Some proofs of undecidability of arithmetic

by

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The aim of this paper is to present the well known Gödel-Rosser's Theorem on the undecidability of arithmetic as a consequence of the representability in arithmetic of recursively enumerable non-general recursive sets of positive integers.

The new features of this paper are: 1. the systematization of the proof by formulation of Lemmas 2-4 concerning representability and 2. the application of two recursively enumerable non-general recursive sets, namely of a diagonal set and of a simple set.

1. General lemmas. A theory T is said to be *decidable* if there exists an effective method (called the *decision procedure*) which allows us to decide after a finite set of simple operations whether a given sentence is or is not a theorem of T. The notion of the method is not quite exact. A strict metamathematical definition of decidability is obtained for example by mapping the set of expressions of a formalized theory onto the set of positive integers. Any formalized theory T can be considered as containing only a finite set of primitive signs: $\beta_1, \beta_2, \ldots, \beta_l$. Each expression Γ can be considered as a finite sequence $\beta_{n_1} \ldots \beta_{n_k}$ of these signs. The function $\mathrm{No}(\Gamma)$, which establishes the enumeration of the expressions, can be defined for example as follows:

$$\text{If} \quad \Gamma = \beta_{n_1}\beta_{n_2}\ldots\beta_{n_k}, \quad \text{then} \quad \mathrm{No}(\Gamma) = 2^{n_1}3^{n_2}\ldots p_k^{n_k}.$$

A theory T is called decidable if the set of numbers representing the theorems of T is computable (generally recursive). A theory which is not decidable is called *undecidable*. If a theory T is consistent and each consistent extension of T is undecidable, then T is said to be *essentially undecidable*. We shall apply the following Lemma on essential undecidability established by Tarski in [10]:

LEMMA 1. *A theory T is essentially undecidable if and only if T is consistent and each consistent and recursively enumerable extension of T is incomplete* (see [10], p. 15, Theorem 2).

Many proofs of essential undecidability can be considered as the applications of this Lemma. In this paper we shall give some simple examples of such proofs.

In the following we shall consider the theories containing the identity-sign, the quantifiers, the minimum operation or the ι-operation and containing the individual constants of the lowest type (terms). A sequence $\{a_n\}$ of terms is regarded as computable if the sequence of numbers $\{\mathrm{No}(a_n)\}$ is computable. The propositional formulas with free variables will be denoted by $\Phi(x)$, $\Psi(x,y)$ etc. The substitutions of the constants in such formulas will be denoted by $\Phi(a_n)$, $\Psi(a_k,a_l)$. The quantifiers in the system will be written as (x), $(\exists x)$ and in the metasystem as $\prod_x$, $\sum_x$. The expression "$\ulcorner\Phi(a_n)\urcorner \in T$"[1]) in the metasystem will be an abbreviation of the proposition "$\Phi(a_n)$ is a theorem of T".

We shall say that a set X of positive integers is *represented in T* by the formula $\Phi(x)$ with respect to the sequence $\{a_n\}$ of terms provided that the following equivalence is satisfied:

$$(1) \qquad \prod_n n \in X \equiv \ulcorner\Phi(a_n)\urcorner \in T.$$

We shall say that a set X is *strongly represented* in T by a formula $\Phi(x)$ with respect to the sequence $\{a_n\}$ of terms if and only if the set X is represented in T by the formula $\Phi(x)$ with respect to $\{a_n\}$ and the complement $-X$ of the set X is represented by the negation $\sim\Phi(x)$ of the formula $\Phi(x)$ with respect to $\{a_n\}$.

The notion of strong representation is identical with the notion of T-definability introduced by Mostowski in [7].

Similarly a relation R is represented in T by a formula $\Psi(x,y)$ with respect to $\{a_n\}$ if the following equivalence is satisfied:

$$(2) \qquad \prod_{n,k} R(n,k) \equiv \ulcorner\Psi(a_n,a_k)\urcorner \in T.$$

A relation R is strongly represented if it is represented by a formula $\Psi(x,y)$ and the relation $\sim R(n,k)$ is represented by the formula $\sim\Psi(x,y)$.

We shall suppose in the continuation that the sequence $\{a_n\}$ is computable, and fixed in all considerations.

A function f is said to be represented by a formula $\Psi(x,y)$ if the relation $k=f(n)$ is represented by the formula $\Psi(x,y)$.

[1]) I shall use the symbols $\ulcorner\ \urcorner$ not strictly in the manner proposed by Quine, but only for distinguish between the names of the expressions of the theory and the expressions of the metatheory.

A function f is said to be strongly represented by a formula $\Psi(x,y)$ if the relation $k=f(n)$ is represented by the formula $\Psi(x,y)$ and the formula

$$(3) \qquad (x,y,z)\big((\Psi(x,y)\wedge\Psi(x,z))\to y=z\big)$$

belongs to T.

We shall make use of some properties of representability. It is obvious that if T is consistent and the function f is strongly represented in T, then the relation $k=f(n)$ is strongly represented in T, provided that the sequence $\{a_n\}$ is discriminable in T (this means that for $n\neq k$, $\ulcorner\sim(a_n=a_k)\urcorner\in T$).

Namely if $k\neq f(n)$ then $\ulcorner\sim(a_k=a_{f(n)})\urcorner\in T$, on the other hand $\ulcorner\Psi(a_n,a_{f(n)})\urcorner\in T$ because the relation $k=f(n)$ is represented in T. Hence according to (3) $\ulcorner\sim\Psi(a_n,a_k)\urcorner\in T$. Conversely if $\ulcorner\sim\Psi(a_n,a_k)\urcorner\in T$ and T is consistent, then $\ulcorner\Psi(a_n,a_k)\urcorner\notin T$ and $k\neq f(n)$. Thus the relation $k\neq f(n)$ is represented by the formula $\sim\Psi(x,y)$.

If the system T contains the minimum operation $(\mu x)[...]$ (or the ι-operation $(\iota x)[...]$) producing the function-formula $\ulcorner\gamma(y,...)\urcorner=\ulcorner(\mu x)[\Psi(x,y,...)]\urcorner$ from the sentential formula $\Psi(x,y,...)$, then the function f is strongly represented in T if and only if there exists a function-formula $\gamma(x)$ such that

$$(4) \qquad \prod_n \ulcorner\gamma(a_n)=a_{f(n)}\urcorner\in T\,.$$

Indeed, if the condition (4) is satisfied then the function f is strongly represented by the formula $\gamma(x)=y$. Conversely if the function f is strongly represented by the formula $\Psi(x,y)$, then setting $\ulcorner\gamma(x)\urcorner=\ulcorner(\mu y)[\Psi(x,y)]\urcorner$ (or $\ulcorner\gamma(x)\urcorner=\ulcorner(\iota y)[\Psi(x,y)]\urcorner$ we obtain the condition (4).

To simplify the proofs in the case of the arithmetic of positive integers we shall assume that the considered theory contains the minimum operator and the function formulas.

We say that a theory T is recursively enumerable if the set of numbers: $\mathrm{No}(\Gamma)$ for $\Gamma\in T$, is recursively enumerable. If the set of axioms is recursively enumerable, and the rules of inference are general recursive, then the whole theory is recursively enumerable. If the theory T is recursively enumerable, then each set X represented in T is recursively enumerable too. This follows immediately from the equivalence (1). According to a well known theorem of E. Post, if the set X and its complement $-X$ are both recursively enumerable, then X and $-X$ are computable. Hence if X and $-X$ are represented in a recursively enumerable theory T, then X and $-X$ are computable. Thus according to our definitions, if the set X is strongly represented in a recursively enumerable theory T, then X is computable. It is evident that the same is true for the relation on two or more arguments.

If two functions f and g are strongly represented by the formulas $\varphi(x)$ and $\gamma(x)$, then the superposition $f(g(x))$ is strongly represented by the substitution $\varphi(\gamma(x))$. If the theory T is the arithmetic of positive integers, then it is well known that if a relation $R(n,k)$ is strongly represented in T by the formula $\Psi(x,y)$, and for each n there exists such k that $R(n,k)$, then the function $(\mu k)[R(n,k)]$ is strongly represented by the formula $(\mu y)[\Psi(x,y)]$.

These two observations are the inductive steps of the proof of the strong representability of all computable functions and relations in arithmetic. Hence the class of computable functions is identical with the class of functions strongly representable in arithmetic. As arithmetic we shall understand the theory Ar described later.

We shall formulate the following lemmas useful in proving essential undecidability.

Lemma 2. *If a theory T is consistent and recursively enumerable, and a non-computable set X is represented in T, then T is not complete.*

Proof. Let X be represented in T by the formula $\Phi(x)$ with respect to the computable sequence $\{a_n\}$ of constants. Hence the equivalence (1) is true. If T is consistent, then

$$(5) \qquad \prod_n \ulcorner\Phi(a_n)\urcorner\notin T\vee\ulcorner\sim\Phi(a_n)\urcorner\notin T\,.$$

If T were complete, the following condition would be satisfied:

$$(6) \qquad \prod_n \ulcorner\Phi(a_n)\urcorner\in T\vee\ulcorner\sim\Phi(a_n)\urcorner\in T\,.$$

From (1), (5) and (6) it follows that

$$(7) \qquad \prod_n n\in -X\equiv\ulcorner\sim\Phi(a_n)\urcorner\in T\,.$$

But (7) means that the set $-X$ is represented by the formula $\sim\Phi(x)$. Hence X and $-X$ would be represented in T, and thus they would be recursively enumerable and computable according to the above mentioned theorem of Post, and that would contradict the supposition of our Lemma.

Lemma 3. *If a non-computable set X is represented in each consistent and recursively enumerable extension of a theory T, then T is essentially undecidable.*

Proof. From Lemmas 1 and 2.

In order to obtain the shape of the undecidable sentences we can formulate the following

LEMMA 4. *If a formula $\Psi(x,y)$ strongly represents in a consistent and recursively enumerable theory T a (computable) relation $R(n,k)$, and if the formula $\Phi(x)$ satisfying the condition*

$$\ulcorner\Phi(x)\urcorner = \ulcorner(\exists y)\Psi(x,y)\urcorner \tag{8}$$

represents in T a non-computable set X, then there exists such a number n that

$$\ulcorner(x)\sim\Psi(a_n,x)\urcorner \notin T \quad \text{and} \quad \ulcorner\sim(x)\sim\Psi(a_n,x)\urcorner \notin T \tag{9}$$

and for any k $\ulcorner\sim\Psi(a_n,a_k)\urcorner \in T$.

Proof. In the same manner as in the proof of Lemma 2 we find that if the condition (6) were satisfied, then the set X would be computable. Hence if X is not computable, then there exists a number n such that $\ulcorner\Phi(a_n)\urcorner \notin T$ and $\ulcorner\sim\Phi(a_n)\urcorner \in T$. This means according to (8) that (9). Now suppose that for some k, $\ulcorner\sim\Psi(a_n,a_k)\urcorner \notin T$. Moreover, the formula $\Psi(x,y)$ strongly represents the relation $R(n,k)$. Hence if $\ulcorner\sim\Psi(a_n,a_k)\urcorner \notin T$, then $R(n,k)$ and $\ulcorner\Psi(a_n,a_k)\urcorner \in T$. Thus $\ulcorner(\exists y)\Psi(a_n,y)\urcorner \in T$ according to the rules of quantifiers. But this contradicts the fact that $\ulcorner\Phi(a_n)\urcorner \notin T$ with regard to (8).

2. Applications to arithmetic. In the following considerations we shall apply these lemmas to the proof of the essential undecidability of arithmetic. The applications of the above lemmas have been involved in the reasonings of some authors, for example in the reasoning of Kleene in [4] or of Mostowski in [6]. Mostowski for example has proved that if all computable relations are strongly represented in T and T is ω-consistent, then each recursively enumerable set is representable in T. Namely if X is a recursively enumerable set, then there exists a computable relation R such that

$$\prod_n n \in X \equiv \sum_k R(n,k).$$

Let R be strongly represented by the formula $\Psi(x,y)$; hence it is evident that X is represented by the formula $(\mathrm{E}y)\Psi(x,y)$. This theorem immediately implies by means of our lemmas Gödel's incompleteness Theorem for ω-consistent theories, because there are many non-computable, recursively enumerable sets.

If we do not assume the ω-consistency, then we cannot repeat the above argumentation of Mostowski. It is an interesting problem whether there exists a non-computable recursively enumerable set representable in any theory T in which all computable relations are strongly represented. But for some formalized theories T of the arithmetic of positive integers it is possible to prove that there exist some non-computable recursively enumerable sets represented in T. Let Ar be the formalized theory of arithmetic considered by Mostowski in [7]. We shall prove the representability in Ar of two non-computable and recursively enumerable sets: X_1^{Ar} and X_2^{Ar}. X_1^{Ar} is a diagonal set (strictly defined later) obtained from the universal relation for computable sets. X_2^{Ar} is a simple set (in the sense of Post) defined by Janiczak.

The notion of diagonal set is used in such a meaning that if R is a relation and for each computable set X there exists such k that $n \in X$ if and only if $R(n,x)$, then the set $Z = \underset{n}{E}[\sim R(n,n)]$ is diagonal. Thus a diagonal set can be non-recursively enumerable.

The notion of simple set is used also in a wide sense. Each set Z intersecting any recursively enumerable infinite set and having an infinite complement is a simple set. Hence a simple set can also be non-recursively enumerable. From these definitions it follows immediately that each diagonal set as well as each simple set is not computable.

We shall use in Ar our notation mentioned at the beginning. The sequence $\{a_n\}$ will represent the numerals of the theory Ar, *i. e.* the names of the numbers $1,2,3,\ldots$ Let $a_1 = \ulcorner 1\urcorner$, $a_2 = \ulcorner 2\urcorner$, $a_3 = \ulcorner 3\urcorner$.

THEOREM 1. *If the theory Ar is consistent, then there exists a formula $\Psi(x,y)$ strongly representing in Ar a relation and such that the formula $\Phi(x)$, satisfying the condition (8), represents a non-computable diagonal set X_1^T in any consistent extension T of Ar.*

Proof. According to the well known theorem of Kleene each computable function f can be presented in the canonical form. This fact we can formulate as follows. For each computable function f there exists a number k such that

$$f(n) = E((\mu t)[G(n,t,k)=1]) \tag{10}$$

and

$$\prod_n \sum_t G(n,t,k)=1 \tag{11}$$

where $E(x)$ is one of the pairing functions, (*e. g.* $E(x) = x - [\sqrt{x}]^2$), and G is a computable function universal for the primitive recursive functions in two arguments [2]).

The functions E and G, being computable, are strongly represented in the arithmetic Ar by some function-formulas $\varepsilon(x)$ and $\gamma(x,y,z)$.

[2]) The function G can be elementary recursive in the sense of Kalmar. Cf. [2], p. 42.

It is easy to prove that for each computable function f there exists such a number k that the function-formula

$$\varepsilon\big((\mu y)[\gamma(x,y,a_k)=1]\big) \tag{12}$$

strongly represents the function f in Ar.

Indeed, if $\gamma(x,y,z)$ strongly represents the function $G(n,t,k)$, then the formula $\gamma(x,y,a_k)$ strongly represents the function $F(n,t)=G(n,t,k)$ for k constant. Afterwards if the condition (11) is satisfied, then the formula $(\mu y)[\gamma(x,y,a_k)=1]$ strongly represents the function $(\mu t)[F(n,t)=1]$. Finally the substitution (12) of the formula $(\mu y)[\gamma(x,y,a_k)=1]$ in the formula $\varepsilon(x)$ strongly represents the superposition $E\big((\mu t)[F(n,t)=1]\big)$ of the considered functions. Thus the function f satisfying (10) is strongly represented by the formula (12).

For each consistent extension T of Ar let us set

$$\prod_n n\in X_1^T \equiv \ulcorner\varepsilon\big((\mu y)[\gamma(a_n,y,a_n)=1]\big)\neq 1\urcorner \in T. \tag{13}$$

The set X_1^T, as defined in (13), is represented in T by the formula $\varepsilon\big((\mu y)[\gamma(a_n,y,a_n)=1]\big)\neq 1$. We shall prove that X_1^T is not computable. Suppose that X_1^T is computable. Thus there exists a computable function f such that

$$\prod_n n\in X_1^T \equiv f(n)=1. \tag{14}$$

For the function f there exists a number k such that the formula (12) strongly represents the function f in Ar. Hence also the formula (12) strongly represents the function f in any consistent extension T of Ar. This means according to (4) that

$$\prod_n \ulcorner\varepsilon\big((\mu y)[\gamma(a_n,y,a_k)=1]\big)=a_{f(n)}\urcorner \in T. \tag{15}$$

From (14) and (15) it follows that

$$\prod_n n\notin X_1^T \equiv \ulcorner\varepsilon\big((\mu y)[\gamma(a_n,y,a_k)=1]\big)\neq 1\urcorner \in T. \tag{16}$$

Namely if $n\notin X_1^T$, then according to (14) $f(n)\neq 1$. Hence $\ulcorner a_{f(n)}\neq a_1\urcorner \in T$ because the relation $n=m$ is strongly represented in T by the formula $x=y$. Thus according to (15) and $a_1=\ulcorner 1\urcorner$ we find that

$$\ulcorner\varepsilon\big((\mu y)[\gamma(a_n,y,a_k)=1]\big)\neq 1\urcorner \in T.$$

Conversely, if $\ulcorner\varepsilon\big((\mu y)[\gamma(a_n,y,a_k)=1]\big)\neq 1\urcorner \in T$, then according to (15) the consistency of the theory T implies $\ulcorner a_{f(n)}=1\urcorner \notin T$. Hence $f(n)\neq 1$ and with respect to (14) $n\notin X_1^T$.

From (13) and (16) for $n=k$ we obtain the contradiction $k\in X_1^T \equiv k\notin X_1^T$. Therefore the set X_1^T cannot be computable. It is easy to show that the formula representing the set X_1^T in T can be written in the form $(\exists y)\Psi(x,y)$ where $\Psi(x,y)$ strongly represents a computable relation. Namely: $\ulcorner\varepsilon(1)=1\urcorner\in Ar$ what implies that the formula representing X_1^T can be written in the following manner:

$$(\exists y)\big(\varepsilon(y)\neq 1 \wedge \gamma(x,y,x)=1 \wedge (z)(z\leqslant y \to \gamma(x,z,x)\neq 1)\big).$$

Notice that the set X_1^{Ar} is recursively enumerable; however, if the theory T is not recursively enumerable, then the set X_1^T may also be not recursively enumerable. The same remark is true with regard to the set X_2^T of the next proof.

THEOREM 2. *If the theory Ar* [3] *is consistent, then there exists a formula $\Psi(x,y)$ strongly representing a relation in Ar and such that the formula $\Phi(x)$ satisfying* (8) *represents in any consistent extension T of Ar a non-computable simple set X_2^T.*

Proof. We start from the following definition of the simple set due to Janiczak [4]. Let $G_n(t)$ be a computable function of two variables universal for the primitive recursive functions. A simple set S can be defined as the set of values of the partially recursive function f,

$$f(n)=G_n\big(\mu(t)[G_n(t)>3n]\big), \tag{17}$$

defined over the elements of the recursively enumerable set Z:

$$Z=\mathop{E}_n\Big[\sum_t G_n(t)>3n\Big]. \tag{18}$$

Let h be a computable function enumerating the set Z. Hence the set S of values of the function f is identical with the set of values of the computable function $f\big(h(n)\big)$. Thus S can be defined as follows:

$$k\in S\equiv \sum_n k=f\big(h(n)\big). \tag{19}$$

The set S is a simple set. It is recursively enumerable according to (19). It intersects each recursively enumerable infinite set. Namely with the set enumerated by the function G_n the set S has the common element $f(n)$. And the complement $-S$ is infinite, because

$$f(n)>3n \tag{20}$$

[3]) For proving the Theorem 2 it is convenient to suppose that the theory Ar contains the constant 0. Such a modification of the theory Ar makes no difficulties.

[4]) See [3]. The notion of the simple set was introduced by Post in [8].

according to (17) and (18). Thus between the numbers n and $3n$ there are n numbers which belong to $-S$.

The functions $G_n(t)$ and $h(n)$, being computable, are strongly represented in Ar by some formulas $\gamma(x,y)$ and $\chi(x)$. Hence the computable function $f(h(n))$ is, according to (17), strongly represented in Ar by the formula $\varphi(\chi(x))$, where

(21) $$\ulcorner\varphi(x)\urcorner = \ulcorner\gamma(x,(\mu y)[\gamma(x,y)>3x])\urcorner.$$

Let us set

(22) $$\ulcorner\Phi(x)\urcorner = \ulcorner(\exists z)\, x=\varphi(\chi(z))\urcorner,$$

(23) $$\prod_k k\in X_2^T \equiv \ulcorner\Phi(a_k)\urcorner\in T.$$

The set X_2^T is *ex definitione* represented in T by the formula $\Phi(x)$. We shall prove that the set X_2^T is not computable. As can easily be shown, t su ffices to prove that 1. $S\subset X_2^T$ and 2. the set $-X_2^T$ is infinite, because each set X containing a simple set and having an infinite complement is non-computable. (Namely if X is computable and $-X$ is infinite, then $-X$, being computable, is recursively enumerable and infinite. Thus $S\cap -X\neq 0$, but this is impossible if $S\subset X$.)

1. $S\subset X_2^T$. Indeed if $k\in S$, then according to (19) there exists a number n such that $k=f(h(n))$. The relation $k=f(h(n))$ is representable by the formula $x=\varphi(\chi(y))$. Hence $\ulcorner a_k=\varphi(\chi(a_n))\urcorner\in Ar$ and according to the rules of quantifiers $\ulcorner(\exists z)\, a_k=\varphi(\chi(z))\urcorner\in Ar$. Thus by (22) and (23) $k\in X_2^T$ if $Ar\subset T$.

2. The definition of the set S depends on the choice of the functions $G_n(t)$ and $h(n)$. In order to prove that the set $-X_2^T$ is infinite we choose those particular functions $G_n(t)$ and $h(n)$, for which $\ulcorner(x)\varphi(\chi(x))>3\chi(x)\urcorner\in T$.

Let $G'_n(t)$ be another function universal for the primitive recursive functions in one argument. We suppose that

$$G_n(t)=\begin{cases} G'_{n/2}(t) & \text{if } n \text{ is even,}\\ 3n+1 & \text{if } n \text{ is odd.}\end{cases}$$

Obviously $G_{2n}(t)=G'_n(t)$. Hence the function $G_n(t)$ is universal for primitive recursive functions if $G'_n(t)$ is such. Let $\varrho(x,y)$ strongly represent in Ar the function $G'_n(t)$. Thus the function formula $\gamma(x,y)$ strongly representing the function $G_n(t)$ can be written in the form:

(24) $$\ulcorner\gamma(x,y)\urcorner=\ulcorner\varrho([x/2],y)\big(1\dot{-}(x\dot{-}2[x/2])\big)+(3x+1)(x\dot{-}2[x/2])\urcorner.$$

It is easy to prove that

(25) $$\ulcorner(x)\big(x\dot{-}2[x/2]=1 \vee (x+1)\dot{-}2[(x+1)/2]=1\big)\urcorner\in Ar.$$

Hence if $Ar\subset T$, then by (24) and (25)

(26) $$\ulcorner(x,y)\big(\gamma(x,y)=3x+1 \vee \gamma(x+1,y)=3(x+1)+1\big)\urcorner\in T$$

and

(27) $$\ulcorner(x)\big(\gamma(x,x)>3x \vee \gamma(x+1,x)>3(x+1)\big)\urcorner\in T.$$

Let $\varkappa(x)$ and $\lambda(x)$ be the formulas strongly representing in Ar the pairing functions $K(n)=E(n)$ and $L(n)=[\sqrt{n}]-K(n)$. Formula (27) implies that

(28) $$\ulcorner(x)(\exists z)\big(z\geqslant x\wedge\gamma(\varkappa(z),\lambda(z))>3\varkappa(z)\big)\urcorner\in T.$$

Hence the function formula $\psi(x)$ satisfying

(29) $$\ulcorner\psi(x)\urcorner=\ulcorner(\mu z)\big[z\geqslant x\wedge\gamma(\varkappa(z),\lambda(z))>3\varkappa(z)\big]\urcorner$$

strongly represents in T the computable function

$$j(n)=(\mu u)\big[u\geqslant n\wedge G_{K(u)}(L(u))>3K(u)\big],$$

and the substitution $\varkappa(\psi(x))$ strongly represents the function $K(j(n))$. The set of values of the function $K(j(n))$ is identical with the set Z. Namely if $n\in Z$, then by (18) for some t, $G_n(t)>3n$. Hence

$$G_{K(P(n,t))}\big(L(P(n,t))\big)>3\big(K(P(n,t))\big) \quad \text{where} \quad P(n,t)=(n+t)^2+n.$$

Thus $P(n,t)=j(P(n,t))$, and $n=K\big(j(P(n,t))\big)$. Conversely if $n=K(j(m))$ then $G_n\big(L(j(m))\big)>3n$ and $n\in Z$ according to (18). Thus we can suppose that

(30) $$h(n)=K(j(n))$$

and

(31) $$\ulcorner\chi(x)\urcorner=\ulcorner\varkappa(\psi(x))\urcorner.$$

From (28) and (29) we find that

(32) $$\ulcorner(x)\Big(\gamma\big(\varkappa(\psi(x)),\lambda(\psi(x))\big)>3\varkappa(\psi(x))\Big)\urcorner\in T.$$

Formulas (31), (32) and (21) imply that

(33) $$\ulcorner(x)\big(\varphi(\chi(x))>3\chi(x)\big)\urcorner\in T.$$

When (33) is obtained we prove that $-X_2^T$ is infinite in the following manner. We shall prove that for each n there exists such a number k that $n \leqslant k \leqslant 3n$ and $k \in -X_2^T$. This means according to (23) that $\ulcorner \Phi(a_k) \urcorner \notin T$. Indeed, supposing that

$$\ulcorner \Phi(a_n) \wedge \Phi(a_{n+1}) \wedge \ldots \wedge \Phi(a_{3n}) \urcorner \in T \tag{34}$$

we obtain a contradiction. Namely if (34), then according to (22)

$$\ulcorner (\exists x_n, \ldots, x_{3n})\, a_n = \varphi\chi(x_n) \wedge a_{n+1} = \varphi\chi(x_{n+1}) \wedge \ldots \wedge a_{3n} = \varphi\chi(x_{3n}) \urcorner \in T\,. \tag{35}$$

For $x_n, \ldots, x_{3n}$ according to (33) we can prove in T that

$$a_n > 3\chi(x_n) \wedge a_{n+1} > 3\chi(x_{n+1}) \wedge \ldots \wedge a_{3n} > 3\chi(x_{3n})\,. \tag{36}$$

Hence

$$\chi(x_n) < a_n \wedge \chi(x_{n+1}) < a_n \wedge \ldots \wedge \chi(x_{3n}) < a_n\,. \tag{37}$$

There exist only n numerals a_i such that $a_i < a_n$ and we have $2n$ formulas $\chi(x_i)$ for $n \leqslant i \leqslant 3n$, thus from (37) it can be proved in T that

$$\begin{aligned} \chi(x_n) = \chi(x_{n+1}) \vee \chi(x_n) = \chi(x_{n+2}) \vee \ldots \vee \chi(x_n) = \chi(x_{3n}) \vee \ldots \vee \chi(x_i) \\ = \chi(x_j) \vee \ldots \vee \chi(x_{3n-1}) = \chi(x_{3n}) \quad \text{for} \quad i \neq j \text{ and } n \leqslant i,\ j \leqslant 3n. \end{aligned} \tag{38}$$

From (35) and (38) it follows that

$$\ulcorner a_n = a_{n+1} \vee \ldots \vee a_i = a_j \vee \ldots \vee a_{3n-1} = a_{3n} \urcorner \in T \quad \text{for} \quad i \neq j \quad \text{and} \quad n \leqslant i, j \leqslant 3n. \tag{39}$$

But this is impossible if T is consistent and $Ar \subset T$ because $\ulcorner a_i \neq a_j \urcorner \in Ar$ for $i \neq j$.

I am unable to repeat this proof with respect to theories narrower than Ar, *e. g.* to the arithmetic of Robinson, considered in [10]. There are difficulties in proving (25).

It is an interesting problem whether it is possible to prove the representability in arithmetic of other well known recursively enumerable non-computable sets. The proof of the undecidability of arithmetic given by Kleene in [5], p. 308 can be considered as the proof of representability in arithmetic of recursively non-separable sets [5]).

The proof of Uspenski in [11] has a similar character.

From Theorem 1 or 2 using Lemmas 3 and 4 we can obviously obtain the following

[5]) The application of theory of computable functions to the proof of the undecidability of the arithmetic was suggested by Kleene already in [4].

THEOREM 3. *If the theory Ar is consistent, then it is essentially undecidable and there exists such a formula $\Psi(x,y)$ strongly representing a computable relation in Ar that for each recursively enumerable and consistent extension T of Ar there are infinitely many such numbers n_i that*

$$\ulcorner (x)\Psi(a_{n_i}, x) \urcorner \in T \quad \text{and} \quad \ulcorner \sim (x)\Psi(a_{n_i}, x) \urcorner \in T$$

and for any k $\ulcorner \Psi(a_{n_i}, a_k) \urcorner \in T$.

The argument of this paper can be formulated in another form by means of the notion of models. For example in Lemma 3 we can say that a theory T is essentially undecidable provided that there exists in T a formula which defines in any computable model of T a non computable set.

In comparision with the proof of undecidability due to Rosser [9], the method exposed in this paper is less constructive. The proofs of Lemmas 2 and 4 are non intuintionistical. Hence the numbers n_i in Theorem 3 are not "given" in the intuitionistic sense.

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