Some cofinality theorems on ordered sets

by

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Let α be an ordinal number. Then $\mathrm{cf}(\alpha)$ is defined to be the smallest ordinal number such that there exists an increasing sequence of ordinal numbers $(\beta_\xi)_{\xi<\omega_{\mathrm{cf}(\alpha)}}$ with

$$\lim_{\xi<\omega_{\mathrm{cf}(\alpha)}} \beta_\xi = \omega_\alpha .$$

For the properties of $\mathrm{cf}(\alpha)$ that we shall make use of in what follows, the reader is referred to [2, p. 185]; for set-theoretical matters in general, to [1].

The cardinal number of a set A and of an ordinal number α will be denoted by $|A|$, $|\alpha|$, respectively. We recall that $W(\alpha)$ denotes the well-ordered set of all ordinal numbers less than α.

We shall be concerned mainly with characterizing $\mathrm{cf}(\alpha)$ in terms of the cardinal number of the intersection of two sets, one being an arbitrary subset, of power $\aleph_\alpha$, of a certain basic set M, and the other being any representative of a certain simple class of subsets of M. In § 1, the set M under consideration is well-ordered; in § 2, it is power-homogeneous.

1. Well-ordered sets. Lemma. *Let $|Q|<\aleph_\alpha$, $\mathbf{t}_q<\aleph_\alpha$ for every $q\in Q$, and*

$$\sum_{q\in Q}\mathbf{t}_q=\aleph_\alpha . \tag{1}$$

Then there exists a set $P\subset Q$ such that $|P|=\aleph_{\mathrm{cf}(\alpha)}$ and

$$\sum_{p\in P}\mathbf{t}_p=\aleph_\alpha . \tag{2}$$

Proof. An immediate consequence of the definition of $\mathrm{cf}(\alpha)$ is that there exists a sequence $(\mathbf{n}_\xi)_{\xi<\omega_{\mathrm{cf}(\alpha)}}$ with

$$\mathbf{n}_\xi<\aleph_\alpha \quad (\xi<\omega_{\mathrm{cf}(\alpha)}), \qquad \sum_{\xi<\omega_{\mathrm{cf}(\alpha)}}\mathbf{n}_\xi=\aleph_\alpha . \tag{3}$$

Suppose that $\xi<\omega_{\mathrm{cf}(\alpha)}$ and that for every $\mu<\xi$ we have defined $q_\mu\in Q$ in such a manner that the elements q_μ $(\mu<\xi)$ are distinct, and $\mathbf{n}_\mu<\mathbf{t}_{q_\mu}$ for every $\mu<\xi$. We define an element $q_\xi\in Q$ as follows. Put $Q_\xi=\{q_\mu\}_{\mu<\xi}$. Then there exists an element $q\in Q-Q_\xi$ such that

$$\mathbf{n}_\xi<\mathbf{t}_q ; \tag{4}$$

otherwise, we should have $\mathbf{t}_q\leqslant\mathbf{n}_\xi$ for every $q\in Q-Q_\xi$, and consequently

$$\sum_{q\in Q}\mathbf{t}_q=\sum_{\mu<\xi}\mathbf{t}_{q_\mu}+\sum_{q\in Q-Q_\xi}\mathbf{t}_q\leqslant\sum_{\mu<\xi}\mathbf{t}_{q_\mu}+|Q|\cdot\mathbf{n}_\xi<\aleph_\alpha ,$$

which contradicts (1). Letting q_ξ be any $q\in Q-Q_\xi$ satisfying (4), the set $\{q_\xi\}_{\xi<\omega_{\mathrm{cf}(\alpha)}}$, which we shall denote by P, is thus well-defined by transfinite induction. Now it is obvious that $P\subset Q$, $|P|=\aleph_{\mathrm{cf}(\alpha)}$, and the relations

$$\aleph_\alpha=\sum_{\xi<\omega_{\mathrm{cf}(\alpha)}}\mathbf{n}_\xi\leqslant\sum_{\xi<\omega_{\mathrm{cf}(\alpha)}}\mathbf{t}_{q_\xi}\Big(=\sum_{p\in P}\mathbf{t}_p\Big)\leqslant\aleph_\alpha$$

yield (2), so that the lemma is proved.

Corollary. *Let φ and α be ordinal numbers, with*

$$\mathrm{cf}(\varphi)\neq\mathrm{cf}(\alpha) . \tag{5}$$

Suppose that

$$(\mathbf{m}_\xi)_{\xi<\omega_{\mathrm{cf}(\varphi)}} \tag{6}$$

is a sequence of cardinal numbers such that

$$\sum_{\xi<\omega_{\mathrm{cf}(\varphi)}}\mathbf{m}_\xi=\aleph_\alpha . \tag{7}$$

Then there exists a $\tau<\omega_{\mathrm{cf}(\varphi)}$ such that

$$\sum_{\xi<\tau}\mathbf{m}_\xi=\aleph_\alpha . \tag{8}$$

Proof. If $\mathrm{cf}(\varphi)<\mathrm{cf}(\alpha)$, there exists a $\gamma<\omega_{\mathrm{cf}(\varphi)}$ such that $\mathbf{m}_\gamma=\aleph_\alpha$, since otherwise (7) would contradict the definition of $\mathrm{cf}(\alpha)$; (8) then holds with $\tau=\gamma\dot{+}1$.

If $\mathrm{cf}(\alpha)<\mathrm{cf}(\varphi)$ and there is a $\gamma<\omega_{\mathrm{cf}(\varphi)}$ such that $\mathbf{m}_\gamma=\aleph_\alpha$, then again (8) holds with $\tau=\gamma+1$. Suppose, however, that every term of (6) is less than $\aleph_\alpha$. If there is a $\mu<\omega_{\mathrm{cf}(\varphi)}$ such that $\mathbf{m}_\xi=0$ for $\mu\leqslant\xi<\omega_{\mathrm{cf}(\varphi)}$, then (8) holds with $\tau=\mu$; if, however, no such μ exists, then the subsequence of non-zero terms of (6) is cofinal with the sequence (6), and, since $\omega_{\mathrm{cf}(\varphi)}$ is regular, this subsequence is of type $\omega_{\mathrm{cf}(\varphi)}$. We may therefore assume, without loss of generality, that, for every $\xi<\omega_{\mathrm{cf}(\varphi)}$, we have $\mathbf{m}_\xi\neq0$. It then follows immediately from (7) that $\mathrm{cf}(\varphi)\leqslant\alpha$, and since $\operatorname{cf}(\varphi)=\alpha$ would imply that $\operatorname{cf}(\varphi)=\operatorname{cf}(\operatorname{cf}(\varphi))=\operatorname{cf}(\alpha)$, which contradicts our hypothesis (5), we must have $\operatorname{cf}(\varphi)<\alpha$. Now an application of the lemma immediately completes the proof of the corollary.

THEOREM 1. *Let $\beta=\omega_\alpha\varphi+\varrho$, with $\varphi>0$ and $\varrho<\omega_\alpha$. Then a necessary and sufficient condition that there exist a set $M\subset W(\beta)$, with $|M|=\aleph_\alpha$, such that $|M\cap W(\gamma)|<\aleph_\alpha$ for every $\gamma<\beta$, is that $\varrho=0$ and either φ be isolated or $\operatorname{cf}(\varphi)=\operatorname{cf}(\alpha)$.*

Proof. We shall first make some preliminary remarks. In case φ is a limit number, there exists (as follows readily from the definition of $\operatorname{cf}(\varphi)$) an increasing sequence of ordinal numbers

(9) $$(\varphi_\xi)_{\xi<\omega_{\operatorname{cf}(\varphi)}}, \quad \varphi_0=0,$$

such that

$$\lim_{\xi<\omega_{\operatorname{cf}(\varphi)}} \varphi_\xi=\varphi.$$

Consequently, in case we also have $\varrho=0$, then

(10) $$\lim_{\xi<\omega_{\operatorname{cf}(\varphi)}} \omega_\alpha\varphi_\xi=\beta;$$

setting $S_\xi=W(\omega_\alpha\varphi_{\xi+1})-W(\omega_\alpha\varphi_\xi)$ for every $\xi<\omega_{\operatorname{cf}(\varphi)}$, it is evident that the sets S_ξ $(\xi<\omega_{\operatorname{cf}(\varphi)})$ are mutually exclusive, and that

(11) $$W(\beta)=\bigcup_{\xi<\omega_{\operatorname{cf}(\varphi)}} S_\xi.$$

Moreover, since the sequence (9) is increasing, we have $|S_\xi|\geqslant\aleph_\alpha$ for every $\xi<\omega_{\operatorname{cf}(\varphi)}$.

Turning now to the theorem, let us first prove the necessity of the condition. If $\varrho\neq 0$, then $\omega_\alpha\varphi<\beta$, and since $|\varrho|<\aleph_\alpha$, we have $|M\cap W(\gamma)|=\aleph_\alpha$ for $\gamma=\omega_\alpha\varphi$. If, however, $\varrho=0$ but φ is a limit number with $\operatorname{cf}(\varphi)\neq\operatorname{cf}(\alpha)$, then (11) holds, and

$$M=\bigcup_{\xi<\omega_{\operatorname{cf}(\varphi)}}(M\cap S_\xi),$$

so that

$$\aleph_\alpha=\sum_{\xi<\omega_{\operatorname{cf}(\varphi)}}|M\cap S_\xi|.$$

According to the above corollary, there exists a $\tau<\omega_{\operatorname{cf}(\varphi)}$ such that

$$\aleph_\alpha=\sum_{\xi<\tau}|M\cap S_\xi|,$$

which implies that $|M\cap W(\gamma)|=\aleph_\alpha$ for $\gamma=\omega_\alpha\varphi_{\tau+1}$; and clearly $\omega_\alpha\varphi_{\tau+1}<\beta$. This establishes the necessity of the condition.

To prove the sufficiency, note first that if $\varrho=0$ and φ is isolated, then $\beta=\omega_\alpha(\varphi-1)+\omega_\alpha$, and it suffices to take $M=W(\omega_\alpha\varphi)-W(\omega_\alpha(\varphi-1))$. If, however, $\varrho=0$ and φ is a limit number with $\operatorname{cf}(\varphi)=\operatorname{cf}(\alpha)$, then (11) may be written as

$$W(\beta)=\bigcup_{\xi<\omega_{\operatorname{cf}(\alpha)}} S_\xi.$$

Let $(\mathbf{n}_\xi)_{\xi<\omega_{\operatorname{cf}(\alpha)}}$ be as in (3), and, for every $\xi<\omega_{\operatorname{cf}(\alpha)}$, choose $N_\xi\subset S_\xi$ with $|N_\xi|=\mathbf{n}_\xi$. Then it suffices to take $M=\bigcup_{\xi<\omega_{\operatorname{cf}(\alpha)}} N_\xi$. The proof of the theorem is now complete.

2. Power-homogenous sets. If M is an ordered set, and b and c are distinct elements of M, with b preceding c, then the ordered subset of M consisting of b, c, and all the elements of M that succeed b and precede c, is called, as usual, an *interval* of M. If, for a fixed ordinal number α, every interval of M consists of $\aleph_\alpha$ elements, then M is said to be $\aleph_\alpha$*-homogeneous.*

For every pair of (not necessarily distinct) ordinal numbers α and β let $T(\alpha,\beta)$ be the lexicographically ordered set of all sequences

(12) $$t=(\tau_\xi)_{\xi<\omega_\beta},$$

where $\tau_\xi\in W(\omega_\alpha)$ for every $\xi<\omega_\beta$, not every τ_ξ is 0, and only a finite number are different from 0. It is easily seen that if $\alpha\geqslant\beta$, then $T(\alpha,\beta)$ is an $\aleph_\alpha$-homogeneous set of power $\aleph_\alpha$. For any $t\in T(\alpha,\beta)$, we shall denote by $R(t)$ the set of all elements of $T(\alpha,\beta)$ that succeed t.

THEOREM 2. *Suppose that $\beta\leqslant\operatorname{cf}(\alpha)$. Then a necessary and sufficient condition that $\beta=\operatorname{cf}(\alpha)$ is that there exist a set $M\subset T(\alpha,\beta)$, with $|M|=\aleph_\alpha$, such that $|M\cap R(t)|<\aleph_\alpha$ for every $t\in T(\alpha,\beta)$.*

Proof. The condition is necessary. For if $\beta=\operatorname{cf}(\alpha)$, there exists an increasing sequence of ordinal numbers $(\zeta_\xi)_{\xi<\omega_\beta}$, with $2\leqslant\zeta_\xi<\omega_\alpha$ for every $\xi<\omega_\beta$, such that

(13) $$\lim_{\xi<\omega_\beta}\zeta_\xi=\omega_\alpha.$$

For every $\chi<\omega_\beta$, and every μ satisfying $0<\mu<\zeta_\chi$, let $t_{\chi\mu}$ be that element (12) of $T(\alpha,\beta)$ such that $\tau_\xi=0$ for every $\xi<\omega_\beta$ with $\xi\neq\chi$, whereas $\tau_\chi=\mu$; denote by M the set of all elements $t_{\chi\mu}$ thus defined. Because of (13), $|M|=\aleph_\alpha$. The subset of M consisting of all elements $t_{\chi 1}$ with $\chi<\omega_\beta$ is coinitial with $T(\alpha,\beta)$, and, since every element of M is succeeded by less than $\aleph_\alpha$ elements of M, it is clear that $|M\cap R(t)|<\aleph_\alpha$ for every $t\in T(\alpha,\beta)$.

The condition is sufficient. For let k be a natural number, and let

(14) $$\xi_1<\xi_2<\ldots<\xi_k<\omega_\beta.$$

Consider the set of all elements (12) of $T(\alpha,\beta)$ such that $\tau_\xi=0$ if, and only if, $\xi\neq\xi_1, \xi\neq\xi_2,\dots,\xi\neq\xi_k$; this is a well-ordered subset, of type ω_α^k, of $T(\alpha,\beta)$. Given k, there are $\aleph_\beta$ distinct ways of choosing a set of ordinal numbers $\xi_1,\xi_2,\dots,\xi_k$ so as to satisfy (14), and any two distinct such choices lead, in the manner just described, to two mutually exclusive well-ordered subsets, each of type ω_α^k, of $T(\alpha,\beta)$. Letting k range over the natural numbers, we thus obtain a decomposition

(15) $$T(\alpha,\beta)=\bigcup_{\xi<\omega_\beta} T_\xi,$$

where for every $\xi<\omega_\beta$, T_ξ is a well-ordered set of type ω_α^k for some natural number k, and the sets T_ξ $(\xi<\omega_\beta)$ are mutually exclusive.

Now let $M\subset T(\alpha,\beta)$, with $|M|=\aleph_\alpha$, and suppose that $\beta\neq \mathrm{cf}(\alpha)$. Then, because of our hypothesis that $\beta\leqslant \mathrm{cf}(\alpha)$, we must have

(16) $$\beta<\mathrm{cf}(\alpha).$$

In view of (15), $M=\bigcup_{\xi<\omega_\beta}(M\frown T_\xi)$ and

(17) $$\aleph_\alpha=\sum_{\xi<\omega_\beta}|M\frown T_\xi|.$$

From (16) and (17), and the definition of $\mathrm{cf}(\alpha)$, we infer that there exists a $\psi<\omega_\beta$ such that

(18) $$|M\frown T_\psi|=\aleph_\alpha.$$

It is evident, however, from the definition of T_ψ, that there exists a $t\in T(\alpha,\beta)$ such that $T_\psi\subset R(t)$, and hence $|M\frown R(t)|=\aleph_\alpha$. This completes the proof of Theorem 2.

THEOREM 3. *If $\beta<\mathrm{cf}(\alpha)$, and $M\subset T(\alpha,\beta)$, with $|M|=\aleph_\alpha$, then M contains a well-ordered subset of power $\aleph_\alpha$; furthermore, $T(\alpha,\beta)$ contains $2^{\aleph_\alpha}$ subsets that are similar to M.*

Proof. The first part of Theorem 3 follows immediately from (18).

To prove the second part, let $\{m_\xi\}_{\xi<\omega_\alpha}$ be a well-ordered subset of M, of type ω_α, indexed so that $\xi<\xi'<\omega_\alpha$ implies that m_ξ precedes $m_{\xi'}$ in M. For every $\xi<\omega_\alpha$, denote by I_ξ the ordered subset of $T(\alpha,\beta)$ consisting of m_ξ and all the elements of $T(\alpha,\beta)$ that succeed m_ξ and precede $m_{\xi+1}$ in $T(\alpha,\beta)$. Then, if $\xi<\xi'<\omega_\alpha$, every element of I_ξ precedes every element of $I_{\xi'}$. It is clear from the definition of $T(\alpha,\beta)$ that between any two of its elements there is a subset of $T(\alpha,\beta)$ similar to $T(\alpha,\beta)$. It is also easy to see that the sequence $(I_\xi)_{\xi<\omega_\alpha}$ has $2^{\aleph_\alpha}$ subsequences $(I'_\xi)_{\xi<\omega_\alpha}$ such that, if $\xi<\xi'<\omega_\alpha$, then every element of I'_ξ precedes every element of $I'_{\xi'}$. For every such subsequence $(I'_\xi)_{\xi<\omega_\alpha}$, for every $\xi<\omega_\alpha$, let M'_ξ be a subset of I'_ξ similar to $M\frown I_\xi$; then the set

$$M'=\Big[M\frown\Big(T(\alpha,\beta)-\bigcup_{\xi<\omega_\alpha}I_\xi\Big)\Big]\smile\Big[\bigcup_{\xi<\omega_\alpha}M'_\xi\Big]$$

is evidently a subset of $T(\alpha,\beta)$ similar to M, and the $2^{\aleph_\alpha}$ subsets M' constructed in this manner are obviously distinct. The proof of Theorem 3 is now complete.

Remark 1. If $\beta=\mathrm{cf}(\alpha)$, and $M\subset T(\alpha,\beta)$, with $|M|=\aleph_\alpha$, then M need not contain a well-ordered subset of type ω_α: witness the set M defined in the sufficiency part of the proof of Theorem 2.

Remark 2. It is of interest to compare Theorem 3 with the following situation. Denote the initial number of $Z(2^{\aleph_0})$ by ω_λ. Then, as is well known, $\mathrm{cf}(\lambda)>0$. Let M be any subset, of power $\aleph_\lambda=2^{\aleph_0}$, of the linear continuum C. Then M does not have any well-ordered subset of type ω_λ, and C contains only $\aleph_\lambda$ subsets that are similar to M.

THEOREM 4. *Let $\mathrm{cf}(\alpha)<\alpha$. Suppose that M is an $\aleph_\alpha$-homogeneous set, and*[1]) *that every interval of M contains both a well-ordered subset of type $\omega_{\mathrm{cf}(\alpha)}$ and an inversely well-ordered subset of type $\omega^*_{\mathrm{cf}(\alpha)}$. Then there exists a set $E\subset M$, with $|E|=\aleph_\alpha$, such that E has neither a well-ordered subset of type ω_α nor an inversely well-ordered subset of type ω_α^*.*

Proof. If some interval of M contains neither a subset of type ω_α nor a subset of type ω_α^*, then there is nothing further to prove, because M is $\aleph_\alpha$-homogeneous. Let us assume, then, that every interval of M contains either a subset of type ω_α or a subset of type ω_α^*.

According to the definition of $\mathrm{cf}(\alpha)$, we have

$$\aleph_\alpha=\sum_{\xi<\omega_{\mathrm{cf}(\alpha)}}\aleph_{\varkappa_\xi},$$

where $\varkappa_\xi<\alpha$ for every $\xi<\omega_{\mathrm{cf}(\alpha)}$. Hence, if

$$v=\sum_{\xi<\omega_{\mathrm{cf}(\alpha)}}\omega^*_{\varkappa_\xi},$$

then $|v|=|v^*|=\aleph_\alpha$, and an ordered set of type v or v^* obviously contains neither a subset of type ω_α nor one of type ω_α^*. We shall complete the proof by showing that M has either a subset of type v or one of type v^*.

Since M, by hypothesis, contains a well-ordered subset of type $\omega_{\mathrm{cf}(\alpha)}$, there exist mutually exclusive intervals J_ξ $(\xi<\omega_{\mathrm{cf}(\alpha)})$ of M such that, if $\xi<\xi'<\omega_{\mathrm{cf}(\alpha)}$, every element of J_ξ precedes every element of $J_{\xi'}$.

[1]) The condition that follows is automatically satisfied in case $\mathrm{cf}(\alpha)=0$.

If each of these intervals contains a subset of type ω_a^*, then their union contains a subset of type v. If, however, one of these intervals, say J_{ξ_0}, does not contain a subset of type ω_a^*, there exist mutually exclusive intervals K_ξ $(\xi < \omega_{\mathrm{cf}(a)})$ of J_{ξ_0} such that, if $\xi < \xi' < \omega_{\mathrm{cf}(a)}$, every element of K_ξ succeeds every element of $K_{\xi'}$; and, of course, no K_ξ $(\xi < \omega_{\mathrm{cf}(a)})$ contains a subset of type ω_a^*, so that, by our assumption, each one must contain a subset of type ω_a. Consequently, their union contains a subset of type v^*, and the proof is complete.

COROLLARY. *Let $a > 0$. Then a necessary and sufficient condition that $\mathrm{cf}(a) > 0$ is that there exist an $\aleph_a$-homogeneous set such that every one of its subsets of power $\aleph_a$ contains a well-ordered subset of power $\aleph_a$.*

Proof. The necessity of the condition is furnished by Theorem 3, with $\beta = 0$; the sufficiency is a consequence of Theorem 4 (see footnote [1])).

References

[1] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914.

[2] A. Tarski, *Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires*, Fund. Math. 16 (1930), p. 181-304.

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Decompositions of a sphere

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1. Introduction. A classical result of F. Hausdorff ([5], p. 469-472) states that – disregarding a denumerable set – "a half" and "a third" of a sphere S – defined by $x^2+y^2+z^2=1$ – may be congruent to each other (using only rotations). This result is exploited to a great extent in a number of problems by S. Banach and A. Tarski, J. von Neumann, W. Sierpiński, R. M. Robinson and others; for references cf. [9] and [11]. The results of this paper generalize those of Robinson [9] in several directions. One notable result of Robinson's is this: S can be divided into two pieces, each of which can be divided into two pieces congruent to itself (by rotation). From this he derives the following "paradoxical" result: it is possible to cut the solid unit sphere $(x^2+y^2+z^2 \leqslant 1)$ into five (but not less than five) mutually disjoint pieces (one of which is a point) and to reassemble these pieces (using only rotations and translations) so that two solid unit spheres are formed [1]). From our results (cf. decomposition theorem below) and the material contained in [9], p. 256, 257, it follows in particular that these pieces may have the additional properties of *connectedness* and *local connectedness*. So the "pieces" are really pieces and not necessarily some kind of "scattered" sets.

The main result of Robinson ([9], p. 252) runs as follows: it is possible to decompose S into n mutually disjoint, non-empty pieces $A_1, A_2, \ldots, A_n$ satisfying a given (finite) system of congruences, each having the form

$$\bigcup_{i=1}^{r} A_{k_i} \cong \bigcup_{j=1}^{s} A_{l_j} \quad (0<r<n,\ 0<s<n), \tag{1}$$

if and only if none of the given congruences and no congruence obtainable from them by taking complements (in S) or by using transitivity (to derive new congruences from the given system) asserts the congruence of two complementary portions of S. Moreover, starting from an arbi-

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[1]) Banach and Tarski proved that a finite number of pieces suffices; von Neumann cut the number of pieces down to 9 (without proof), Sierpiński to 8.