_{q-2}\big(\Omega^2(S^m,y^m)\big)\approx H_{q-1}\big(\Omega(S^m,y^m)\big)$ for $2<q<2m-2$, etc. or we have for $k<m$ that $H_r\big(\Omega^k(S^m,y^m)\big)\approx H_{r+k}(S^m)$ for $0<r<2(m-k)$. Thus we see that the only non-zero homology group in the range of dimensions being considered is in dimension $m-k$. Therefore to complete the proof, it is necessary only to observe that φ is an isomorphism if $r=m-k$.

LEMMA 4. *If (X,A) and (Y,B) are compact pairs, and $f\colon (X,A)\to(Y,B)$ is a relative homeomorphism* ([3], p. 266), *then f induces a homeomorphism between $E(Y,B)$ and $E(X,A)$.*

THEOREM 3. *If k is an integer less than m, (X,A) is a k-dimensional compact pair, A is non-empty and $H^0(X,A)=0$, then*

(1) *$E(X,A)$ is connected.*

(2) *$\varphi\colon\ H_q\big(E(X,A)\big)\to H^{m-q}(X,A)$ is an isomorphism for $0<q<\min\{2(m-k),m\}$,*

(3) *$H_q\big(E(X,A)\big)=0$ for $m\leqslant q<2(m-k)$.*

Proof. By Lemma 4 it suffices to assume A is a point. Further, since we are using Čech cohomology, it suffices to prove the theorem in case X is a finite connected complex. Therefore, assume A is a point, and X is a k-dimensional finite connected complex. Let X^0 be a maximal tree in X containing A, and let X^i be the i-skeleton of X for $i>0$. Now X^0 is contractible, and consequently $E(X^0,A)$ is contractible. Therefore, the theorem is true for the pair (X^0,A). Suppose now the theorem is proved for the pairs (X^i,A) for $i\leqslant j$. We may apply Theorem 2 to the triple (X^{j+1},X^j,A). The inductive step in the proof then follows from the inductive hypothesis, and Lemma 3.

Extract from a letter to K. Borsuk

The fact that the map p in proof of Theorem 1 is a fibre map, and that in proof of Theorem 2 $E(X,B)\to E(A,B)$ is a fibre map is a corollary of the following theorem which is I believe essentially only a rewording of a theorem of yours.

THEOREM. Suppose

(1) *Y is an absolute neighborhood retract, $y_0\in Y$,*

(2) *X is a compact space, and A,B are closed subspaces of X, then the map $p\colon\ (Y,y_0)^{(X,B)}\to(Y,y_0)^{(A,A\cap B)}$ given by $p(f)=f|A$ is a fibre map.*

Proof. To show that p is a fibre map, we must show that for every finite complex K, and commutative diagram

$$\begin{array}{ccccc} & & & \overset{f}{\nearrow} & (Y,y_0)^{(X,B)} \\ & & & & \downarrow p \\ K & \xrightarrow{r_0} & K\times I & \xrightarrow{F} & (Y,y_0)^{(A,A\cap B)} \end{array}$$

where I is the unit interval, and $r_0(k)=(k,0)$ there exists

$$\widetilde{F}\colon\ K\times I\to(Y,y_0)^{(X,B)}$$

such that $p\widetilde{F}=F$, and $\widetilde{F}r_0=f$. However, having F is equivalent to having $G\colon \big(K\times I\times A, K\times I\times(A\cap B)\big)\to(Y,y_0)$ defined by $G(k,t,a)=F(k,t)(a)$. Further having f corresponds to having $g\colon\ (K\times X, K\times B)\to(Y,y_0)$ defined by $g(k,x)=f(k)(x)$, and since the diagram is commutative $g(k,a)=G(k,0,a)$ for $k\in K$, $a\in A$. In other words we have a map defined on $(K\times\{0\}\times X)\cup(K\times I\times A)$ which sends the subset $(K\times\{0\}\times B)\cup\cup\big(K\times I\times(A\cap B)\big)$ into y_0. First extend to a map of $(K\times\{0\}\times X)\cup\cup(K\times I\times A)\cup(K\times I\times B)$ which sends $K\times I\times B$ into y_0. We then have a map defined on $(K\times\{0\}\times X)\cup\big(K\times I\times(A\cup B)\big)$ extend this to a map G: $K\times I\times X\to Y$ and define $\widetilde{F}$ by $\widetilde{F}(k,t)(x)=G(k,t,x)$. The proof of the existence of $\widetilde{G}$ parallels the proof of lemma 2.2 in [3], p. 298.

Problems

1. Is Theorem 3 valid for arbitrary paracompact spaces?

2. Can the dimension condition in Theorem 3 be replaced by (X,A) is a finite dimensional compact pair such that $H^q(X,A)=0$ for $q>k$?

3. If X is a compact space, does there exist a spectral sequence which relates the cohomology of X and the homology of $E(x)$?

4. If X is a compact space, what can be said about the relationship between the cohomology of X and the homology of the function space of maps of X into an $m-1$ connected space Y?

References

[1] K. Borsuk, *Concerning the homological structure of the functional space S^{m^X}*, Fund. Math. 39 (1952), p. 25-37.

[2] C. H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. 69 (1947), p. 200-242.

[3] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton N. J. 1952.

[4] S. Eilenberg and J. A. Zilber, *On products of complexes*, Amer. J. Math. 75 (1953), p. 200-204.

[5] J. P. Serre, *Homologie singulière des espaces fibrés*, Ann. of Math. 54 (1951), p. 425-505.

[6] E. Spanier, *Borsuk's cohomotopy groups*, Ann. of Math. 50 (1949), p. 203-245.

[7] — and J. H. C. Whitehead, *A first approximation to homotopy theory*, Proc. Nat. Acad. Sci. U. S. A. 39 (1953), p. 655-660.

[8] — — *S-theory and duality*, to appear.

[9] H. Whitney, *The maps of an n-complex into an n-sphere*, Duke Math. J. 3 (1937), p. 51-55.

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