Decompositions of a sphere

by

T. J. Dekker and J. de Groot (Amsterdam)

1. Introduction. A classical result of F. Hausdorff ([5], p. 469-472) states that – disregarding a denumerable set – "a half" and "a third" of a sphere S – defined by $x^2+y^2+z^2=1$ – may be congruent to each other (using only rotations). This result is exploited to a great extent in a number of problems by S. Banach and A. Tarski, J. von Neumann, W. Sierpiński, R. M. Robinson and others; for references cf. [9] and [11]. The results of this paper generalize those of Robinson [9] in several directions. One notable result of Robinson's is this: S can be divided into two pieces, each of which can be divided into two pieces congruent to itself (by rotation). From this he derives the following "paradoxical" result: it is possible to cut the solid unit sphere ($x^2+y^2+z^2\leqslant 1$) into five (but not less than five) mutually disjoint pieces (one of which is a point) and to reassemble these pieces (using only rotations and translations) so that two solid unit spheres are formed [1]). From our results (cf. decomposition theorem below) and the material contained in [9], p. 256, 257, it follows in particular that these pieces may have the additional properties of *connectedness* and *local connectedness*. So the "pieces" are really pieces and not necessarily some kind of "scattered" sets.

The main result of Robinson ([9], p. 252) runs as follows: it is possible to decompose S into n mutually disjoint, non-empty pieces $A_1, A_2, \ldots, A_n$ satisfying a given (finite) system of congruences, each having the form

$$(1)\qquad \bigcup_{i=1}^{r} A_{k_i} \cong \bigcup_{j=1}^{s} A_{l_j} \qquad (0<r<n,\ 0<s<n),$$

if and only if none of the given congruences and no congruence obtainable from them by taking complements (in S) or by using transitivity (to derive new congruences from the given system) asserts the congruence of two complementary portions of S. Moreover, starting from an arbi-

[1]) Banach and Tarski proved that a finite number of pieces suffices; von Neumann cut the number of pieces down to 9 (without proof), Sierpiński to 8.

trary free rotation group (with a finite number of free generators), Robinson proves that the decomposition is possible in such a way that each congruence (1) is effectuated by a free generator of the group.

We extend this theorem in several directions (a summary of results has been published in [2]). First we dispose of the assumption of finiteness; indeed one can use any cardinal that makes sense in our problem, that is any cardinal less than or equal to $\mathfrak{c}$, the cardinal of the set of real numbers. Secondly, we reduce the necessary and sufficient conditions mentioned above to the empty set, *i. e.* any system of congruences (1) can be satisfied. However, sense-reversing mappings cannot be excluded in this case. We learned that the same idea occurred to J. F. Adams [1] (he deals with the finite case). Thirdly it is possible to require connectedness and local connectedness of the sets A_i if the number of sets A_i and relations (1) are both less than $\mathfrak{c}$.

So we can state the following theorem [2]):

DECOMPOSITION THEOREM. *The sphere S may be decomposed into a mutually disjoint, non-empty pieces — a being any cardinal less than or equal to $\mathfrak{c}$ — satisfying any given number $\beta \leqslant \mathfrak{c}$ of congruences between non-empty and non-complete* [3]) *(but otherwise arbitrary, finite or infinite) sums of the pieces mentioned. Moreover, if $a, \beta < \mathfrak{c}$, all pieces can be chosen in such a way that they are connected and locally connected* [4]).

The proof, using the axiom of choice, follows in sections 2 and 3. Section 4 contains some additional results. We conclude this section with a few examples. Other examples (for the finite case) can be found in Robinson [9], and in Adams [1].

Examples. 1. S is the sum of mutually disjoint sets A_i $(i=1,2,\ldots)$ such that all possible sums of sets A_i (the empty sum and the sum equal to S excluded) are congruent to each other. In particular, the A_i themselves are congruent to each other.

2. S is the sum of an increasing well-ordered system (with potency $\mathfrak{c}$) of mutually congruent subsets of S.

3. It is possible to divide S into $a \leqslant \mathfrak{c}$ mutually disjoint pairs of sets B_m, C_m, and to reassemble each pair by congruent mappings such that a copies of S are formed:

$$B_m \cup C_m \underset{2}{=} S.$$

[2]) Mycielski informed us that he also discovered independently this theorem (the last part of it excepted). See [7] and [8], where this theorem is stated (without proof). See also [6] for other results.

[3]) *I. e.* sums unequal to S.

[4]) More precisely: the pieces can be made *totally imperfect* and are therefore, by a well known theorem of Sierpiński, connected and locally connected.

4. The solid sphere is for any infinite cardinal $a \leqslant \mathfrak{c}$ the sum of a mutually disjoint sets, each of which is equivalent by finite decomposition to S (cf. [11], p. 94, and [10]).

Proof. Apply the decomposition theorem in a way similar to the Robinson decomposition ([9], p. 256, 257).

5. There is a system of different subsets $\{M_a\}$ of the sphere such that the set of indices $\{a\}$ is of the order type of the continuum and

$$M_a \cong M_{a'}, \quad M_a \subset M_{a'}, \quad \text{if} \quad a < a'.$$

Some unsolved problems. The number β of congruences may be larger than $\mathfrak{c}$. To what extent does the decomposition theorem remain valid? May the inequality $a, \beta < \mathfrak{c}$ in the second part of the theorem be replaced by $a, \beta \leqslant \mathfrak{c}$? [5]).

2. Preliminaries.

By a *rotation* we shall always mean a rotation of the three-dimensional space which leaves the origin O fixed. If φ is a rotation, then the transform of a point u by φ will be denoted by $u\varphi$, and the transform of a point-set A by φ will be denoted by $A\varphi$. We denote by ω the inversion in O, *i. e.* the transformation which transforms any point (x,y,z) into $(-x,-y,-z)$.

LEMMA I [6]). *The rotation group about O contains a free (non-abelian) subgroup with continuously many free generators.*

This lemma is essential for our proof of the decomposition theorem. A proof, using the axiom of choice, can be found in [3]. It is also possible to define *explicitly* a free subgroup of continuous rank. Indeed, it is proved in [4] that the rotations

$$R_t = A(x_t)\cdot B(x_1)\cdot A^{-1}(x_t) \qquad (0<t<1)$$

with

$$A(x)=\begin{pmatrix} f(x) & -g(x) & 0 \\ g(x) & f(x) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B(x)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & f(x) & -g(x) \\ 0 & g(x) & f(x) \end{pmatrix},$$

$$f(x)=\frac{1-x^2}{1+x^2}, \qquad g(x)=\frac{2x}{1+x^2},$$

$$x_t=\sum_{n=0}^{\infty} 2^{2^{[nt]}-2^{n^2}} \qquad (t>0),$$

[5]) Mycielski proved (in litt.) that $a < \mathfrak{c}$ can be replaced in the theorem by $a \leqslant \mathfrak{c}$.

[6]) Mycielski kindly informed us that Sierpiński already proved a lemma ([10], p. 238), though not stated in terms of group theory, from which Lemma I easily follows. The proof is effective and clearly precedes the proof in [3]. On the other hand these results are simplified and improved in [4].

are the matrix representations of a system of free generators with potency $\mathfrak{c}$.

Let $\psi_b, \psi_{b'}, \ldots$ be β ($\beta \leqslant \mathfrak{c}$) free rotations,

$$\{b, b', \ldots\} = B.$$

Let φ_b be equal to ψ_b for some values of b and equal to $\omega\psi_b$ for the other values of b.

We consider the group Φ of transformations, generated by

$$\varphi_b, \varphi_{b'}, \ldots$$

Lemma II. *The transformations $\varphi_b, \varphi_{b'}, \ldots$ are free generators of a free group Φ.*

Proof. Suppose there is a non-trivial relation:

$$\varphi_{b_1}^{l_1}\varphi_{b_2}^{l_2}\ldots\varphi_{b_n}^{l_n} = 1 \qquad (l_i = \pm 1).$$

Since ω is commutative with any rotation, this relation may be written in the form

$$\omega^m \psi_{b_1}^{l_1}\psi_{b_2}^{l_2}\ldots\psi_{b_n}^{l_n} = 1.$$

Hence m is even, and thus $\omega^m = 1$, which is impossible since the rotations ψ_{b_i} are free generators.

Let us consider the effect of the group Φ on the points of the surface S of the unit sphere. S falls into disjoint classes of equivalent points, that is, points which may be transformed into one another by transformations of the group. A point which is fixed for some transformation $\neq 1$ of the group will be called a *fixed point*. We notice first that any point equivalent to a fixed point is a fixed point. Indeed, if v is fixed for α, then $v\beta$ is fixed for $\beta^{-1}\alpha\beta$. Thus a class of equivalent points consists entirely of fixed points, or entirely of non-fixed points.

Consider any class U of non-fixed points. If some point u of the class is chosen, then, obviously, any point of the class is representable uniquely in the form $u\gamma$, where $\gamma \in \Phi$.

Consider any class V of fixed points. Choose a shortest (*i. e.* having the smallest possible number of factors $\varphi_b^{\pm 1}$) transformation ϑ having a fixed point v in V. Let

$$\vartheta = \varphi_{b_1}^{j_1}\ldots\varphi_{b_s}^{j_s} \qquad (j_\sigma = \pm 1).$$

Since ϑ has a fixed point $v = v\vartheta \in V$, ϑ is a rotation, viz. $\vartheta = \psi_{b_1}^{j_1}\ldots\psi_{b_s}^{j_s}$.

Moreover, $\varphi_{b_1}^{j_1} \neq \varphi_{b_s}^{-j_s}$, for otherwise the transformation $\varphi_{b_1}^{-j_1}\vartheta\varphi_{b_1}^{j_1}$, would be shorter and have a fixed point in V.

Lemma III. *If $v\alpha = v$, then $\alpha = \vartheta^n$, where n is an integer.*

Lemma IV. *Any point $v' \in V$ may be written in the form $v' = v\gamma$, where $\gamma \in \Phi$ and γ does not begin either with the block ϑ or with $\varphi_{b_s}^{-j_s}$. Furthermore this representation (in this specified form) is unique.*

(These lemmas may be proved in the same manner as in Robinson [9], § 2.)

The s points

$$v, v\varphi_{b_1}^{j_1}, v\varphi_{b_1}^{j_1}\varphi_{b_2}^{j_2}, \ldots, v\vartheta = v \tag{2}$$

form a closed cycle of different points in V.

Lemma V. (2) *is the only closed cycle in the class V.*

Proof. Otherwise the representation $v' = v\gamma$, where γ does not begin either with the block ϑ or with $\varphi_{b_s}^{-j_s}$, would not be unique.

3. Proof of the decomposition theorem. Suppose we are given a set A with cardinal number $\alpha \leqslant \mathfrak{c}$ and a set B with cardinal number $\beta \leqslant \mathfrak{c}$; and for any $b \in B$ two non-empty proper subsets P_b and Q_b of A. Each of the sought α pieces of S will be denoted by a^*, where $a \in A$.

Furthermore we define

$$P_b^* = \bigcup_{a \in P_b} a^*.$$

We write the given congruences in the form

$$P_b^* \cong Q_b^* \quad \text{for any} \quad b \in B. \tag{3}$$

The complement $A \setminus P_b$ of P_b will be denoted by $\bar{P}_b$. Hence

$$\bar{P}_b^* = S \setminus P_b^*.$$

Now we have to show that S may be decomposed into α disjoint pieces a^* satisfying the conditions (3).

Any condition of (3) is equivalent to its complementary condition:

$$\bar{P}_b^* \cong \bar{Q}_b^* \quad \text{for any} \quad b \in B. \tag{4}$$

We divide the non-empty proper subsets of A into two "camps" I and II in such a way that if $M \in$ I, then $\bar{M} \in$ II and if $M \in$ II, then $\bar{M} \in$ I.

Starting from an arbitrary system of free rotations $\{\psi_b\}$ with cardinal number β (according to lemma I there exists such a system), we define φ_b as follows:

$\varphi_b = \psi_b$ if P_b and Q_b belong to the same camp,

$\varphi_b = \omega\psi_b$ if P_b and Q_b belong to different camps.

Let Φ be the group generated by the system $\{\varphi_b\}$. Evidently Φ has the properties dealt with in section 2.

Now we shall prove that S may be decomposed into α disjoint pieces a^* satisfying the conditions:

(5) $$P_b^*\varphi_b=Q_b^* \quad \text{for any} \quad b\in B.$$

With respect to Φ, S falls into classes of equivalent points. It is clear that the distribution of points into the α sets a^* is independent for different classes. Thus we need only show how to make this distribution for any class in such a way that the conditions (5) are satisfied.

Consider any class U of non-fixed points. Choose at random a point u of the class. Any point u' of the class can be written uniquely in the form $u'=u\gamma$ where $\gamma\in\Phi$. We assign the points u' of U to the subsets a^* by induction with respect to the number of factors of γ.

Start by assigning u to any set a^*.

After $u\mu$ has been put into some set a^*, if $\mu'=\mu\varphi_b$ where φ_b does not cancel with the last factor of μ, we put $u\mu'$ in some set a'^* such that

$$(a\in P_b \,\&\, a'\in Q_b) \quad \text{or} \quad (a\in\bar{P}_b \,\&\, a'\in\bar{Q}_b)\,.$$

If $\mu'=\mu\varphi_b^{-1}$, then we interchange the roles of P_b and Q_b according to the condition:

$$Q_b^*\varphi_b^{-1}=P_b^*\,.$$

Then in U all conditions are satisfied.

Consider any class V of fixed points. Choose the shortest transformation ϑ of Φ having a fixed point v in V. Let

$$\vartheta=\varphi_{b_1}^{j_1}\varphi_{b_2}^{j_2}\ldots\varphi_{b_s}^{j_s} \quad (j_k=\pm 1)\,;$$

ϑ is a rotation (cf. section 2).

Consider the cycle

(6) $$v,v\varphi_{b_1}^{j_1},v\varphi_{b_1}^{j_1}\varphi_{b_2}^{j_2},\ldots,v\varphi_{b_1}^{j_1}\ldots\varphi_{b_s}^{j_s}=v\vartheta=v\,.$$

If we put

$$v_\sigma=v\varphi_{b_1}^{j_1}\ldots\varphi_{b_\sigma}^{j_\sigma} \quad (\sigma=0,1,\ldots,s,\ v_0=v)\,,$$

the cycle may be written in the form

(6′) $$v_0,v_1,v_2,\ldots,v_s=v_0\,.$$

With

$$P_{b_\sigma}=L_\sigma \quad\text{and}\quad Q_{b_\sigma}=R_\sigma \quad\text{if}\quad j_\sigma=+1,$$
$$P_{b_\sigma}=R_\sigma \quad\text{and}\quad Q_{b_\sigma}=L_\sigma \quad\text{if}\quad j_\sigma=-1$$

the congruences of the factors $\varphi_{b_\sigma}^{j_\sigma}$ of ϑ obtain the form

(7) $$L_1^*\varphi_{b_1}^{j_1}=R_1^*,\ldots,L_s^*\varphi_{b_s}^{j_s}=R_s^*\,.$$

Equivalent to (7) are the complementary conditions:

(8) $$\bar{L}_1^*\varphi_{b_1}^{j_1}=\bar{R}_1^*,\ldots,\bar{L}_s^*\varphi_{b_s}^{j_s}=\bar{R}_s^*\,.$$

If we assign the points v_σ of (6′) to the subsets a_σ^* ($a_\sigma\in A$, $\sigma=0,1,\ldots,s$), then the elements a_σ must satisfy the conditions: $a_0=a_s$ (since $v=v_0=v_s$) and

(9) $$(a_{\sigma-1}\in L_\sigma \,\&\, a_\sigma\in R_\sigma) \quad\text{or}\quad (a_{\sigma-1}\in\bar{L}_\sigma \,\&\, a_\sigma\in\bar{R}_\sigma) \quad (\sigma=1,\ldots,s)\,.$$

We investigate now whether A contains elements $a_0,\ldots,a_s$ satisfying these conditions.

We distinguish two cases:

1° $R_\sigma\neq L_{\sigma+1}$ and $\neq\bar{L}_{\sigma+1}$ for some σ ($1\leqslant\sigma\leqslant s-1$).

Then either

$$L_{\sigma+1}\cap R_\sigma\neq 0 \,\&\, L_{\sigma+1}\cap\bar{R}_\sigma\neq 0\,,$$

or

$$\bar{L}_{\sigma+1}\cap R_\sigma\neq 0 \,\&\, \bar{L}_{\sigma+1}\cap\bar{R}_\sigma\neq 0\,.$$

We may assume that the former is true, for otherwise we interchange in (7) and (8) $L_{\sigma+1}$ and $\bar{L}_{\sigma+1}$, $R_{\sigma+1}$ and $\bar{R}_{\sigma+1}$.

We put $a_{\sigma+1}$ in $R_{\sigma+1}$. Further, we put $a_{\sigma+2},\ldots,a_s=a_0,a_1,\ldots,a_{\sigma-1}$ according to (9).

If $a_{\sigma-1}\in L_\sigma$ we put a_σ in $R_\sigma\cap L_{\sigma+1}$.

If $a_{\sigma-1}\in\bar{L}_\sigma$ we put a_σ in $\bar{R}_\sigma\cap L_{\sigma+1}$.

Evidently, the elements $a_0,a_1,\ldots,a_s=a_0$ satisfy the conditions (9).

2° $R_\sigma=L_{\sigma+1}$ or $=\bar{L}_{\sigma+1}$ for any $\sigma=1,\ldots,s-1$.

We may assume that the former is true for any σ, for otherwise for some values of σ we interchange L_σ and $\bar{L}_\sigma$ and at the same time R_σ and $\bar{R}_\sigma$.

Now the following conditions are satisfied:

$$L_1^*\vartheta=R_s^* \quad\text{and}\quad \bar{L}_1^*\vartheta=\bar{R}_s^*\,.$$

We observe first that $L_1\neq\bar{R}_s$. Indeed, if $L_1=\bar{R}_s$, then L_1 and R_s should belong to different camps, thus ϑ contains an odd number of factors ω, which is impossible, since ϑ is a rotation. Thus $L_1\neq\bar{R}_s$.

Then we have $L_1\cap R_s\neq 0$ or $\bar{L}_1\cap\bar{R}_s\neq 0$. We may assume again that the former is true.

We put a_0 in $L_1\cap R_s$. Further a_σ in $R_\sigma=L_{\sigma+1}$ ($\sigma=1,\ldots,s-1$). Finally a_s must belong to R_s, which is true indeed, since $a_s=a_0\in L_1\cap R_s$.

Again it is evident that the elements $a_0,a_1,\ldots,a_s=a_0$ satisfy the conditions (9).

Now we assign the elements v_σ to a_σ^* ($\sigma=0,1,\ldots,s$).

Since (6) is the only closed cycle in V (lemma V), we may assign the other points of V to the subsets a^* in the same way as in the case of non-fixed points. Then in V all conditions are satisfied.

Since the conditions (5) are all satisfied in all classes of non-fixed points as well as of fixed points, the first part of the theorem has been proved [7]).

To prove the second part of the theorem, assuming $\alpha, \beta < \mathfrak{c}$ we well order the $\mathfrak{c}$ compact indenumerable subsets of S:

$$M_0, M_1, \ldots, M_\omega, \ldots, M_\nu, \ldots \quad (\nu < \omega_\mathfrak{c}). \tag{10}$$

We also well order the points of S:

$$x_0, x_1, \ldots, x_\omega, \ldots, x_\nu, \ldots \quad (\nu < \omega_\mathfrak{c}). \tag{11}$$

Let D be the set of all fixed points. Let $U(p)$ be the class of equivalent points which contains p.

We define p_{00} as the first point of (11), that belongs to $M_0 \setminus D$, and $p_{0\nu}$ as the first point of (11), that belongs to

$$(M_0 \setminus D) \setminus \bigcup_{\sigma<\nu} U(P_{0\sigma}) \quad (\nu = 1, 2, \ldots, \nu < \omega_a).$$

Suppose that $p_{\xi\nu}$ is already defined for all $\nu < \omega_a$ and for all $\xi < \gamma$, then we define $p_{\gamma\nu}$ as follows:

$$\text{if} \quad S_\gamma = \bigcup_{0 \leqslant \xi < \gamma,\ 0 \leqslant \nu < \omega_a} U(p_{\xi\nu}),$$

$p_{\gamma 0}$ is the first point of (11) that belongs to $M_\gamma \setminus (S_\gamma \cup D)$,
$p_{\gamma\nu}$ is the first point of (11) that belongs to

$$M_\gamma \setminus \big(S_\gamma \cup D \cup \bigcup_{\sigma<\nu} U(p_{\gamma\sigma})\big). \tag{12}$$

This is always possible. Indeed, since the potencies of each of the sets S_a, D and $\bigcup_{\sigma<\nu} U(p_{\gamma\sigma})$ are less than $\mathfrak{c}$, the sets (12) are not empty.

We now spread the points of S over the a required pieces A_ν $(0 \leqslant \nu \leqslant \omega_a)$ in the well known way, using Φ of potency $< \mathfrak{c}$, in which we ensure that

$$p_{\gamma\nu} \in A_\nu \quad (0 \leqslant \nu < \omega_a,\ 0 \leqslant \gamma < \omega_\mathfrak{c}).$$

This is possible, because in every class U of non-fixed points we can arbitrarily choose a point u and put this point in an arbitrary A. The given system of congruence conditions is therefore satisfied. A sum occurring in any of these relations does not contain an M_ν, since the complement contains points of M_ν. So all sums are totally imperfect. Since the same is true for the complement of a sum in S, the sums are also connected and microconnected according to a theorem of Sierpiński.

[7]) We still have to prove that the pieces can be made non-empty. This is clear since Φ may be chosen in such a way that there are $\mathfrak{c}$ classes of non-fixed points.

4. Conclusions. 1° If we write down each congruence of (3) $\mathfrak{c}$ times, then the cardinal number of the number of equations of (3) is $\mathfrak{c}$ again.

So even the following theorem is true:

Suppose we are given a system of congruences:

$$P_b^* \cong Q_b^* \quad \textit{for any} \quad b \in B,$$

where the cardinal number of B is less than or equal to $\mathfrak{c}$. Then the sphere S may be decomposed into a disjoint pieces, satisfying the given system of congruences in such a way that each congruence can be effectuated by $\mathfrak{c}$ free congruent transformations.

2° We call the system of congruences (3) "strong" if there is a condition $M^* \cong \overline{M}^*$ belonging to (3) or (4) or obtainable from them by using transitivity of the given congruence relations. Otherwise the system is called "weak". If the given system (3) is weak, we can divide the subsets of A into two camps I and II, defined as above, in such a way that no congruence is required between sets M^* and N^*, where $M \in \mathrm{I}$ and $N \in \mathrm{II}$. In this case the transformations φ_b are equal to ψ_b for any $b \in B$.

Hence we can establish the following theorem:

Suppose we are given a weak system of congruences (3), *having a cardinal number $\beta \leqslant \mathfrak{c}$ and any system of β free rotations $\{\psi_b\}$. Then the sphere S may be decomposed into $a \leqslant \mathfrak{c}$ disjoint pieces a^*, such that $P_b^*\psi_b = Q_b^*$ for any $b \in B$. Each congruence may be effectuated by $\mathfrak{c}$ free rotations of the system.*

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MATHEMATISCH INSTITUUT
UNIVERSITY OF AMSTERDAM

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On a theorem of Borsuk

by

J. C. Moore (Princeton, N. J.)

Recently Borsuk [1] showed that if a compact space of dimension at most k has a positive k-dimensional Betti number, then for $m \geqslant k$ the function space of maps of the original space into the m-sphere has a positive $(m-k)$th Betti number. The case $m=k$ is covered by the Hopf Classification Theorem [9], and the object of this paper is to show that the case $m>k$ may be viewed as a generalization of this theorem, and that there is in fact a sort of duality between the homology of the function space and the cohomology of the original space.

1. The critical dimension

Notation and conventions. In this paper space will always mean Hausdorff space. For any space X, let $H^q(X)$ denote the q-dimensional Čech cohomology group of X with integer coefficients [3], and if A is a subspace of X let $H^q(X,A)$ be the q-dimensional Čech cohomology group of the pair (X,A). Similarly let $H_q(X)$ and $H_q(X,A)$ be the q-dimensional singular homology groups with integer coefficients of the space X and of the pair (X,A).

If X is a space and $x \in X$, denote the q-dimensional homotopy group of X based at the point x by $\pi_q(X,x)$. If X is a function space containing a unique constant map, this map will be taken as the base point, and the preceding notation will be abbreviated to $\pi_q(X)$. Finally, if X has a multiplication with an identity up to homotopy, and an inverse up to homotopy, let $\pi_0(X)$ denote the group of path components of X.

For any non-negative integer k, let S^k denote the k-dimensional sphere, and let y^k be a point of S^k. Let I denote the closed interval of real numbers from -1 to $+1$.

Definition. If X is a space, define $s(X)$ to be the identification space of $X\times I$ obtained by the following identifications: $(x,1)$ is identified with $(x',1)$, and $(x,-1)$ is identified with $(x',-1)$ for $x,x' \in X$. Define $s_+(X)$ to be the image of those pairs (x,t) such that $t\geqslant 0$, define $s_-(X)$