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Continuous functions considered from the standpoint of Dini's conditions *

by

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Introduction

Let $w(t)$ denote a continuous function defined and not equal to 0 for $t>0$, monotone, non-decreasing and tending to zero for $t\to 0$. By $W(\tau)$ we understand

(1) $$W(\tau)=\int_{\tau}^{1}\frac{dt}{w(t)}$$

and suppose that everywhere

(2) $$\lim_{\tau\to+0} W(\tau)=\infty .$$

Moreover, we suppose that the functions $f(x)$ always mean continuous, defined and bounded functions in the interval $(-\infty, \infty)$.

The object of this paper is the examination of classes of functions $f(x)$ with regard to their satisfying the generalized Dini condition, *i. e.*

(3) $$\int_{0}^{1}\frac{|f(x+t)-f(x)|}{w(t)}\,dt\leqslant M\ ^{1)} .$$

Let D_w denote a class of functions $f(x)$ satisfying (3) for every x with a certain constant M. We shall suppose that the function $w(t)$ satisfies condition (2).

From the inequality $w_1(t)\leqslant w_2(t)$, for $0<t<a$ and with a certain constant a, follows $D_{w_1}\subset D_{w_2}$. This permits a classification of $f(x)$ according to $w(t)$. *E. g.*, taking in (3) $w(t)=t^{1+\delta}|\log t|^{\gamma}$ we obtain a logarithmic-

* This paper was presented to the Poznań University as a part of Doctor's Thesis of the author in December 1951. The author wishes to thank Professor W. Orlicz for having suggested the subject to him, and for his advice and criticism made while the paper was being written.

[1]) For simplicity we denote the upper limit of integration as 1. It is obvious that it can be replaced by a constant positive a arbitrarily chosen for $f(x)$ and $w(t)$.

-power scale and for $\gamma=0$ a power scale classifying $f(x)$ with regard to Dini's condition.

Let D_w^∞ denote a class of functions $f(x)$ satisfying

$$\int_0^1 \frac{|f(x+t)-f(x)|}{w(t)}\,dt=\infty \text{ [1]}$$

for every x. We shall suppose that $w(t)$ satisfies (2). It will be observed, to begin with, that $D_w^\infty \subset C - D_w$, where C denotes the set of all continuous functions $f(x)$, and that in the case of the inequality $w_1(t) \leqslant w_2(t)$ satisfied for $0<t<a$ with a certain constant a, we shall have $D_{w_2}^\infty \subset D_{w_1}^\infty$.

In this paper we shall formulate the necessary and sufficient condition which must be satisfied by $w_1(t)$ and $w_2(t)$, in order that $f(x)$ belong simultaneously to classes D_{w_1} and $D_{w_2}^\infty$. These results are given under additional conditions regarding $w_1(t)$ and $w_2(t)$, formulated as follows:

$$(*) \qquad \int_0^1 \frac{t}{w(t)}\,dt<\infty,$$

$$(**) \qquad \lim_{t\to+0} W(t)\cdot\frac{w(t)}{t}=g>1,$$

$$({}^{*}_{**}) \qquad \overline{\lim_{t\to+0}}\,\frac{w(2t)}{w(t)}=s<\infty .$$

The first two of these conditions concern $w_1(t)$, the first and third concern $w_2(t)$.

In this paper we examine functions of the type O, which we understand functions $f(x)$ of the following form:

$$f(x)=\sum_{n=1}^{\infty} a_n\varphi(b_n,x),$$

where $a_n>0$, $0<b_n<b_{n+1}$, $b_n\to\infty$, $\sum_{n=1}^{\infty} a_n<\infty$, the function $\varphi(x)$ being defined for every x, non-constant, periodic with period l and satisfying Lipschitz's condition [2]).

We shall also consider functions of the type W, *i. e.* functions of the type O where $a_n=a^n$, $b_n=b^n$ and $0<a<1$, $ab\geqslant 1$.

We denote by $D(\delta,\gamma)$ and $D^\infty(\delta,\gamma)$ the classes D_w and D_w^∞ respectively in the logarithmic-power scale, *i. e.* in the case of

$$w(t)=t^{1+\delta}|\log t|^\gamma .$$

[2]) In order that $W(l/b_n)$ be defined by formula (1) for every n, we shall assume for simplicity of notation, $b_1>l$ and, independently of that, $l\geqslant 1$.

Taking into account suppositions (2) and (*) [3]) and the necessary and sufficient conditions of the existence of a function $f(x)$ belonging simultaneously to classes D_{w_1} and $D_{w_2}^\infty$, we can indicate the domain of validity of the logarithmic-power scale (domain of the values of the parameters δ,γ) and also the containing-direction of classes $D(\delta,\gamma)$ with regard to the values of the parameters δ and γ, the classes being proper parts of each other for different values of pairs of those parameters. For this scale, the sufficient conditions for the coefficients a_n, b_n in order that the function $f(x)$ of type O belong simultaneously to classes $D(\delta_1,\gamma_1)$ and $D^\infty(\delta_2,\gamma_2)$, have been established. These conditions made possible the construction of various examples, of which worthy of attention is the example of a universal function $f(x;\delta,\gamma)$ of type O. For the whole of the logarithmic-power scale, the values of the parameters δ,γ uniquely define the narrowest class $D(\delta_1,\gamma_1)$ of this scale to which this function still belongs. Moreover, if $D(\delta_2,\gamma_2)$ defines any class, that is a proper part of class $D(\delta_1,\gamma_1)$, the function belongs simultaneously to $D^\infty(\delta_2,\gamma_2)$.

A closer examination of the above conditions leads to a comparison of two classifications in the logarithmic-power scale, one according to Dini's conditions, the other according to Hölder's conditions, of which the latter has been examined in a former paper of mine [6]. It turns out that the two classifications are in a certain sense complementary.

Reference to the paper mentioned requires the repetition of definitions used therein:

H_ω denotes the class of functions $f(x)$ satisfying, for every x and for every h, the condition

$$|f(x+h)-f(x)|\leqslant M\omega(|h|),$$

where M is a constant, dependent only on $f(x)$. Function $\omega(h)$ is defined and different from zero for $h>0$, monotone non-decreasing and tending to zero for $h\to 0$. We suppose additionally that

$$(4) \qquad \lim_{h\to+0}\left(\sup_{0<t\leqslant h}\frac{t}{\omega(t)}\right)<\infty .$$

H_ω^∞ denotes a class of functions $f(x)$ satisfying, for each x, the condition

$$\overline{\lim_{h\to+0}}\,\frac{|f(x+h)-f(x)|}{\omega(|h|)}=\infty ,$$

where $\omega(h)$ satisfies condition (4).

[3]) These suppositions are explained in Remark of § 2.

In the case of the logarithmic-power scale defined by Hölder's conditions, *i. e.* when

$$\omega(h) = h^{\delta}|\log h|^{\gamma},$$

the classes H_ω are denoted $H(\delta,\gamma)$ and the classes H_ω^∞ by $H^\infty(\delta,\gamma)$.

Apart from the above notation

$$f(x),\ \varphi(x),\ w(t),\ W(\tau),\ D_w,\ D_w^\infty,\ D(\delta,\gamma),\ D^\infty(\delta,\gamma),$$

$$\omega(h),\ H_\omega,\ H_\omega^\infty,\ H(\delta,\gamma),\ H^\infty(\delta,\gamma)$$

the following symbols are repeatedly used:

a_n, b_n coefficients used in the definition of functions of type O,
a, b coefficients used in the definition of functions of type W,
K Lipschitz' constant of the function $\varphi(x)$,
l period of the function $\varphi(x)$,
D oscillation of the function $\varphi(x)$ in the interval $0 \leqslant x < l$.

§ 1. Lemmas

LEMMA 1. *If $w(t)$ satisfies the suppositions $(*)$ and $(**)$, then the inequality*

$$\frac{\int_0^{\tau} \big(t/w(t)\big)\,dt}{\tau W(\tau)} \leqslant L \tag{5}$$

holds for every τ, where $0<\tau\leqslant\tau_0<1$, and L is a positive constant dependent on τ_0.

Proof. If the lemma did not hold, there would exist a sequence $\{\tau_i\}$ such that $\tau_i\to 0$ and

$$\frac{\int_0^{\tau_i} \big(t/w(t)\big)\,dt}{\tau_i W(\tau_i)} \to \infty. \tag{6}$$

Taking into account $(*)$ we should have $\int_0^{\tau_i}\big(t/w(t)\big)\,dt\to 0$ and thus, by (6) also $\tau_i W(\tau_i)\to 0$. From $(**)$ we should obtain

$$\lim_{i\to\infty}\frac{\int_0^{\tau_i}\big(t/w(t)\big)\,dt}{\tau_i W(\tau_i)} = \lim_{\tau\to+0}\frac{\tau}{W(\tau)\,w(\tau)-\tau} = \frac{1}{g-1},$$

contrary to (6). This result is obtained by replacing the limit of the quotient of the functions for $\tau=\tau_i\to 0$ by the limit of the quotient of their derivatives [4].

LEMMA 2. *If $(\overset{*}{**})$ is satisfied, then*

$$\overline{\lim_{\tau\to+0}}\frac{W(\tau)}{W(2\tau)} \leqslant \frac{s}{2}. \tag{7}$$

Proof. From $(\overset{*}{**})$ follows

$$\frac{1}{w(t)} \leqslant \frac{s+\varepsilon}{w(2t)}$$

true for every $\varepsilon>0$ and for every t satisfying $0<t\leqslant a$, where a is a constant suitably chosen for ε.

Integrating, we obtain

$$\int_{\tau}^{a/2}\frac{dt}{w(t)} \leqslant (s+\varepsilon)\int_{\tau}^{a/2}\frac{dt}{w(2t)} = \frac{s+\varepsilon}{2}\int_{2\tau}^{a}\frac{dt}{w(t)},$$

and hence

$$W(\tau)-W\left(\frac{a}{2}\right) \leqslant \frac{s+\varepsilon}{2}\big(W(2\tau)-W(a)\big).$$

From the last inequality and (2) the truth of the lemma is obvious.

LEMMA 3. *If $\lim_{t\to+0}\big(w_2(t)/w_1(t)\big)=0$, then $\lim_{\tau\to+0}\big(W_1(\tau)/W_2(\tau)\big)=0$.*

Proof. Since

$$\frac{1}{w_1(t)} \leqslant \frac{\varepsilon}{w_2(t)} \quad \text{for} \quad 0<t\leqslant\tau_0,$$

[4]) In fact, if there exists a sequence $\tau_i\to 0$ $(\tau_i>0)$ such that $g(\tau_i)\to 0$, $\psi(\tau_i)\to 0$ $(\psi(\tau_i)\neq 0)$ and $g(\tau)$, $\psi(\tau)$ have derivatives in a certain right-sided neighbourhood of $\tau=0$, and $\lim_{\tau\to+0}\big(g'(\tau)/\psi'(\tau)\big)$ exists, then there exists a limit of the quotient $g(\tau_i)/\psi(\tau_i)$ for $\tau_i\to 0$ and

$$\lim_{i\to\infty}\frac{g(\tau_i)}{\psi(\tau_i)} = \lim_{\tau\to+0}\frac{g'(\tau)}{\psi'(\tau)}.$$

The proof immediately follows from the fact that the inequality

$$\left|\frac{g(\tau_i)-g(\tau_{ij})}{\psi(\tau_i)-\psi(\tau_{ij})}-\frac{g(\tau_i)}{\psi(\tau_i)}\right| < \frac{1}{i},$$

is true for every i, where τ_{ij} is a term of the sequence $\{\tau_i\}$, suitably chosen for τ_i, and moreover satisfying $\tau_{ij}<\tau_i$.

then the integration of both expressions over the interval $\langle\tau,\tau_0\rangle$ gives

$$W_1(\tau)-W_1(\tau_0)\leqslant\varepsilon\big(W_2(\tau)-W_2(\tau_0)\big),$$

which according to (2) proves the truth of the lemma.

§ 2. Sufficient conditions for a function of type O to belong to one of the classes D_w, D_w^∞

THEOREM 1. *Let $w(t)$ satisfy* (*), (**). *If the coefficients a_n, b_n of a function $f(x)$ of type O satisfy the condition*

$$\sum_{n=1}^{\infty}a_nW(l/b_n)<\infty, \tag{8}$$

then $f(x)$ belongs to D_w.

Proof. The proof results from joining the inequalities

$$a_n\int_{l/b_n}^{1}\frac{|\varphi(b_n(x+t))-\varphi(b_nx)|}{w(t)}\,dt\leqslant Da_nW\Big(\frac{l}{b_n}\Big),$$

$$a_n\int_{0}^{l/b_n}\frac{|\varphi(b_n(x+t))-\varphi(b_nx)|}{b_nt}\cdot\frac{b_nt}{w(t)}\,dt\leqslant L_0Kla_nW\Big(\frac{l}{b_n}\Big),$$

the second inequality resulting directly from Lemma 1. As the constant L_0 we assume the lower bound of numbers L satisfying inequality (5) for $\tau_0=l/b_1<1$ [2]).

THEOREM 2. *Let $w(t)$ satisfy* (*), (**), ($\overset{*}{**}$). *If the coefficients a_n, b_n of a function $f(x)$ of type O satisfy simultaneously the conditions*

$$\varlimsup_{n\to\infty}a_nW(l/b_n)=\infty, \tag{9}$$

$$\varlimsup_{n\to\infty}\frac{1}{a_nW(l/b_n)}\sum_{i=1}^{n-1}a_iW(l/b_i)<\Theta\frac{2d}{(D+L_0Kl)sl}, \tag{10}$$

$$\varlimsup_{n\to\infty}\frac{1}{a_n}\sum_{i=n+1}^{\infty}a_i<(1-\Theta)\frac{2d}{Dsl}, \tag{11}$$

then $f(x)$ belongs to class D_w^∞.

The constants s, d, L_0 in (10), (11) are defined as follows: the constant s is defined by ($\overset{*}{**}$), the constant d by (14), and the constant L_0 is the lower bound of numbers L satisfying (5) for $\tau_0=l/b_1<1$ and $0<\Theta<1$.

Proof. We introduce the following notation:

$$I_n=\int_{l/b_n}^{1}\frac{|f(x+t)-f(x)|}{w(t)}\,dt,$$

$$A_n=\int_{l/b_n}^{1}\sum_{i=1}^{n-1}a_i\frac{|\varphi(b_i(x+t))-\varphi(b_ix)|}{w(t)}\,dt,$$

$$A=\int_{0}^{1}\sum_{i=1}^{n-1}a_i\frac{|\varphi(b_i(x+t))-\varphi(b_ix)|}{w(t)}\,dt,$$

$$B_n=a_n\int_{l/b_n}^{1}\frac{|\varphi(b_n(x+t))-\varphi(b_nx)|}{w(t)}\,dt,$$

$$C_n=\int_{l/b_n}^{1}\sum_{i=n+1}^{\infty}a_i\frac{|\varphi(b_i(x+t))-\varphi(b_ix)|}{w(t)}\,dt.$$

Let us consider the expressions of the right side of the inequality

$$I_n\geqslant B_n-A-C_n, \tag{12}$$

the truth of which can easily be verified.

a. Using the same estimation of the expression A as in proof of Theorem 1, we obtain

$$A\leqslant(D+L_0Kl)\sum_{i=1}^{n-1}a_iW(l/b_i). \tag{13}$$

b. Substituting $b_nt=u$, we obtain

$$B_n=\frac{a_n}{b_n}\int_{l}^{b_n}\frac{|\varphi(b_nx+u)-\varphi(b_nx)|}{w(u/b_n)}\,du,$$

$$B_n\geqslant\frac{a_n}{b_n}\sum_{k=1}^{m-1}\int_{kl}^{(k+1)l}\frac{|\varphi(b_nx+u)-\varphi(b_nx)|}{w(u/b_n)}\,du,$$

where $m=[b_n/l]$.

Considering

$$\int_{kl}^{(k+1)l}\frac{|\varphi(b_nx+u)-\varphi(b_nx)|}{w(u/b_n)}\,du\geqslant\frac{1}{w\big((k+1)l/b_n\big)}\int_{kl}^{(k+1)l}|\varphi(b_nx+u)-\varphi(b_nx)|\,du$$
$$\geqslant\frac{d}{w\big((k+1)l/b_n\big)},$$

where

$$d = \min_{x} \int_0^l |\varphi(x+u) - \varphi(x)|\, du, \tag{14}$$

and $d > 0$, we obtain

$$B_n \geqslant \frac{da_n}{b_n} \sum_{k=1}^{m-1} \frac{1}{w\big((k+1)l/b_n\big)} \geqslant \frac{da_n}{l} \int_{2l/b_n}^{(m+1)l/b_n} \frac{dt}{w(t)} > \eta \frac{da_n}{l} W\left(\frac{2l}{b_n}\right).$$

In the last inequality $\eta = 1 + 2\varepsilon/s$, where ε is a suitably chosen, sufficiently small positive number and s is defined by $(\overset{*}{**})$.

From Lemma 2 we obtain for sufficiently distant terms b_n

$$B_n > \frac{2d}{sl} a_n W\left(\frac{l}{b_n}\right). \tag{15}$$

c. The following inequality is also true:

$$C_n \leqslant DW(l/b_n) \sum_{i=n+1}^{\infty} a_i .$$

From (13), (15) and the last inequality, inequality (12) becomes

$$I_n \geqslant a_n W\left(\frac{l}{b_n}\right)\left(\frac{2d}{sl} - \frac{D + L_0 K l}{a_n W(l/b_n)} \sum_{i=1}^{n-1} a_i W\left(\frac{l}{b_i}\right) - \frac{D}{a_n} \sum_{i=n+1}^{\infty} a_i\right), \tag{16}$$

from which we directly obtain Theorem 2.

In proving inequality (13) and Theorem 1 we used both $(*)$ and $(**)$. It turns out that $(**)$ can be omitted by taking into account

$$A = \sum_{i=1}^{n-1} a_i \int_0^1 \frac{|\varphi(b_i x + b_i t) - \varphi(b_i x)|}{b_i t} \cdot \frac{b_i t}{w(t)}\, dt \leqslant G \sum_{i=1}^{n-1} a_i b_i , \tag{17}$$

where

$$G = K \int_0^1 \frac{t}{w(t)}\, dt , \tag{18}$$

and in view of $(*)$, we have $G < \infty$. In this case, however, (10) should be replaced by

$$\overline{\lim_{n\to\infty}} \frac{1}{a_n W(l/b_n)} \sum_{i=1}^{n-1} a_i b_i < \Theta \frac{2d}{Gsl}. \tag{19}$$

Thus, instead of Theorem 2, we should obtain the following

THEOREM 2*. *Let a function $w(t)$ satisfy $(*)$ and $(\overset{*}{**})$. If the coefficients a_n, b_n of the function $f(x)$ of type O satisfy simultaneously* (9), (19), (11), *then $f(x)$ belongs to class D_w^∞.*

The constants G, d, s in (19) and (11) are defined by (18), (14) and $(\overset{*}{**})$ respectively.

Remark. Analyzing the conditions imposed on $w(t)$ we come to the following conclusions:

a. If $w(t)$ did not satisfy $(*)$, then for example a function $f(x)$, periodic with period 2, defined for $-1 \leqslant x < 1$ by $f(x) = |x|$, would belong to D_w^∞.

b. If $w(t)$ did not satisfy condition (2), then every function $f(x)$ (*i. e.* a continuous and bounded function) would belong to D_w, and thus, no function $f(x)$ would belong to D_w^∞.

§ 3. Necessary and sufficient conditions for the existence of a function belonging simultaneously to classes D_{w_1} and $D_{w_2}^\infty$

THEOREM 3. *If $f(x)$ belongs to D_{w_1}, and the condition*

$$\lim_{t\to+0} \frac{w_2(t)}{w_1(t)} > 0 \tag{20}$$

is satisfied, then $f(x)$ belongs simultaneously to D_{w_2}.

To prove this let us observe that if (20) were true, we should have

$$\frac{w_1(t)}{w_2(t)} \leqslant M ,$$

where $M > 0$ and $0 < t \leqslant t_0$. This would give

$$\int_0^{t_0} \frac{|f(x+t) - f(x)|}{w_2(t)}\, dt \leqslant M \int_0^{t_0} \frac{|f(x+t) - f(x)|}{w_1(t)}\, dt ,$$

which proves the proposition. Theorem 3 gives directly

THEOREM 3*. *The necessary condition for the existence of $f(x)$ belonging simultaneously to D_{w_1} and $D_{w_2}^\infty$ is*

$$\lim_{t\to+0} \frac{w_2(t)}{w_1(t)} = 0 . \tag{21}$$

THEOREM 4. *If*

$$\lim_{t\to+0}\frac{w_2(t)}{w_1(t)}=0\,, \tag{22}$$

where $w_1(t)$ satisfies $(*)$, $(**)$ and $w_2(t)$ satisfies $(*)$, $(\overset{*}{**})$, then, given a function $\varphi(x)$, we can choose such coefficients a_n and b_n of the function $f(x)$ of type O that $f(x)$ will belong simultaneously to D_{w_1} and $D^{\infty}_{w_2}$.

Proof. By Theorems 1 and 2* it is sufficient to show how to choose the coefficients a_n and b_n in order to satisfy simultaneously condition (8) for $w(t)=w_1(t)$ and conditions (9), (19), (11) for $w(t)=w_2(t)$. As will be seen below, we can also choose an integer b_n so as to make $f(x)$ periodic.

Instead of the above-mentioned conditions, we shall consider the following:

$$\sum_{n=1}^{\infty}a_nW_1(l/b_n)<\infty\,, \tag{23}$$

$$\lim_{n\to\infty}a_nW_2(l/b_n)=\infty\,, \tag{24}$$

$$\lim_{n\to\infty}\frac{1}{a_nW_2(l/b_n)}\sum_{i=1}^{n-1}a_ib_i=0\,, \tag{25}$$

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{i=n+1}^{\infty}a_i=0\,. \tag{26}$$

We shall write

$$k_n=a_nW_2(l/b_n)$$

and assume, for example, $k_1=1$ and $b_1=2$. Defining by induction we shall suppose that k_i, b_i are already defined for $i<n$. We define k_n so as to satisfy simultaneously the inequalities

$$\frac{1}{k_n}\sum_{i=1}^{n-1}a_ib_i<\frac{1}{2^n}\quad\text{ and }\quad k_n\geqslant n\,,$$

and hence simultaneously (24), (25).

Now we choose b_n large enough to satisfy

$$\frac{k_n}{W_2(l/b_n)}<\frac{1}{2^{n-1}}\cdot\frac{k_{n-1}}{W_2(l/b_{n-1})}\,, \tag{27}$$

which gives

$$\sum_{i=n+1}^{\infty}\frac{k_i}{W_2(l/b_i)}<\frac{k_n}{W_2(l/b_n)}\left(\frac{1}{2^n}+\frac{1}{2^{n+(n+1)}}+\dots\right)<\frac{k_n}{W_2(l/b_n)}\cdot\frac{1}{2^{n-1}}\,,$$

and hence condition (26).

Thus, without the aid of (22), we have constructed a function $f(x)$ of type O whose coefficients satisfy simultaneously (24), (25) and (26). Thus, in virtue of Theorem 2*, $f(x)$ belongs to $D^{\infty}_{w_2}$.

Suppose now that (22) is satisfied. From Lemma 3 it follows that for a fixed k_n we can choose b_n sufficiently large to satisfy simultaneously (27) and the inequality

$$\frac{k_n}{W_2(l/b_n)}W_1(l/b_n)<\frac{1}{k_n^2}\,.$$

This means, however, that (23) is satisfied, and thus, according to Theorem 1, $f(x)$ belongs simultaneously to D_{w_1} and $D^{\infty}_{w_2}$.

In this way Theorem 4 is proved.

Joining Theorems 3* and 4 we obtain both the necessary condition (21) and the sufficient condition (22) for the existence of a function of type O (periodic in the particular case) belonging simultaneously to D_{w_1} and $D^{\infty}_{w_2}$.

Remark. From the purport of the proof and from the Remark in § 2 it follows that if $w(t)$ satisfies $(\overset{*}{**})$ there exists a function of type O in each class D^{∞}_{w} [5]).

§ 4. Logarithmic-power scale

Suppose that the limit g^* of the quotient $w_2(t)/w_1(t)$ exists for $t\to+0$. Now, if $w_1(t)\leqslant w_2(t)$ for $0<t<a$ with a certain a, then $D_{w_1}\subset D_{w_2}$. From Theorems 3 and 4 it follows that for $g^*=0$ class D_{w_1} is a proper part of class D_{w_2}, and for $g^*>0$ both classes coincide. It follows hence that, under the assumption of the existence of limit g^*, if both $w_1(t)$ and $w_2(t)$ satisfy (2), $(*)$, $(**)$ and $(\overset{*}{**})$, the condition (22) can be used to establish the scale of classification of functions $f(x)$.

Let

$$w(t)=t^{1+\delta}|\log t|^{\gamma}\quad(\delta>0;\ \text{and for }\delta=0,\ \gamma\leqslant1)$$

for $0<t\leqslant a$. In this interval, with a suitable a, the function $w(t)$ is increasing. For $t>a$ let $w(t)$ be so defined as to be monotone, non-decreasing and remain continuous. This leads, on the basis of (22), to the definition of a logarithmic-power scale and, for $\gamma=0$, to the definition of a power scale.

For this kind of classes D_w and D^{∞}_w we have used the notation $D(\delta,\gamma)$ and $D^{\infty}(\delta,\gamma)$ respectively. $D(\delta_2,\gamma_2)$ is a proper part of $D(\delta_1,\gamma_1)$, if $\delta_1<\delta_2$,

[5]) The proof of the existence of functions of class D^{∞}_{w} is dealt with by S. Kaczmarz in the paper [3], where the particular case $w(t)=t$ is considered.

and for $\delta_1=\delta_2$ if $\gamma_1>\gamma_2$. An examination of the validity of (2) and (*) for the function $w(t)$ defined above leads to the definition of the limits of the values of the parameters δ,γ for which those suppositions are still satisfied. The parameters δ,γ can take the following values:

$$0\leqslant\delta\leqslant 1 \quad \text{and} \quad \gamma\begin{cases}>1 & \text{if } \delta=1,\\ \text{arbitrary} & \text{if } 0<\delta<1,\\ \leqslant 1 & \text{if } \delta=0.\end{cases}$$

For these values of δ,γ, the conditions (**) and (⁎⁎⁎) are also satisfied.

Using (8), (9), (10) and (11) and taking into account (16), we obtain in the case of a logarithmic-power scale, after suitable computations, on the basis of Theorems 1 and 2 the following theorems:

THEOREM 5. *A function $f(x)$ of type O belongs to $D(\delta,\gamma)$ for $0<\delta\leqslant 1$ if its coefficients a_n,b_n satisfy the inequality*

$$\sum_{n=1}^{\infty} a_n b_n^{\delta}(\log b_n)^{-\gamma}<\infty. \tag{28}$$

THEOREM 6. *The function $f(x)$ of type O belongs to $D^{\infty}(\delta,\gamma)$ for $0<\delta\leqslant 1$ if its coefficients a_n,b_n satisfy simultaneously the conditions*

$$\lim_{n\to\infty}\frac{a_{n-1}b_{n-1}^{\delta}(\log b_{n-1})^{-\gamma}}{a_n b_n^{\gamma}(\log b_n)^{-\gamma}}=0, \tag{29}$$

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{i=n+1}^{\infty}a_i<\frac{2d}{Dsl}, \tag{30}$$

where the constants d and s are defined by (14) *and* (⁎⁎⁎), *d depends on the choice of $\varphi(x)$, $s=2^{\delta}$.*

THEOREM 7. *The function $f(x)$ of type O belongs to $D(0,\gamma)$ if its coefficients a_n,b_n satisfy*

(a) *for $\gamma<1$ the inequality*

$$\sum_{n=1}^{\infty}a_n(\log b_n)^{1-\gamma}<\infty, \tag{31}$$

(b) *for $\gamma=1$ the inequality*

$$\sum_{n=1}^{\infty}a_n\log(\log b_n)<\infty. \tag{32}$$

THEOREM 8. *The function $f(x)$ of type O belongs to $D^{\infty}(0,\gamma)$ if its coefficients a_n,b_n satisfy*

(a) *for $\gamma<1$ the condition*

$$\lim_{n\to\infty}\frac{a_{n-1}(\log b_{n-1})^{1-\gamma}}{a_n(\log b_n)^{1-\gamma}}=0 \tag{33}$$

and condition (30), *where $s=1$,*

(b) *for $\gamma=1$ the condition*

$$\lim_{n\to\infty}\frac{a_{n-1}\log(\log b_{n-1})}{a_n\log(\log b_n)}=0 \tag{34}$$

and condition (30), *where $s=1$.*

THEOREM 9. *If function $f(x)$ is of type W, then it belongs to class $D(\delta,\gamma)$, where $\gamma>1$ and δ is defined by $ab^{\delta}=1$, where $0\leqslant\delta\leqslant 1$.*

Thus Theorem 9 results directly from Theorem 5.

THEOREM 10. *For any arbitrarily chosen function $\varphi(x)$ and numbers δ_1,δ_2 satisfying the inequality $0<\delta_1<\delta_2\leqslant 1$ we can always find such coefficients a,b of a function $f(x)$ of type W that the function $f(x)$ will belong simultaneously to the classes $D(\delta_1,0)$ and $D^{\infty}(\delta_2,0)$.*

Proof. Taking $w_1(t)=t^{1+\delta_1}$ and $w_2(t)=t^{1+\delta_2}$ let us apply Theorem 1 to the first of these functions and Theorem 2 to the second. It will be observed that

$$a_nW\left(\frac{l}{b_n}\right)=\frac{1}{\delta l^{\delta}}(ab^{\delta})^n-\frac{1}{\delta}a^n,$$

where for δ we take δ_1 in the first case, and δ_2 in the second.

The right sides of inequalities (10), (11), defined by a given function $\varphi(x)$ and arbitrarily chosen values for $\Theta(0<\Theta<1)$, will be denoted respectively by A and B $(A>0, B>0)$. Under the assumption that $ab^{\delta_2}>1$ the inequalities (10) and (11) become

$$\frac{1}{ab^{\delta_2}-1}<A, \qquad \frac{a}{1-a}<B.$$

We shall write $(1+A)/A=M>1$ and $B/(1+B)=N<1$. If the coefficients a,b are so chosen that they simultaneously satisfy the inequalities

$$b^{\delta_1}\geqslant 1/N, \qquad b^{\delta_2-\delta_1}>M, \qquad \text{and} \qquad M/b^{\delta_2}<a<1/b^{\delta_1},$$

then conditions (9), (10), (11) and also the inequality $ab^{\delta_1}<1<ab^{\delta_2}$ will simultaneously be satisfied. This means that function $f(x)$ belongs simultaneously to classes $D(\delta_1,0)$ and $D^{\infty}(\delta_2,0)$.

Taking an integer b, we obtain a periodic $f(x)$ satisfying the required condition.

The relation of H_ω to D_w in the case of a logarithmic-power scale is explained by the following Theorem:

THEOREM 11. *Class* $H(\delta_1,\gamma_1)$ *is contained in class* $D(\delta_2,\gamma_2)$, *if* $\delta_2<\delta_1$, *and for* $\delta_2=\delta_1$ *if* $\gamma_2>\gamma_1+1$.

Proof. Suppose that $\omega(t)$ is continuous and the inequality

$$\int_0^a \frac{\omega(t)}{w(t)}\,dt \leqslant C \tag{35}$$

is satisfied for certain positive constants C and a. It is obvious that in this case H_ω is contained in D_w.

If

$$\omega(t)=t^{\delta_1}|\log t|^{\gamma_1}, \qquad w(t)=t^{1+\delta_2}|\log t|^{\gamma_2},$$

the inequality (35) is satisfied for $\delta_1=\delta_2$, and thus the Theorem is valid for this case. The same is true for $\delta_2<\delta_1$ since

$$H(\delta_1,\gamma_1)\subset D(\delta_1,\gamma_1+2)\subset D(\delta_2,\gamma_2).$$

As follows from Theorem 11 and from the examples in § 5, the logarithmic-power scales according to Hölder's conditions, and according to Dini's conditions, are complementary. The widest class for both scales is $D(0,1)$, every other class of both logarithmic-power scales being its proper part, the narrowest class for both scales is the Lipschitz class $H(1,0)$, which is a proper part of every other class of both logarithmic-power scales (see § 5, Example 5).

The cases in which, at every point, at least one of the derived numbers is infinite or the expression

$$\int_0^1 \frac{|f(x+t)-f(x)|}{t}\,dt \tag{36}$$

is infinite for every x, appear here as particular cases of the function $f(x)$ belonging to class $H^\infty(\delta,\gamma)$ for $\delta=1$, $\gamma=0$ or to class $D^\infty(\delta,\gamma)$ for $\delta=0$, $\gamma=0$.

The relation of the classes is as follows:

$$H(1,0)\subset H(1,\gamma_1)\subset D(1,\gamma_2)\subset D(\delta,\gamma_3)$$

where $\gamma_1>0$, $\gamma_2>\gamma_1+1$, $1>\delta\geqslant 0$, γ_3 arbitrary.

$$H(\delta_1,\gamma_1)\subset H(\delta_2,\gamma_2)\subset H(\delta_2,\gamma_3)\subset D(\delta_2,\gamma_4)\subset D(\delta_2,\gamma_5)\subset D(\delta_3,\gamma_6),$$

where $1>\delta_1>\delta_2>\delta_3\geqslant 0$, $\gamma_3>\gamma_2, \gamma_4>\gamma_3+1, \gamma_5>\gamma_4$ and γ_1,γ_6 are arbitrary real numbers.

$$H(0,\gamma_1)\subset D(0,\gamma_2)\subset D(0,0)\subset D(0,\gamma_3),$$

where $\gamma_1<-1$, $\gamma_1+1<\gamma_2<0$, $0<\gamma_3\leqslant 1$.

$$H(0,\gamma_1)\subset D(0,\gamma_2)\subset D(0,1),$$

where $\gamma_1<0$, $\gamma_1+1<\gamma_2<1$.

Each of the classes written down in the above way is a proper part of that class in which it is contained. For Hölder's classes this results from the contents of another paper of mine (Theorem 7 in [6]) and for Dini's classes the above results from Theorem 4. For Hölder's classes contained in the respective Dini's classes examples of functions belonging to Dini's classes and not belonging to the respective Hölder's classes are given in § 5.

§ 5. Examples

The problem of the existence of a continuous function for which the integral (36) takes only infinite values for every x, has been the subject of investigations for a long time (see Steinhaus [5]). Examples of continuous functions satisfying that condition are known [6]). These investigations suggest the problem of carrying out an analysis not only with regard to the singularities which a given function shows but also with regard to its positive properties of the same type as the examined singularity. According to the notation used in this paper the problem is reduced to the question to which classes $D(\delta_1,\gamma_1)$, $D^\infty(\delta_2,\gamma_2)$, for example, the function in question can simultaneously belong. Theorems 3 and 4 make both such an analysis and the construction of suitable examples actually possible.

The examples given below are considered from this point of view. In this manner we obtain a more complete picture, since it turns out to be possible to establish both the narrowest class to which the examined function in this scale belongs and the degree of the examined singularity. The examples concern exclusively the logarithmic-power scale.

The relation of classes $H(\delta,\gamma)$ and $D(\delta,\gamma)$ has also been examined. This makes the universal example found for Hölder's classes (Tarnawski [6]) possible in the whole domain of the scale. It can also be used in the case of a classification according to Dini's conditions. In other examples

[6]) An example of such a function is given by S. Kaczmarz im his paper [3], p. 196-198. Another example can be found in [7], p. 78.

several known functions have been analyzed regardless of the question whether other authors have given then as examples with regard to Dini's or Hölder's conditions. A closer analysis of these examples (G. Faber, A. Zygmund, W. Orlicz) has permitted their generalization and simultaneously the fixing of their place in both classifications in question. An example of a function of type O which, belonging to $D^{\infty}(0,1)$, for this reason does not belong to the widest class of both classifications, $D(0,1)$, has also been constructed.

In the examples considered functions of type W have not been taken into account. The proof of Theorem 10, however, shows the method of such constructions.

The examples particularly concern periodic functions with properties required in the given example, because in the construction of each of the examples below, also integer values, can be taken for the coefficients b_n.

Example 1. We give an example of a function of type O belonging simultaneously to classes $D(0,1)$ and $D^{\infty}(0,\gamma)$, where $\gamma<1$. This function $f(x)$ is characterized by the coefficients

$$a_n=A^{-an}, \qquad b_n=B^{(n!)^{\beta}},$$

where $A>1$, $B>1$, $\beta>0$, and a satisfies the inequality

$$A^{a}>1+\frac{Dl}{2d}, \tag{37}$$

d being defined by (14).

Using Theorems 7 and 8 we shall find out whether (32), (33) and (30) are satisfied. It is easy to see that conditions (32) and (33) are satisfied. Condition (30) is satisfied in view of (37) since

$$\frac{1}{a_n}\sum_{i=n+1}^{\infty}a_i=\frac{1}{A^{a}-1}<\frac{2d}{Dl}.$$

Thus $f(x)$ has the required properties.

We shall take in particular $B=2$, $\beta=1$, which gives

$$f(x)=\sum_{n=1}^{\infty}A^{-an}\varphi(2^{n!}x).$$

For $\varphi(x)=\cos x$ we obtain $D=2$, $l=2\pi$, $d=4$ and $A^{a}>1+\pi/2$.

For $\varphi(x)=\min_p|x-p|$ (p—integer) we obtain $D=1/2$, $l=1$, $d=1/8$ and $A^{a}>3$.

Taking in the last case $A=10$, $a=1$, we obtain the example of a function given by G. Faber ([1] and [2]). Thus for this function con-

dition (37) is satisfied and it therefore belongs simultaneously to classes $D(0,1)$ and $D^{\infty}(0,\gamma)$ for every $\gamma<1$.

In another paper ([6], § 6, Example 3) I have shown that Faber's function belongs to class $H^{\infty}(0,\gamma)$ for every $\gamma<0$, and thus to none of the classes $H(0,\gamma)$. Considering that $H(0,\gamma)\subset D(0,1)$ for every $\gamma<0$, classes $H(0,\gamma)$ are consequently proper parts of class $D(0,1)$. Simultaneously we have found, in the logarithmic-power scale with regard to both Dini's and Hölder's conditions, the narrowest class (class $D(0,1)$) to which Faber's function belongs.

Example 2. We shall examine a function $f(x)$ of type O with the coefficients

$$a_n=\frac{1}{(p_1p_2\dots p_n)^{a}}, \qquad b_n=B^{(p_1p_2\dots p_{n+k})^{\beta}},$$

where $0<p_n\to\infty$, $p_{n+1}\leqslant 2p_n$, $\beta\geqslant a>0$, $B>1$, $k\geqslant 0$.

We shall show that this function belongs simultaneously to classes $D(0,\gamma)$ and $D^{\infty}(0,\gamma_1)$, where

$$0\leqslant(\beta-a)/\beta<\gamma\leqslant 1, \qquad \gamma_1<(\beta-a)/\beta$$

and simultaneously to classes $H(0,\gamma)$ and $H^{\infty}(0,\gamma_1)$, where

$$-a/\beta<\gamma<0, \qquad \gamma_1<-a/\beta.$$

For this purpose we shall use Theorems 7 and 8 and find out whether the coefficients a_n, b_n satisfy (31), (30) and (33), replacing in the last condition γ by γ_1.

Condition (30) is obviously satisfied. Conditions (31) and (33) are satisfied for γ and γ_1 respectively since

$$\frac{a_n(\log b_n)^{1-\gamma}}{a_{n-1}(\log b_{n-1})^{1-\gamma}}=\frac{p_{n+k}^{\beta(1-\gamma)}}{p_n^{a}}\leqslant\frac{2^{k\beta(1-\gamma)}}{p^{a-\beta(1-\gamma)}}.$$

The function $f(x)$ also has the required properties with regard to Hölder's classes in view of

$$\frac{a_{n-1}b_{n-1}}{a_nb_n}=\frac{p_n^{a}}{B^{(p_1p_2\dots p_{n-1+k})^{\beta}(p_{n+k}^{\beta}-1)}}\to 0\ ^{7)}.$$

We shall now take $p_n=n$, $B=2$, $k=0$, $\beta=2$, $a=1$. We obtain an example of the function

$$\sum_{n=1}^{\infty}\frac{1}{n!}\varphi(2^{(n!)^2}x),$$

[7] This results from Theorems 8 and 9 (case a) of the paper [6].

given by A. Zygmund [7] as a function for which integral (36) is infinite for every x, *i. e.*, according to our notation, as a function belonging to $D^{\infty}(0,0)$. It turns out that this function belongs simultaneously to classes $D(0,\gamma)$, where $\gamma>1/2$, and $D^{\infty}(0,\gamma_1)$, where $\gamma_1<1/2$ (and thus, in particular, to class $D^{\infty}(0,0)$).

We shall now take $p_n=n, B=2$, $k=1$, $\alpha=\beta=1$. We obtain an example of the function

$$\sum_{n=1}^{\infty}\frac{1}{n!}\varphi(2^{(n+1)!}x),$$

given by W. Orlicz ([4], p. 37) as a function belonging to $H^{\infty}(\delta,0)$ for every $\delta>0$. It turns out that this function belongs simultaneously to classes $D(0,\gamma)$, where $\gamma>0$, and $D^{\infty}(0,\gamma_1)$, where $\gamma_1<0$, and, in the case of Hölder's classes, simultaneously to classes $H(0,\gamma)$ where $\gamma>-1$, and $H^{\infty}(0,\gamma_1)$ where $\gamma_1<-1$ (and thus, in particular, to classes $H^{\infty}(\delta,0)$ for $\delta>0$).

Example 3. An example of a function of type O belonging to classes $D^{\infty}(0,1)$, and thus to none of the classes $D(\delta,\gamma)$ (*i. e.*, to none of the classes $H(\delta,\gamma)$) of the logarithmic-power scale, is the function $f(x)$ with the coefficients

$$a_n=\frac{1}{(p_1p_2\dots p_n)^{\alpha}},\qquad \log b_n=B^{(p_1p_2\dots p_n)^{\beta}},$$

where $B>1$, $0<p_n\to\infty$, $\beta>\alpha>0$.

Combining conditions (33) and (34) of Theorem 8, we immediately see that $f(x)$ belongs to $D^{\infty}(0,1)$ for the same reason that (considering $\beta>\alpha$) the function in Example 2 belongs to $D^{\infty}(0,0)$.

Example 4. An example of a function of type O belonging simultaneously to classes $D(0,0)$ and $D^{\infty}(0,\gamma)$ for every $\gamma<0$ is the function

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{(n!)^{\alpha}}\varphi(B^{((n-1)!)^{\alpha}}x),$$

where $\alpha>1$, $B>1$.

The coefficients of $f(x)$ obviously satisfy condition (31), in which we must take $\gamma=0$, and also satisfy condition (33) for every $\gamma<0$ and condition (30).

The function $f(x)$ belongs also to each of the classes $H^{\infty}(0,\gamma)$ where $\gamma<-1$ since

$$\frac{a_nb_n}{a_{n+1}b_{n+1}}=(n+1)^{\alpha}B^{((n-1)!)^{\alpha}(1-n^{\alpha})}\to 0\ ^{8}).$$

[8]) This results from the comparison of conditions (33) and (30) (Theorem 8) of this paper with conditions (34), (37) and (38) of Theorem 9 (case a) of the paper [6].

Thus it turns out that each of the classes $H(0,\gamma)$, where $\gamma<-1$, is a proper part of class $D(0,0)$.

Example 5. In a former paper of mine (Examples 5 and 6 in [6]) I give the universal examples of functions $f(x)$ of type O, which, according to the values of parameters δ,γ, belong to the respective classes of the logarithmic-power scale according to Hölder's condition. The coefficients of the function $f(x)$ were defined as follows:

$$a_n=A^{-\delta\cdot 2^{\lambda(n)}}\cdot 2^{\gamma\lambda(n)}\cdot\frac{1}{n^2},\qquad b_n=A^{2^{\lambda(n)}}\qquad (A>1).$$

The function $f(x)$, in the case of $\lambda(n)=n$, belongs to each of the classes $H(\delta,\gamma)$, $H^{\infty}(\delta,\gamma_1)$, where $\gamma_1<\gamma$, and $\delta\neq 0$ and $\delta\neq 1$.

For $\lambda(n)=n^2$ the function has a universal character and thus concerns the cases formerly excluded ($\delta=0$ and $\delta=1$).

Taking, in the definition of a_n, the expression $1/(\lambda(n))^k$ $(k>0)$ instead of $1/n^2$ and everywhere $\lambda(n)=n!$, we obtain an example of a universal function concerning all classes, including the case of $\delta=0$, $\gamma=0$, *i. e.*, of a function $f(x)$ of type O belonging to $H^{\infty}(0,\gamma_1)$ for every $\gamma_1<0$.

In the case of the classification according to Dini's conditions, if the example is to cover the whole domain of the scale, we take $k>1$ and in particular $k=2$. Thus

$$a_n=A^{-\delta\cdot 2^{\lambda(n)}}\cdot 2^{\gamma\lambda(n)}\frac{1}{(\lambda(n))^2},\qquad b_n=A^{2^{\lambda(n)}}\qquad (A>1).\tag{38}$$

The function $f(x)$ of type O thus defined will belong:

(a) for $\lambda(n)=n$ simultaneously to each of the classes $D(\delta,\gamma)$, $D^{\infty}(\delta,\gamma_1)$ and also to each of the classes $H(\delta,\gamma)$, $H^{\infty}(\delta,\gamma_1)$, where $\gamma_1<\gamma$ and $0<\delta<1$;

(b) for $\lambda(n)=n!$ and $\delta>0$ simultaneously to each of the classes $D(\delta,\gamma)$, $D^{\infty}(\delta,\gamma_1)$, where $\gamma_1<\gamma$;

and for $\lambda(n)=n!$ and $\delta=0$ simultaneously to each of the classes $D(0,\gamma+1)$, $D^{\infty}(0,\gamma_1+1)$, where $\gamma_1<\gamma\leqslant 0$.

In case (b) $f(x)$ will also belong to each of the classes $H(\delta,\gamma)$, $H^{\infty}(\delta,\gamma_1)$, where $\gamma_1<\gamma$.

Thus case (b) concerns all values of δ,γ in the logarithmic-power scale for both classifications in question.

From Theorem 11 it follows that $H(\delta,\gamma-\varepsilon)\subset D(\delta,\gamma+1)$, where ε is an arbitrary positive number. However small ε is chosen the first of these classes is a proper part of the second, since $f(x)$, whose coefficients have been defined by (38), belongs to $D(\delta,\gamma+1)$ and simultaneously to $H^{\infty}(\delta,\gamma-\varepsilon/2)$. Thus this function does not belong to class $H(\delta,\gamma-\varepsilon/2)$, and therefore neither to class $H(\delta,\gamma-\varepsilon)$.

Taking $\delta=1$, $\gamma=\varepsilon$, we obtain class $\mathrm{H}(1,0)$ as the narrowest, and taking $\delta=0$ and $\gamma=0$ we obtain class $\mathrm{D}(0,1)$ as the widest in the whole of the logarithmic-power scale of both classifications in question.

In order to show that $f(x)$, whose coefficients are defined by (38), has the required properties, it is sufficient to compare the conditions of Theorems 5, 6, 7 and 8 of this paper with the conditions of the Theorems 8 and 9 of the paper [6] and to consider Examples 5 an 6 therein. The conditions concerned coincide, with the exception of conditions (29), (33), and (32). Examining them we find that condition (29) is satisfied for every $\gamma_1<\gamma$ since

$$\frac{a_{n-1}b_{n-1}^{\delta}(\log b_{n-1})^{-\gamma_1}}{a_n b_n^{\delta}(\log b_n)^{-\gamma_1}}=2^{-(\gamma-\gamma_1)(\lambda(n)-\lambda(n-1))}\left(\frac{\lambda(n)}{\lambda(n-1)}\right)^2\to 0\,.$$

We can identically examine condition (33) for $\delta=0$. Condition (32) is satisfied, since taking in (38) $\delta=0$, $\gamma=0$, we obtain

$$\sum_{n=1}^{\infty}a_n\log(\log b_n)=\sum_{n=1}^{\infty}\frac{1}{\lambda^2(n)}[\lambda(n)\log 2+\log(\log A)]<\infty\,.$$

Thus $f(x)$ possesses the required properties.

This example does not concern the case of a function of type O belonging to class $\mathrm{D}^{\infty}(0,1)$ and thus not belonging to the widest class of the considered logarithmic-power scale, namely to class $\mathrm{D}(0,1)$. In order to obtain such an example, it is sufficient to take $\log b_n=A^{2^{\lambda(n)}}$, and in (38) $\delta=\gamma=0$. Both (30) and (34) are then satisfied.

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Reçu par la Rédaction le 19.1.1954

A constructivist theory of plane curves

by

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Introduction. This paper develops a theory of p-curves, which are finite matrices with binary fraction elements. Roughly speaking, a p-curve is a finite assemblage of points in serial order on a grid, the jump from one point to the next being of fixed amount in one or other of two "directions". The concept of a plane curve is then introduced in terms of sequences of p-curves. The emphasis throughout the paper is on the strictly finitist character of the proof processes.

The present work on *analysis situs* is a preliminary to a study of curvilinear integrals.

Definitions. We denote integers by $i, j, k, l, m, n, \mu, \nu, p, q, r, s, t, \varrho, \sigma, \tau$ with or without suffixes, and binary fractions $m/2^p$ by $a, b, c, d, x, y, \xi, \eta$ with or without suffixes or affixes; more specifically, for a given p we write x^p, etc., for $m/2^p$. The ordered pair (x,y) is called a *point*, and the ordered pair $\langle x_1,y_2\rangle$ an *interval*; the ordered pair of intervals $\langle x_1,x_2\rangle\langle y_1,y_2\rangle$ (where $x_1<x_2$, $y_1<y_2$) is called a *rectangle* with *vertices* (x_r,y_s), $r=1,2$ and $s=1,2$. If

$$2^p a_r^p\,,\quad 0\leqslant r\leqslant\mu_p,\quad 2^p b_s^p,\quad 0\leqslant s\leqslant\nu_p$$

are the integers from $2^p x_1^p$ to $2^p x_2^p$ and from $2^p y_1^p$ to $2^p y_2^p$, respectively, then the points

$$(a_r^p,b_s^p),\quad 0\leqslant r\leqslant\mu_p,\quad 0\leqslant s\leqslant\nu_p,$$

are called the *lattice points of the network*

$$F_p\begin{pmatrix}x_1^p & x_2^p\\ y_1^p & y_2^p\end{pmatrix}$$

in the rectangle $\langle x_1^p,x_2^p\rangle\langle y_1^p,y_2^p\rangle$; the rectangles $\langle a_r^p,a_{r+1}^p\rangle\langle b_s^p,b_{s+1}^p\rangle$, $0\leqslant r\leqslant\mu_p$, $0\leqslant s\leqslant\nu_p$ are called the *p-cells* of the rectangle $\langle x_1^p,x_2^p\rangle\langle y_1^p,y_2^p\rangle$ or of the network

$$F_p\begin{pmatrix}x_1^p & x_2^p\\ y_1^p & y_2^p\end{pmatrix}.$$