On the spaces of functions satisfying Dini's condition

by

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We shall always suppose that functions denoted by $f(x)$ are continuous, periodic with period $l=1$, finite and defined for every value of the real variable x.

In this paper we shall denote by $w(t)$, $w_1(t)$ functions defined and differing from zero for $t>0$, monotonic, non-decreasing and tending to zero for $t\to 0$.

We introduce the following notation:

$$W(\tau)=\int_{\tau}^{1}\frac{dt}{w(t)} \quad \text{and respectively} \quad W_1(\tau)=\int_{\tau}^{1}\frac{dt}{w_1(t)}.$$

As a general supposition we take

$$\lim_{\tau\to+0} W_1(\tau)=\infty. \tag{1}$$

We shall suppose additionally that

$$(*) \qquad \int_0^1 \frac{t}{w(t)}\,dt<\infty \quad \text{and respectively} \quad \int_0^1 \frac{t}{w_1(t)}\,dt<\infty,$$

$$(**) \qquad \lim_{t\to+0} W(t)\cdot\frac{w(t)}{t}=g>1,$$

$$({*}{*}{*}) \qquad \overline{\lim_{t\to+0}}\,\frac{w_1(2t)}{w_1(t)}=s<\infty.$$

We denote by D_w the space of all functions $f(x)$ satisfying Dini's generalized condition, *i. e.* the inequality

$$\int_0^1 \frac{|f(x+t)-f(x)|}{w(t)}\,dt\leqslant 1$$

for every x. The distance ϱ between two elements of this space we define by

$$\varrho(f_1,f_2)=\max_{0\leqslant x<l}|f_1(x)-f_2(x)|.$$

Space D_w is a complete space.

Let S denote a set of functions $f(x)$ belonging to space D_w and satisfying, for every x, the condition

$$\int_0^1 \frac{|f(x+t)-f(x)|}{w_1(t)}\,dt=\infty . \tag{2}$$

Assuming that functions $w(t)$ and $w_1(t)$ satisfy the supposition (1), $(*)$ [1]), $(**)$, $({}^{**}_{*})$, and further, condition (3), we shall prove that S is a residual set in the space D_w.

The Theorem remains valid if space D_w is replaced by the space of all continuous and periodic functions $f(x)$ with period $l=1$. In this case with regard to $w_1(t)$ we suppose only that it satisfies $({}^{**}_{*})$. In this manner we obtain the generalization of Kaczmarz's Theorem [2]).

Lemma 1 [3]). *If $w(t)$ satisfies $(*)$ and $(**)$, then for every τ, where $0<\tau\leqslant\tau_0<1$ and L is a certain positive constant dependent on τ_0, the inequality*

$$\int_0^\tau \frac{t}{w(t)}\,dt\leqslant L\tau W(\tau)$$

is valid.

Lemma 2. *If $w_1(t)$ satisfies $({}^{**}_{*})$, then*

$$\overline{\lim_{\tau\to+0}}\frac{W_1(\tau)}{W_1(2\tau)}\leqslant\frac{s}{2}.$$

Lemma 3. If $\lim\limits_{t\to+0}\dfrac{w_1(t)}{w(t)}=0$, then $\lim\limits_{\tau\to+0}\dfrac{W(\tau)}{W_1(\tau)}=0$.

Lemma 4. *Given a function $f(x)$ belonging to space D_w we can find a sequence $\{f_n(x)\}$ of functions also belonging to space D_w satisfying Lipschitz's condition and tending uniformly to $f(x)$.*

Proof. For the proof it suffices to take $f_n(x)$ defined by the formula

$$f_n(x)=n\int_x^{x+1/n} f(u)\,du$$

and show that $f(x)$ belong to D_w. The latter results from

$$\int_0^1\frac{|f_n(x+t)-f_n(x)|}{w(t)}\,dt\leqslant n\int_0^{1/n}d\tau\int_0^1\frac{f(x+\tau+t)-f(x+\tau)}{w(t)}\,dt\leqslant 1 .$$

[1]) In the spaces satisfying (12) the supposition $(*)$ can be omitted for both $w(t)$ and $w_1(t)$ (cf. Theorem 1*).

[2]) Cf. Kaczmarz [1]. In this work the proof concerns the case $w_1(t)=t$.

[3]) The proof of this and of the next two Lemmas can be found in my paper [2].

We shall consider the functional space D_w previously defined which, as is easy to prove, is a complete space, and the set S of functions $f(x)$ belonging to D_w and satisfying (2) for every x. We have assumed, in general, that $w_1(t)$ satisfies (1) [4]). Making the additional suppositions $(*)$, $(**)$, $({}^{**}_{*})$ for $w(t)$ and $w_1(t)$ we express the following Theorem for set S:

Theorem 1. *If $w_1(t)$ satisfies the condition*

$$\lim_{t\to+0}\frac{w_1(t)}{w(t)}=0 , \tag{3}$$

then set S is a set residual in space D_w.

Proof. We denote by Z_n a set of functions $f(x)$ of space D_w satisfying

$$\int_0^1\frac{|f(x+t)-f(x)|}{w_1(t)}\,dt\leqslant n \tag{4}$$

for a certain x. Let

$$Z=\sum_{n=1}^{\infty}Z_n .$$

Then from the definition of set S it follows that

$$S=D_w-Z .$$

For the proof of the Theorem it suffices to show that each of the sets Z_n is non-dense in space D_w.

Suppose, for the proof, that for a certain value n_0 the set Z_{n_0} is not non-dense in space D_w. Since each of the sets is closed in space D_w, there would exist in space D_w a sphere $K_{\varrho_0}(f_0)$ with centre $f_0(x)$ and radius ϱ_0, belonging completely to Z_{n_0}.

On the basis of Lemma 4, there exists in D_w a sequence of functions $\{f_n(x)\}$ tending uniformly to $f_0(x)$, whose expressions satisfy Lipschitz's condition. We can therefore find in space D_w a sphere $K_{\varrho_1}(y_1)$ of centre $y_1(x)$ and radius ϱ_1 such that

$$K_{\varrho_1}(y_1)\subset K_{\varrho_0}(f_0)\subset Z_{n_0}$$

with $y_1(x)=\theta f_N(x)$, where $0<\theta<1$ and $f_N(x)$ is a sufficiently distant expression of the sequence $\{f_n(x)\}$. Function $y_1(x)$ thus satisfies Lipschitz's condition and belongs to space D_w. Consequently, the inequalities

$$\int_0\frac{|y_1(x+t)-y_1(x)|}{w(t)}\,dt\leqslant\theta<1 \tag{5}$$

[4]) If we did not accept supposition (1), set S would be empty (also in the space C of continuous functions).

and, with regard to (*),

$$\int_0^1 \frac{|y_1(x+t)-y_1(x)|}{w_1(t)}\,dt \leqslant C \tag{6}$$

are valid for every x, where

$$C = k_1 \int_0^1 \frac{t}{w_1(t)}\,dt$$

and k_1 is the Lipschitz constant of function $y_1(x)$.

Let us now examine the function $y_2(x) = a\varphi(bx)$, where $\varphi(x)$ is a certain non-constant function, periodic with period l, satisfying Lipschitz's condition. We shall choose the constants a, b later, postulating that b is an integer greater than $2l$. By D we shall denote the oscillations of $\varphi(x)$ in the interval $0 \leqslant x < l$, by k its Lipschitz constant.

On account of

$$a \int_{l/b}^1 \frac{|\varphi(b(x+t)) - \varphi(bx)|}{w(t)}\,dt \leqslant aD \int_{l/b}^1 \frac{dt}{w(t)} = aDW\left(\frac{l}{b}\right)$$

on the basis of Lemma 1, taking therein $\tau_0 = 1/2$, and by

$$a \int_0^{l/b} \frac{|\varphi(b(x+t)) - \varphi(bx)|}{bt} \cdot \frac{bt}{w(t)}\,dt \leqslant kalLW\left(\frac{l}{b}\right),$$

we find, for every x, that

$$\int_0^1 \frac{|y_2(x+t)-y_2(x)|}{w(t)}\,dt \leqslant GaW\left(\frac{l}{b}\right), \tag{7}$$

where G is a constant independent of b.

Putting $m = [b/l]$ and substituting $bt = u$ we obtain the inequalities

$$\int_{l/b}^1 \frac{|y_2(x+t)-y_2(x)|}{w_1(t)}\,dt \geqslant \frac{a}{b} \sum_{i=1}^{m-1} \int_{il}^{(i+1)l} \frac{|\varphi(bx+u) - \varphi(bx)|}{w_1(u/b)}\,du$$

$$\geqslant \frac{ad}{b} \sum_{i=1}^{m-1} \left(w_1\left(\frac{(i+1)l}{b}\right)\right)^{-1} \geqslant \frac{ad}{l} \int_{2l/b}^{(m+1)l/b} \frac{dt}{w_1(t)} \geqslant \frac{ad}{l} W_1\left(\frac{2l}{b}\right),$$

where $d = \min_{0 \leqslant x < l} \int_0^l |\varphi(x+u) - \varphi(x)|\,du > 0$.

Taking into account Lemma 2, we shall obtain the inequality

$$\int_{l/b}^1 \frac{|y_2(x+t)-y_2(x)|}{w_1(t)}\,dt > BaW_1\left(\frac{l}{b}\right) \tag{8}$$

valid for every x, where B is a certain constant greater than zero and independent of b.

Besides assuming that b is an integer greater than $2l$, let us take for b a value so large as to satisfy the inequalities

$$\frac{1}{W_1(l/b)} < \frac{\varrho_1 B}{(C+n_0)\max_{0 \leqslant x < l} |\varphi(x)|} \tag{9}$$

and

$$\frac{W(l/b)}{W_1(l/b)} < \frac{B(1-\theta)}{(C+n_0)G}, \tag{10}$$

which is possible considering (3) and Lemma 3.

Having fixed the value of b, let us take

$$a = \frac{C+n_0}{BW_1(l/b)}. \tag{11}$$

We have thus finally chosen the coefficients a, b and thus defined the function $y_2(x)$.

We write $f^*(x) = y_1(x) + y_2(x)$.

From (7), (10), (11) it follows that

$$\int_0^1 \frac{|y_2(x+t)-y_2(x)|}{w(t)}\,dt \leqslant 1-\theta \quad \text{for every} \quad x$$

and, considering (5), that $f^*(x)$ belongs to D_w.

Considering (9) and (11) we should have

$$\varrho(f^*, y_1) = \max_{0 \leqslant x < l} |a\varphi(bx)| < \varrho_1,$$

whence it would follow that $f^*(x)$ belongs to the set Z_{n_0}.

On the other hand, comparing (8) and (11), we should obtain for every x

$$\int_0^1 \frac{|y_2(x+t)-y_2(x)|}{w_1(t)}\,dt > C + n_0,$$

and hence, considering (6), we should have

$$\int_0^1 \frac{|f^*(x+t)-f^*(x)|}{w_1(t)}\,dt > n_0 \quad \text{for every} \quad x,$$

contrary to the fact that $f^*(x)$ belongs to Z_{n_0}.

The supposition that Z_{n_0} is non-dense would lead to contradictions and thus Theorem 1 has been completely proved.

We note that the supposition

$$\int_0^1 \frac{t}{w_1(t)}\,dt<\infty$$

can be omitted if we impose on space D_w the condition (12) and also evidently supposition (*) for $w(t)$.

THEOREM 1*. *Let $w_1(t)$ satisfy* (1) *and* ($\overset{*}{*}\!*$), *and $w(t)$ the supposition* (**). *If with a certain γ_0, where $\gamma_0>1$, we have*

$$\lim_{t\to+0}\frac{t^2|\log t|^{\gamma_0}}{w(t)}=0 \tag{12}$$

and if $w_1(t)$ satisfies (3), *then set S is residual in space D_w.*

Proof. Function $t^2|\log t|^{\gamma_0}$ $(\gamma_0>1)$ is increasing for $0<t<t_0$. Let us define the function $w_1^*(t)$ as follows:

$$w_1^*(t)=\begin{cases}\max\left(w_1(t_0),t_0^2|\log t_0|^{\gamma_0}\right) & \text{if}\quad t\geqslant t_0,\\ \max\left(w_1(t),t^2|\log t|^{\gamma_0}\right) & \text{if}\quad 0<t<t_0.\end{cases}$$

We note that

$$\int_0^1 \frac{t}{w_1^*(t)}\,dt<\infty$$

and, moreover, on the basis of the suppositions concerning $w_1(t)$ and on account of (12), $w_1^*(t)$ satisfies both ($\overset{*}{*}\!*$) and (3). By Lemma 3 it satisfies also (1).

Applying Theorem 1 to $w_1^*(t)$ and noting the inequality

$$\int_0^1 \frac{|f(x+t)-f(x)|}{w_1^*(t)}\,dt\leqslant\int_0^1 \frac{|f(x+t)-f(x)|}{w_1(t)}\,dt \tag{13}$$

valid for every x, we have proved Theorem 1*.

We note that the suppositions of Theorem 1* are satisfied, for example, by $w(t)=t^{1+\delta}|\log t|^{\gamma}$ for $\delta=1$, with $\gamma>1$ and for $0\leqslant\delta<1$ with an arbitrary value of γ.

Theorem 1 remains valid if we replace D_w by the space C of all continuous functions. In that case it has the following form:

THEOREM 2. *If $w_1(t)$ satisfies suppositions* (1) *and* ($\overset{*}{*}\!*$), *then the set S of functions satisfying condition* (2) *for every x is a residual set in the space C of all continuous and periodic functions $f(x)$.*

Proof. The Theorem is proved with suppositions (1), (*) and ($\overset{*}{*}\!*$) concerning $w_1(t)$, in the same way as Theorem 1. The proof is simplified since formulas (5), (7), (10) and (3) can be omitted. Also Lemmas 1 and 3 are superfluous, just as suppositions (*) and (**) are superfluous for $w(t)$, since it is easy to see, without the need of referring to those formulas, that $f^*(x)\in C$.

We shall now prove Theorem 2 with suppositions (1), ($\overset{*}{*}\!*$) and the supposition contrary to (*) for $w_1(t)$.

We define the function $w_1^*(t)$ as follows:

$$w_1^*(t)=\max\left(t,w_1(t)\right)\quad \text{for every}\quad t.$$

Evidently $w_1^*(t)$ satisfies ($\overset{*}{*}\!*$) and (*). We shall prove that $w_1^*(t)$ satisfies also (1) and thus that Theorem 2 is valid for $w_1^*(t)$.

If the inequality $w_1^*(t)=t\geqslant w_1(t)$ is valid with $0<t<t_0$ for certain t_0, then $w_1^*(t)$ evidently satisfies (1).

On the contrary, considering $0<t<e^{-1}$, it suffices to compare the integrals of the functions $t/w_1(t)$ and $1/w_1^*(t)$ in those intervals in which $w_1^*(t)=t\geqslant w_1(t)$ everywhere. And either the inequality $t/w_1(t)\leqslant 1/w_1^*(t)$ is valid for every t in almost all intervals, and thus $w_1^*(t)$ satisfies (1), or we have $t_n'\leqslant t_n^2<t_n$ for an infinite number of intervals $t_n'\leqslant t\leqslant t_n$ $(n=1,2,\ldots)$. In the last case $w_1^*(t)$ satisfies also (1).

Thus Theorem 2 is true for $w_1^*(t)$. By inequality (13), valid for $w_1^*(t)$, Theorem 2 is true for $w_1(t)$ also in the case contrary to (*).

Thus Theorem 2 is completely proved.

References

[1] S. Kaczmarz, *Integrale vom Dini'schen Typus*, Stud. Math. 3 (1931), p. 189-199.

[2] E. Tarnawski, *Continuous functions considered from the standpoint of Dini's conditions*, this volume, p. 3-22.

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