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On a concept of dependence for continuous mappings *

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The present note is concerned with a concept of dependence of mappings. This concept, belonging to the homotopy theory, is extremely elementary, but it quickly leads to hard problems. Our knowledge of the relations between the dependence of mappings and other concepts of the algebraic topology is meagre. Only in the classical case (of H. Hopf), concerning the mappings of a compactum of dimension $\leqslant n$ into the Euclidean n-sphere, the concept of dependence is partly reduced to the homology theory. In other cases only partial results are obtained.

1. Only metric spaces will be considered. We denote by $Y_0^{X_0}$ the set of all continuous mappings of a space X_0 into subsets of another space Y_0. If X_0 is a compactum, we define a metric in $Y_0^{X_0}$ by setting

$$\varrho(f,g)=\sup_{x\in X_0}\varrho\big(f(x),g(x)\big) \quad \text{for every} \quad f,g\in Y_0^{X_0}.$$

In this paper we shall restrict ourselves to the case where X_0 is a compactum and Y_0 is an ANR set (= absolute neighbourhood retract). Then the space $Y_0^{X_0}$ is locally connected (even locally contractible). The component of $Y_0^{X_0}$ containing a given function $f\in Y_0^{X_0}$ will be denoted by $[f]$ and called the *homotopy class* of f. Two functions $f,g\in Y_0^{X_0}$ belonging to the same homotopy class are said to be *homotopic*.

A function $f\in Y_0^{X_0}$ is said to be *extendable* over a space $X\supset X_0$ (with respect to Y_0) provided that there exists a function $f'\in Y_0^{X}$ (called *extension* of f) satisfying the condition

$$f'(x)=f(x) \quad \text{for every} \quad x\in X_0.$$

It is known (see for instance [9], p. 86) that if f is extendable over X and $g\in[f]$, then g is also extendable over X. The functions $f\in Y_0^{X_0}$ extendable over every compactum $X\supset X_0$ will be called *zero-functions* (cf. [4]). Evidently, they coincide with the functions homotopic with functions mapping X_0 onto singular points of Y_0.

* Most of the results were published without proof in a preliminary report [5].

Let Φ be a subset of the space $Y_0^{X_0}$. A function $g \in Y_0^{X_0}$ is said to be *dependent on the set* Φ provided for every compact space $X \supset X_0$ the existence of an extension over X (with respect to Y_0) for every function $f \in Y_0^{X_0}$ implies the existence of an extension over X (with respect to Y_0) for g. The set of all functions $g \in Y_0^{X_0}$ dependent on the set Φ will be denoted by $\mathfrak{R}(\Phi)$. In the case where Φ contains only a finite number of functions $f_1, f_2, \ldots, f_k$ we shall write $\mathfrak{R}(f_1, f_2, \ldots, f_k)$ instead of $\mathfrak{R}(\Phi)$.

Obviously, $f \in \mathfrak{R}(\Phi)$ implies $[f] \subset \mathfrak{R}(\Phi)$. On the other hand, if $[\Phi]$ denotes the union of all homotopy classes $[f]$ with $f \in \Phi$, then $\mathfrak{R}([\Phi]) = \mathfrak{R}(\Phi)$. Consequently, the relation of the dependence $g \in \mathfrak{R}(\Phi)$ is in fact a relation between the homotopy class $[g]$ and the homotopy classes $[f]$, where $f \in \Phi$.

The following relations are immediate consequences of the definition of the $\mathfrak{R}(\Phi)$:

$$\Phi \subset \mathfrak{R}(\Phi) = \mathfrak{R}(\mathfrak{R}(\Phi)),$$
$$\mathfrak{R}(\Phi_1 \cup \Phi_2) \supset \mathfrak{R}(\Phi_1) \cup \mathfrak{R}(\Phi_2),$$
$$\mathfrak{R}(\Phi_1 \cap \Phi_2) \subset \mathfrak{R}(\Phi_1) \cap \mathfrak{R}(\Phi_2).$$

Moreover, let us observe that $\mathfrak{R}(0)$ coincides with the set of all zero-functions belonging to $Y_0^{X_0}$ and that $\mathfrak{R}(f)$ is the same as the set of all functions $g \in Y_0^{X_0}$ such that for every compact space $X \supset X_0$ the extendability of f over X (with respect to Y_0) implies the extendability of g over X. The functions g with the last property are called *multiples* of f and f is said to be a *divisor* of g (symbolically: f/g; cf. [4]).

Example. Let Y_0 be a topological group. Then $Y_0^{X_0}$ is also a topological group. For every $f, g \in Y_0^{X_0}$ let us denote by $f \circ g$ the product of f and g (considered as elements of the group $Y_0^{X_0}$) and f^{-1} and g^{-1} the inverses of the elements f and g. Let us observe that for every set $\Phi \subset Y_0^{X_0}$ the set $\mathfrak{R}(\Phi)$ is a subgroup of $Y_0^{X_0}$. In fact, if $f, g \in \mathfrak{R}(\Phi)$ and if every function belonging to Φ is extendable over X (with respect to Y_0) then there exist also the extensions $f^*, g^* \in Y_0^{X_0}$ of f and g over X. But Y_0^X is also a topological group and the functions $f^* \circ g^*$ and f^{*-1}, g^{*-1} are extensions over X of $f \circ g$, f^{-1} and g^{-1}, respectively. Hence $f \circ g$, f^{-1}, $g^{-1} \in \mathfrak{R}(\Phi)$. It follows that if (Φ) denotes the intersection of all subgroups of $Y_0^{X_0}$ containing Φ, then $(\Phi) \subset \mathfrak{R}(\Phi)$. Moreover, if $\varphi \in (\Phi)$ and $\psi \in Y_0^{Y_0}$, then $\psi\varphi \in \mathfrak{R}(\Phi)$, since the extendability of φ over a space X implies the extendability of $\psi\varphi$ over X. The question whether every function $f \in \mathfrak{R}(\Phi)$ is homotopic with a function of the form $\psi\varphi$ with $\psi \in (\Phi)$ and $\psi \in Y_0^{Y_0}$ remains open.

2. It is sometimes convenient to modify a little the concept of dependence. We shall say that a function $g \in Y_0^{X_0}$ is *dependent on the set* $\Phi \subset Y_0^{X_0}$ *in the dimension* m if for every space $X \supset X_0$ satisfying the condition

$$\dim(X - X_0) \leqslant m$$

the existence of an extension over X (with respect to Y_0) for every function $f \in \Phi$ implies the existence of an extension over X of the function g. The set of all functions $g \in Y_0^{X_0}$ dependent on Φ in the dimension m will be denoted by $\mathfrak{R}_m(\Phi)$; if Φ contains only a finite number of functions $f_1, f_2, \ldots, f_k$ we shall write $\mathfrak{R}_m(f_1, f_2, \ldots, f_k)$ instead of $\mathfrak{R}_m(\Phi)$.

Evidently, the relation $g \in \mathfrak{R}_m(\Phi)$ of dependence in the dimension m is in fact a relation between the homotopy class $[g]$ and the homotopy classes $[f]$, where $f \in \Phi$. The functions belonging to $\mathfrak{R}_m(0)$ are said to be *zero-functions in the dimension* m, and the functions belonging to $\mathfrak{R}_m(f)$ — *multiples of* f *in the dimension* m. If $g \in \mathfrak{R}_m(f)$, then f is said to be a *divisor of* g *in the dimension* m (symbolically: f_m/g; see [1]). Moreover, we have:

$$\Phi \subset \mathfrak{R}_m(\Phi) = \mathfrak{R}_m[\mathfrak{R}_m(\Phi)],$$
$$\mathfrak{R}_m(\Phi_1 \cup \Phi_2) \supset \mathfrak{R}_m(\Phi_1) \cup \mathfrak{R}_m(\Phi_2),$$
$$\mathfrak{R}_m(\Phi_1 \cap \Phi_2) \subset \mathfrak{R}_m(\Phi_1) \cap \mathfrak{R}_m(\Phi_2),$$
$$\mathfrak{R}_m(\Phi) \supset \mathfrak{R}_{m+l}(\Phi) \supset \mathfrak{R}(\Phi) \quad \text{for every} \quad m, l = 0, 1, 2, \ldots$$

It is easy to show by examples that for $m \leqslant \dim X_0$ the sets $\mathfrak{R}_m(\Phi)$ differ in general from the set $\mathfrak{R}(\Phi)$. The question whether for $m > \dim X_0$ the set $\mathfrak{R}_m(\Phi)$ may differ from the set $\mathfrak{R}(\Phi)$ remains open.

3. Let $f_1, f_2, \ldots, f_k$ be a finite sequence of functions belonging to $Y_0^{X_0}$. We shall say that these functions are *separate* if there exist a point $y \in Y_0$ and disjoint open subsets $G_1, G_2, \ldots, G_k$ of X_0 such that

$$f_i(x) = y_0 \quad \text{for every} \quad x \in X_0 - G_i.$$

Then we shall say that the point y_0 and the sets $G_1, G_2, \ldots, G_k$ *realize* a separation of the functions $f_1, f_2, \ldots, f_k$.

The functions $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$ are said to be *separable* provided that there exist separate functions $f_1', f_2', \ldots, f_k'$ homotopic respectively to $f_1, f_2, \ldots, f_k$.

Examples. Obviously, one function is always separate. It is easy to observe that in the case where X_0 is a Euclidean sphere and Y_0 is connected every finite set of functions $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$ is separable (cf. [7], p. 10, and also [3], p. 234). Moreover, it is known (see [1], [2], [3] and [11]) that in the case where Y_0 is a Euclidean n-sphere and $\dim X_0 < 2n$ every two functions $f_1, f_2 \in Y_0^{X_0}$ are separable. On the other hand, it is known (see [8] and [2]) that if $X_0 = Y_0 \times Y_0$, where Y_0 denotes a Euclidean

n-sphere and n is even, then the functions $f_1, f_2 \in Y_0^{X_0}$ defined by the formulas

$$f_1(y_1, y_2) = y_1, \quad f_2(y_1, y_2) = y_2$$

for every $(y_1, y_2) \in Y_0 \times Y_0$ are not separable.

Remark. A system of k functions $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$ may be considered as one continuous function

$$\varphi(x) = \big(f_1(x), f_2(x), \ldots, f_k(x)\big)$$

with the range X_0 and the values belonging to the Cartesian product

$$Z = Y_1 \times Y_2 \times \ldots \times Y_k, \quad \text{where} \quad Y_i = Y_0 \quad \text{for} \quad i = 1, 2, \ldots, k.$$

It is clear that if $f_1', f_2', \ldots, f_k' \in Y_0^{X_0}$ and

$$\varphi'(x) = \big(f_1'(x), f_2'(x), \ldots, f_k'(x)\big),$$

then the homotopy of φ and φ' (in Z^{X_0}) is equivalent to the homotopy of f_i to f_i' for every $i = 1, 2, \ldots, k$.

Let us choose a fixed point $y \in Y_0$ and let us denote by $Z(y)$ the subset of Z consisting of all points $(y_1, y_2, \ldots, y_k)$ such that at most one of the coordinates $y_1, y_2, \ldots, y_k$ differs from y. It is manifest that a necessary and sufficient condition for the functions $f_1, f_2, \ldots, f_k$ to be separate is that Y_0 contain a point y_0 such that $\varphi(X_0) \subset Z(y_0)$. Then the separation of $f_1, f_2, \ldots, f_k$ is realized by the point y_0 and the open sets

$$G_i = f_i^{-1}\big(Y_0 - (y_0)\big).$$

It follows that a necessary and sufficient condition for $f_1, f_2, \ldots, f_k$ to be separable is that there exist a point $y_0 \in Y_0$ such that the function φ is homotopic to a function $\varphi' \in Z^{X_0}$ with values belonging to the set $Z(y_0)$.

THEOREM. *Let $f_1, f_2, \ldots, f_k$ and $g_1, g_2, \ldots, g_k$ be functions mapping a compactum X_0 into a connected* ANR *set Y_0. If $f_1, f_2, \ldots, f_k$ are separable and if f_i/g_i, for $i = 1, 2, \ldots, k$, then $g_1, g_2, \ldots, g_k$ are also separable.*

Proof. Without loss of generality we may assume that $f_1, f_2, \ldots, f_k$ are separate and that a point $y_0 \in Y_0$ and the open sets $G_1, G_2, \ldots, G_k$ realize the separation of $f_1, f_2, \ldots, f_k$. By f_i/g_i there exists (see [4], p. 81) a function $\psi_i \in Y_0^{Y_0}$ such that $\psi_i f_i$ is homotopic to g_i. Evidently, ψ_i may be replaced by any function homotopic to itself. Since Y_0 is connected, we may assume that $\psi_i(y_0) = y_0$. Hence

$$\psi_i f_i(x) = \psi_i(y_0) = y_0 \quad \text{for every} \quad x \in X_0 - G_i,$$

i. e., the point y_0 and the sets $G_1, G_2, \ldots, G_k$ realize the separation of the functions $\psi_1 f_1, \psi_2 f_2, \ldots, \psi_k f_k$ homotopic respectively to the functions $g_1, g_2, \ldots, g_k$.

4. Now we shall prove a theorem concerning the separability of functions mapping X_0 in a Euclidean sphere Y_0. We begin with the following

LEMMA 1. *Let Q^n be the Euclidean n-cube consisting of points $(t_1, t_2, \ldots, t_n)$ with $0 \leqslant t_i \leqslant 1$ for $i = 1, 2, \ldots, n$ and let*

$$Q = Q_1 \times Q_2 \times \ldots \times Q_k, \quad \text{where} \quad Q_\nu = Q^n \quad \text{for} \quad \nu = 1, 2, \ldots, k.$$

Let us choose, for $\nu = 1, 2, \ldots, k$, a point b_ν lying in the interior of Q_ν. Let M denote the subset of Q consisting of all points $(x_1, x_2, \ldots, x_k) \in Q_1 \times Q_2 \times \ldots \times Q_k$ such that at most one of the coordinates x_ν belongs to the interior of Q_ν, and let N denote the subset of Q consisting of all points $(x_1, x_2, \ldots, x_k) \in Q_1 \times Q_2 \times \ldots \times Q_k$ such that for some two indices $\mu \neq \nu$ we have $x_\mu = b_\mu$ and $x_\nu = b_\nu$. Then there exists a retraction by deformation [1]) *$r(x, t)$ of the set $Q - N$ to the set M satisfying the condition*

(1) *$r(x, t) = \big(y_1(x, t), y_2(x, t), \ldots, y_k(x, t)\big)$, where $x = (x_1, x_2, \ldots, x_k) \in Q - N$ and if for an index ν the coordinate x_ν of x lies on the boundary of Q_ν, then $y_\nu(x, t) = x_\nu$ for every $0 \leqslant t \leqslant 1$.*

Proof. Let M_l, where $0 \leqslant l < k$, denote the subset of Q consisting of all points $(x_1, x_2, \ldots, x_k)$ such that for at least l indices ν the point x_ν belongs to the boundary of Q_ν. Hence $M_0 = Q$ and $M_{k-1} = M$. Let us prove that

(2) *If $l \leqslant k-2$ then there exists a retraction by deformation $r(x, t)$ of the set $M_l - N$ to the set $M_{l+1} - N$ satisfying condition* (1).

Let $\nu_1, \nu_2, \ldots, \nu_l$ be a system of l different natural indices $\leqslant k$. Consider the set $M(\nu_1, \nu_2, \ldots, \nu_l) \subset Q$ of all points $(x_1, x_2, \ldots, x_k) \in Q$ such that each of the coordinates x_{ν_j}, $j = 1, 2, \ldots, l$, lies on the boundary of Q_{ν_j}. It is evident that the sets $M(\nu_1, \nu_2, \ldots, \nu_l)$ are closed, that their union coincides with M_l, and that the intersection of any two of them is a subset of M_{l+1}. We infer that to complete the proof of (2) it suffices to show that there exists a retraction by deformation $r(x, t)$ of the set

[1]) A function $r \in X^X$ is said to be a *retraction* of X, if $r(x) = x$ for every $x \in r(X)$. The set of values $r(X)$ of a retraction $r \in X^X$ is said to be a *retract* of X. By a *retraction by deformation* of X to a set $Y \subset X$ we understand a continuous function $r(x, t)$ defined for $x \in X$ and $0 \leqslant t \leqslant 1$ with values belonging to X such that $r(x, 0)$ is the identity and $r(x, 1)$ is a retraction of X to Y. Then the set Y is said to be a *deformation retract* of X.

$M(\nu_1,\nu_2,\ldots,\nu_l)-N$ to the set $M_{l+1}\cap[M(\nu_1,\nu_2,\ldots,\nu_l)-N]$ satisfying condition (1) and the condition

(3) $r(x,t)=x$ for every $x\in M_{l+1}\cap[M(\nu_1,\nu_2,\ldots,\nu_l)-N]$ and $0\leqslant t\leqslant 1$.

To simplify the notation we shall consider the case where the indices $\nu_1,\nu_2,\ldots,\nu_l$ coincide with $1,2,\ldots,l$. Hence $M(\nu_1,\nu_2,\ldots,\nu_l)$ is the set of all points $(x_1',x_2',\ldots,x_l',x_{l+1},\ldots,x_k)\in Q_1\times Q_2\times\ldots\times Q_k$ with x_i' belonging to the boundary of Q_i, for $i=1,2,\ldots,l$, and with x_j belonging to Q_j, for $j=l+1,\ldots,k$. For fixed coordinates $x_1',x_2',\ldots,x_l'$ the points of this form constitute a Euclidean cube $Q(x_1',x_2',\ldots,x_l')$ of dimension $(k-l)n\geqslant 2n$, and the point

$$p(x_1',x_2',\ldots,x_l')=(x_1',x_2',\ldots,x_l',b_{l+1},\ldots,b_k)$$

belongs to the interior of it.

Let $\psi(x)$ denote, for every point $x\in Q(x_1',x_2',\ldots,x_l')-p(x_1',x_2',\ldots,x_l')$, the projection of x from the point $p(x_1',x_2',\ldots,x_l')$ on the boundary of the cube $Q(x_1',x_2',\ldots,x_l')$. It is evident that $\psi(x)$ depends continuously on all coordinates of the point $x=(x_1',x_2',\ldots,x_l',x_{l+1},\ldots,x_k)$, that $\psi(x)\in M_{l+1}$, and that for $x\in M_{l+1}$ we have $\psi(x)=x$. Moreover, let us observe that if $y=(x_1',x_2',\ldots,x_l',y_{l+1},\ldots,y_k)\neq p(x_1',x_2',\ldots,x_l')$ is a point lying on the straight line joining x with $p(x_1',x_2',\ldots,x_n')$, then the jth coordinate y_j of y is equal to b_j if and only if the jth coordinate of the point x is equal to b. It follows that for $x\in M(\nu_1,\nu_2,\ldots,\nu_l)-N$ the segment $\overline{x\psi(x)}$ lies in the set $M(\nu_1,\nu_2,\ldots,\nu_l)-N$. Hence setting

$$r(x,t)=\text{point which divides the segment } \overline{x\psi(x)} \text{ in the ratio } t:(1-t),$$

we obtain a retraction by deformation of the set $M(\nu_1,\nu_2,\ldots,\nu_l)-N$ to the set $M_{l+1}\cap[M(\nu_1,\nu_2,\ldots,\nu_l)-N]$ and this retraction satisfies conditions (1) and (3).

Thus (2) is proved. Starting from the set $M_0-N=Q-N$ we infer by a finite induction that there exists a retraction by deformation $r(x,t)$ of the set $Q-N$ to the set $M_{k-1}-N=M-N=M$ satisfying condition (1), *i. e.* the proof of the lemma is finished.

5. Since the Euclidean n-sphere may be obtained from the Euclidean n-cube by identification of all points lying on its boundary we obtain from lemma 1 the following

LEMMA 2. *Let $T=S_1\times S_2\times\ldots\times S_k$, where S_ν is a Euclidean n-sphere and let a_ν,b_ν be two different points belonging to S_ν, for $\nu=1,2,\ldots,k$. The subset A of T consisting of all points $(x_1,x_2,\ldots,x_k)\in S_1\times S_2\times\ldots\times S_k$ such that at most one of the coordinates x_ν differs from a_ν, is a deformation retract of the set $T-B$, where B is the subset of T consisting of all points $(x_1,x_2,\ldots,x_k)\in S_1\times S_2\times\ldots\times S_k$ such that for some two indices $\mu\neq\nu$ we have $x_\mu=b_\mu$, $x_\nu=b_\nu$.*

Proof. We can assume that S_ν is obtained from a Euclidean n-cube Q_ν by the identification of its boundary with the point a_ν. Hence b_ν is a point lying in the interior of Q_ν. It follows that the set T may be considered as the set obtained from the cube $Q=Q_1\times Q_2\times\ldots\times Q_k$ by the identification in every term Q_ν of the boundary with the point a_ν. By this identification the sets A and B correspond, respectively, to the sets M and N considered in section 4. To the retraction by deformation $r(x,t)$ of the set $Q-N$ to the set M, satisfying condition (1), corresponds the retraction by deformation of the set $T-B$ to the set A. Thus the proof of the lemma 2 is finished.

Remark. It is easy to observe that the set T is a (kn)-dimensional manifold which can be triangulated in such a manner that the sets A and B of the statement of lemma 2 are subcomplexes of this triangulation.

6. LEMMA 3. *Let X be a compact k-dimensional space and P — an l-dimensional rectilinear polytope lying in the Euclidean n-space E_n with $n>k+l$, and let Q be a closed subset of E_n disjoint with P. For every function $f\in E_n^X$ and every $\varepsilon>0$ there exists a function $f_0\in E_n^X$ satisfying the following conditions:*

(4) $\varrho(f(x),f_0(x))<\varepsilon$ *for every* $x\in X$,

(5) $f_0(X)\subset E_n-P$,

(6) *if* $f(x)\in Q$, *then* $f_0(x)=f(x)$.

Proof. Since Q is closed and $P\subset E_n-Q$ compact, there exists a positive η such that

(7) $0<\eta<\frac{1}{4}\varepsilon$, $0<\eta<\frac{1}{4}\varrho(p,q)$ for every $p\in P$ and $q\in Q$.

It is known (see, for instance, [10], p. 207) that there exists a function $f_1\in E_n^X$ satisfying the conditions:

(8) $\varrho(f(x),f_1(x))<\eta$ for every $x\in X$,

(9) $f_1(X)$ is a subset of a k-dimensional polytope $W\subset E_n$.

Since $\dim W=k$, $\dim P=l$ and $n>k+l$, we infer that there exists an isometric mapping α of the space E_n onto itself satisfying the conditions:

(10) $\varrho(\alpha(p),p)<\eta$ for every $p\in E_n$,

(11) $\alpha(W)\subset E_n-P$.

Setting

(12) $f_2(x)=\alpha[f_1(x)]$ for every $x\in X$,

we infer by (8)-(12) that

(13) $\varrho(f(x),f_2(x))<2\eta$ for every $x\in X$,

(14) $f_2(X)\subset E_n-P$.

Consider now the 2η-neighbourhood of the set Q, *i. e.*, the set U of all points $p\in E_n$ with $\varrho(p,Q)<2\eta$. The set $G=f_2^{-1}(U)$ is an open subset of X and

(15) $\varrho(f_2(x),Q)<2\eta$ for every $x\in G$.

By (13) the compact set $f^{-1}(Q)$ is contained in G.

Consider the function β defined in the compact set $f_2^{-1}(Q)\cup(X-G)$ by the formulas

(16) $\beta(x)=f(x)-f_2(x)$ for every $x\in f^{-1}(Q)\subset G$,

(17) $\beta(x)=0$ for every $x\in X-G$.

By (13), (16) and (17) we have $|\beta(x)|<2\eta$ for every $x\in f_2^{-1}(Q)\cup(X-G)$. We infer that there exists an extension $\beta'\in E_n^X$ of β satisfying the inequality

(18) $|\beta'(x)|\leqslant 2\eta$ for every $x\in X$.

Let us set:

(19) $f_0(x)=f_2(x)+\beta'(x)$ for every $x\in X$.

It follows by (13) and (18) that $\varrho(f(x),f_0(x))<4\eta$ for every $x\in X$. We infer by (7) that condition (4) is satisfied.

Moreover, by (18), (19) and (15) we have for every $x\in G$:

$$\varrho(f_0(x),Q)\leqslant\varrho(f_0(x),f_2(x))+\varrho(f_2(x),Q)<2\eta+2\eta=4\eta.$$

It follows by (7) that

(20) $f_0(x)\in E_n-P$ for every $x\in G$.

For $x\in X-G$, we infer by (17) and (19) that $f_0(x)=f_2(x)$. It follows by (14) that $f_0(x)\in E_n-P$. With respect to (20) we conclude that condition (5) is satisfied.

Finally, we infer by (16) and (19) that the relation $f(x)\in Q$ implies $f_0(x)=f(x)$, hence condition (6) is also satisfied.

7. A subset B of an n-dimensional manifold T is said to be *locally polytopical* if for every point $p\in B$ there exist an open neighbourhood

G of p in T, a compact neighbourhood $P\subset G$ of p in B, and a homeomorphism h mapping G onto the Euclidean n-space E_n in such a manner that $h(P)$ is a rectilinear polytope.

LEMMA 4. *Let B be a locally polytopical, l-dimensional, compact subset of an n-dimensional manifold T and A a compact subset of $T-B$. If X is a compact space with dimension $<n-l$ and $\varphi\in T^X$, then there exists a function $\varphi'\in T^X$ homotopic to φ and satisfying the conditions:*

(21) $\varphi'(X)\cap B=0$,

(22) *if* $\varphi(x)\in A$, *then* $\varphi'(x)=\varphi(x)$.

Proof. For every point $p\in B$ there exist an open neighbourhood $G_p\subset T-A$ of p in T, a compact neighbourhood $P_p\subset B\cap G_p$ of p in B and a homeomorphism h mapping G_p onto the Euclidean n-space E_n in such a manner that $h(P_p)$ is a rectilinear polytope. Let V_p be an open neighbourhood of the set P_p in T such that the closure $\overline{V}_p$ of V_p lies in G_p. Since B is compact there exists a finite system of points $p_1,p_2,\ldots,p_k\in B$ such that

(23) $$B=\bigcup_{\nu=1}^{k}P_{p_\nu}.$$

Let us set

(24) $B_0=0$, $B_i=\bigcup_{\nu=1}^{i}P_{p_\nu}$ for $i=1,2,\ldots,k$.

We shall show by induction that for every positive ε and $i=0,1,\ldots,k$ there exists a function $\varphi_i\in T^X$ satisfying the following conditions:

(25_i) $\varphi_i(X)\cap B_i=0$,

(26_i) $\varrho(\varphi(x),\varphi_i(x))\leqslant\varepsilon-\varepsilon/2^i$,

(27_i) if $\varphi(x)\in A$ then $\varphi_i(x)=\varphi(x)$.

In the case of $i=0$ we set $\varphi_0=\varphi$ and then propositions (25_i), (26_i) and (27_i) are obvious. Assume that $0\leqslant i<k$ and that φ_i is a function satisfying (25_i), (26_i) and (27_i). Since the sets B_i and $\varphi_i(X)$ are compact, we infer by (25_i) that there exists a positive η such that

(28_i) if $x\in X$ and $y\in B_i$, then $\varrho(\varphi_i(x),y)>\eta$.

Applying lemma 3 to the function φ_i considered only in the set $X_i=\varphi_i^{-1}(\overline{V}_{p_{i+1}})$, and to the sets $P=P_{p_{i+1}}$, $Q=G_{p_{i+1}}-V_{p_{i+1}}$ lying in the set $G_{p_{i+1}}$ homeomorphic with E_n, we infer that there exists a continuous function φ_{i+1} defined in the set X_i and satisfying the conditions:

(29) $\varrho(\varphi_{i+1}(x),\varphi_i(x))<\min(\eta,\varepsilon/2^{i+1})$,

(30) $\varphi_{i+1}(x)\in G_{p_{i+1}}-P_{p_{i+1}}$,

(31) if $\varphi_i(x)\in G_{p_{i+1}}-V_{p_{i+1}}$, then $\varphi_{i+1}(x)=\varphi_i(x)$.

Setting $\varphi_{i+1}(x)=\varphi_i(x)$ for every $x\in X-X_i$, we infer by (31) that the mapping φ_{i+1}, extended in this manner over X, is continuous and satisfies moreover conditions (29), (30) and (31). Since $G_{p_{i+1}}\cap A=0$, the extended function φ_{i+1} satisfies condition (27_i). Moreover, by (29) and (26_i) it satisfies condition (26_{i+1}). From (28_i) and (29) we infer that $\varphi_{i+1}(x)\notin B_i$ for every $x\in X$. It follows by (24) and (30) that condition (25_{i+1}) is also satisfied.

Thus we have shown that for every $i=0,1,\dots,k$ there exists a function φ_i satisfying conditions (25_i)-(27_i). Consider the function $\varphi'=\varphi_k$. By (23) and (24) it satisfies condition (21) and, by (27_k), condition (22). Moreover, it follows by (26_k) that $\varrho(\varphi(x),\varphi'(x))<\varepsilon$ for every $x\in X$. Since the space T^X is locally connected, we infer that for positive ε sufficiently small the function φ' is homotopic to φ, which completes the proof.

8. THEOREM. *If X_0 is a compact space of dimension $<2n$ and Y_0 — the Euclidean n-sphere, then every finite system of functions $f_1,f_2,\dots,f_k\in Y_0^{X_0}$ is separable.*

Proof. The statement is obvious for $k=1$. Assume now that $k>1$. Let us assign to every point $x\in X_0$ the point

$$\varphi(x)=(f_1(x),f_2(x),\dots,f_k(x))\in S_1\times S_2\times\dots\times S_k,$$

where $S_i=Y_0$ for $i=1,2,\dots,k$. We obtain a function $\varphi\in T^{X_0}$, where T denotes the (nk)-dimensional manifold $S_1\times S_2\times\dots\times S_k$.

Now let us choose an arbitrary point $y_0\in Y_0$. For every $\nu=1,2,\dots,k$ let us denote by a_ν the point y_0 and by b_ν another point of S_ν. Consider the sets A and B defined as in lemma 2. It is clear that B is a locally polytopical $(k-2)n$-dimensional compact subset of the (nk)-dimensional manifold T. Applying lemma 4 we infer that there exists a function $\varphi'\in T^{X_0}$ homotopic to φ and such that $\varphi'(X_0)\cap B=0$. Hence we can assume that the given functions $f_1,f_2,\dots,f_k$ satisfy the condition

$$(f_1(x),f_2(x),\dots,f_k(x))\in T-B \quad \text{for every} \quad x\in X_0.$$

But by lemma 2 the set A is a deformation retract of the set $T-B$. It follows that for every $\nu=1,2,\dots,k$ there exists a function $f'_\nu\in Y_0^{X_0}$, homotopic to f_ν and such that

$$(f'_1(x),f'_2(x),\dots,f'_k(x))\in A \quad \text{for every} \quad x\in X_0.$$

By the remark in section 3 it follows that the point y_0 and the sets $G_1,G_2,\dots,G_k$ realize a separation of the functions $f_1,f_2,\dots,f_k$.

Remark. Applying a theorem of Hopf ([8], p. 436, theorem V), one can easily show that for every even n there exist two non-separable functions mapping the Cartesian product of two Euclidean n-spheres $S\times S$ into S (cf. [2], p. 736). Consequently, the hypothesis $\dim X_0<2n-1$ in the last theorem cannot be dropped.

9. Let $f_1,f_2,\dots,f_k$ be a separable system of functions belonging to $Y_0^{X_0}$. Consider a point $y_0\in Y_0$ and a system $G_1,G_2,\dots,G_k$ of open subsets of X_0 realizing the separation of $f_1,f_2,\dots,f_k$. Hence for every $i=1,2,\dots,k$ there exists a function $f'_i\in Y_0^{X_0}$ homotopic to f_i and such that $f'_i(x)=y_0$ for every $x\in X_0-G_i$, $i=1,2,\dots,k$. If we set

$$f(x)=f'_i(x) \quad \text{for} \quad x\in G_i, \quad i=1,2,\dots,k,$$
$$f(x)=y_0 \quad \text{for} \quad x\in X_0-\bigcup_{i=1}^k G_i,$$

we obtain a function $f\in Y_0^{X_0}$, which we shall call the *join* of the functions $f_1,f_2,\dots,f_k$ (cf. [1], [2], [3] and [11]).

THEOREM. *If f is a join of the functions $f_1,f_2,\dots,f_k\in Y_0^{X_0}$ where Y_0 is connected, then for every true cycle γ of X_0 we have*

(32) $$f(\gamma)\sim f_1(\gamma)+f_2(\gamma)+\dots+f_k(\gamma) \quad in \quad Y_0.$$

Proof. We can assume that the functions $f_1,f_2,\dots,f_k$ are separate and that the point y_0 and the open sets $G_1,G_2,\dots,G_k$ realize their separation. Let $\gamma=\{\gamma_\nu\}$ where γ_ν is a ε_ν-cycle (with arbitrary coefficients) of X_0 and $\lim_{\nu\to\infty}\varepsilon_\nu=0$. One easily sees that γ is homologous in X_0 with a true cycle $\gamma'=\{\gamma'_\nu\}$ such that every simplex of γ'_ν lies either in the set $X_0-\bigcup_{i=1}^k G_i$ or in one of the sets $\bar G_1,\bar G_2,\dots,\bar G_k$. Without loss of generality we may assume that γ satisfies the last condition. Let

$$\gamma_\nu=a_{\nu1}\cdot\Delta_{\nu1}+a_{\nu2}\cdot\Delta_{\nu2}+\dots+a_{\nu m}\cdot\Delta_{\nu m_\nu}.$$

Then

$$f(\gamma_\nu)=a_{\nu1}\cdot f(\Delta_{\nu1})+a_{\nu2}\cdot f(\Delta_{\nu2})+\dots+a_{\nu m}\cdot f(\Delta_{\nu m_\nu}).$$

If $\Delta_{\nu j}$ lies in $\bar G_i$ then $f(\Delta_{\nu j})=f_i(\Delta_{\nu j})$, and if $\Delta_{\nu j}$ lies in $X_0-\bigcup_{i=1}^k G_i$, then $f(\Delta_{\nu j})=y_0$. This gives us homology (32) for true cycles γ of positive dimension. If however γ is of dimension 0, then $f(\gamma)\sim 0$ and also $f_i(\gamma)\sim 0$ for $i=1,2,\dots,k$, and consequently (32) holds also in this case.

Two functions $f,g\in Y_0^{X_0}$ are said to be *homologous* provided that for every true cycle γ of X_0 we have $f(\gamma)\sim g(\gamma)$ in Y_0. It is known that two homotopic mappings are necessarily homologous. In the case where $\dim X_0 \leqslant n$ and Y_0 is the Euclidean n-sphere (we shall quote this case as the *n-dimensional case of Hopf*) the inverse is also true: two homologous mappings are necessarily homotopic.

COROLLARY 1. *All joins of a given system of functions $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$ are homologous.*

Every function $\varphi \in Y_0^{X_0}$ induces a homomorphism $\chi_\varphi^{(\mathfrak{A},n)}$ of the n-dimensional group of homology $H^n(X_0, \mathfrak{A})$ of X_0 with coefficients belonging to an arbitrary Abelian group $\mathfrak{A}$ into the group $H^n(Y_0, \mathfrak{A})$.

COROLLARY 2. *If f is a join of functions $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$ then $\chi_f^{(\mathfrak{A},n)} = \chi_{f_1}^{(\mathfrak{A},n)} + \chi_{f_2}^{(\mathfrak{A},n)} + \ldots + \chi_{f_k}^{(\mathfrak{A},n)}$.*

10. By the theorem of section 8 if $\dim X_0 < 2n$ and Y_0 is the Euclidean n-sphere, then there exists a join for every system $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$. If $\dim X_0 < 2n-1$, then we can prove the following, somewhat stronger theorem:

THEOREM. *If X_0 is a compact space of dimension $<2n-1$ and Y_0 — the Euclidean n-sphere, then for every system $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$ the joins of $f_1, f_2, \ldots, f_k$ constitute one component of $Y_0^{X_0}$.*

Proof. We may assume that $n > 0$ and that there exist a point $y_0 \in Y_0$ and a system of open and disjoint subsets $G_1, G_2, \ldots, G_k$ of X_0 realizing the separation of $f_1, f_2, \ldots, f_k$. Setting

$$f(x) = f_i(x) \quad \text{for every} \quad x \in G_i, \quad i = 1, 2, \ldots, k,$$

$$f(x) = y_0 \quad \text{for every} \quad x \in X_0 - \bigcup_{i=1}^{k} G_i,$$

we obtain a join $f \in Y_0^{X_0}$ of $f_1, f_2, \ldots, f_k$.

Consider now another point y_0' of Y_0. It is evident that we may define a continuous rotation $\vartheta(x,t)$ of the sphere Y_0 such that

$$\vartheta(y, 0) = y \quad \text{for every} \quad y \in Y_0,$$

$$\vartheta(y_0, 1) = y_0'.$$

Setting for every $x \in X_0$ and $0 \leqslant t \leqslant 1$:

$$f^t(x) = \vartheta(f(x), t),$$

$$f_i^t(x) = \vartheta(f(x), t) \quad \text{for} \quad i = 1, 2, \ldots, k,$$

we obtain continuous families of functions f^t, f_i^t belonging to $Y_0^{X_0}$ and such that $f^0 = f$, $f_i^0 = f_i$ for every $i = 1, 2, \ldots, k$, and that f^t is a join of the functions $f_1^t, f_2^t, \ldots, f_k^t$. It follows that f^1 constitutes a join of the functions $f_1^1, f_2^1, \ldots, f_k^1$ and that this join is homotopic to f. Moreover, it is clear that the point y_0' and the sets $G_1, G_2, \ldots, G_k$ realize the separation of the functions $f_1^1, f_2^1, \ldots, f_k^1$.

It follows that to prove another join f' of functions $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$ to be homotopic to the join f we may assume that f' is defined in the following manner:

There exist functions $f_1', f_2', \ldots, f_k'$ homotopic with $f_1, f_2, \ldots, f_k$, respectively, and open disjoint subsets $G_1', G_2', \ldots, G_k'$ of X_0 such that

$$f_i'(x) = y_0 \quad \text{for every} \quad x \in X_0 - G_i' \text{ and } i = 1, 2, \ldots, k,$$

$$f'(x) = f_i'(x) \quad \text{for} \quad x \in G_i' \quad \text{and} \quad i = 1, 2, \ldots, k,$$

$$f'(x) = y_0 \quad \text{for every} \quad x \in X - \bigcup_{i=1}^{k} G_i'.$$

Since f_i and f_i' are homotopic, there exists a continuous function $g_i(x,t)$ defined for $(x,t) \in X_0 \times \langle 0,1 \rangle$, with values belonging to Y_0 and such that

$$g_i(x, 0) = f_i(x); \quad g_i(x,1) = f_i'(x) \quad \text{for every} \quad x \in X_0 \quad \text{and} \quad i = 1, 2, \ldots, k.$$

Setting

$$g(x,t) = (g_1(x,t), g_2(x,t), \ldots, g_k(x,t)) \quad \text{for every} \quad (x,t) \in X_0 \times \langle 0,1 \rangle$$

we obtain a function $g \in T^{X_0 \times \langle 0,1 \rangle}$, where $T = S_1 \times S_2 \times \ldots \times S_k$ and $S_i = Y_0$ for $i = 1, 2, \ldots, k$. Let $a_\nu = y_0$ and $b_\nu \in S_\nu - (y_0)$ for $\nu = 1, 2, \ldots, k$ and consider the subsets A and B of T defined as in the lemma 2 (section 5). By this lemma A is a deformation retract of $T - B$. Since $\dim T = kn$, $\dim B = (k-2)n$, and $\dim(X_0 \times \langle 0,1 \rangle) = \dim X_0 + 1 < 2n$ we infer by lemma 4 that there exists a function $g' \in T^{X_0 \times \langle 0,1 \rangle}$ homotopic to g and such that

$$g'(X_0 \times \langle 0,1 \rangle) \cap B = 0,$$

$$g'(x,0) = g(x,0) \quad \text{and} \quad g'(x,1) = g(x,1)$$

for every $(x,t) \in X_0 \times \langle 0,1 \rangle$. If we apply the retraction by deformation of $T - B$ to A we infer that g' is homotopic to a function $g'' \in T^{X_0 \times \langle 0,1 \rangle}$ satisfying the conditions:

$$g''(x,t) \in A \quad \text{for every} \quad (x,t) \in X_0 \times \langle 0,1 \rangle,$$

$$g''(x,0) = g(x,0) = (f_1(x), f_2(x), \ldots, f_k(x)) \quad \text{for every} \quad x \in X_0,$$

$$g''(x,1) = g(x,1) = (f_1'(x), f_2'(x), \ldots, f_k'(x)) \quad \text{for every} \quad x \in X_0.$$

The points $p \in A$ are of the form $p = (y_1, y_2, \ldots, y_k)$, where there exists at most one index i_p such that $y_{i_p} \neq y_0$. Setting $\psi(p) = y_{i_p}$ we define a continuous mapping $\psi \in Y_0^A$. It is evident that the functions $\varphi_t(x) = \psi g''(x,t)$ constitute a continuous family $\{\varphi_t\} \subset Y_0^{X_0}$ joining the function $\varphi_0(x)$

$=\psi g''(x,0)=\psi[f_1(x),f_2(x),\ldots,f_k(x)]=f(x)$ with the function $\varphi_1(x)=\psi g''(x,1)$ $=\psi[f_1'(x),f_2'(x),\ldots,f_k'(x)]=f'(x)$. Hence f and f' are homotopic and the theorem is completely established.

11. The following theorem states a relation between the notion of join and the notion of dependence of functions:

THEOREM. *If:*

1° *the functions* $f_1,f_2,\ldots,f_k\in Y_0^{X_0}$ *are separable and*

2° $m\geqslant\dim X_0$, $m+1\geqslant\dim Y_0$,

then every function $g\in\mathfrak{K}_{m+1}(f_1,f_2,\ldots,f_k)$ *is a join of some multiples of* $f_1,f_2,\ldots,f_k$.

Proof. Let $y_0\in Y_0$ and let the open sets $G_1,G_2,\ldots,G_k$ realize the separation of the functions $f_1,f_2,\ldots,f_k$. We may assume that

$$f_i(x)=y_0 \quad \text{for every} \quad x\in X_0-G_i,\quad i=1,2,\ldots,k.$$

Consider the Cartesian products $X_1=X_0\times\langle 0,1\rangle$ and $Y_i=Y_0\times(i)$ for $i=1,2,\ldots,k$. We may assume that X_0,X_1 and $Y_0,Y_1,\ldots,Y_k$ are disjoint subsets of a metric space. Let us identify every point $(x,0)\in X_1$ with the point $x\in X_0$, every point $(y_0,i)\in Y_i$ with the point $y_0\in Y_0$, every point $(x,1)\in G_i\times\langle 0,1\rangle\subset X_1$ with the point $(f_i(x),i)\in Y_i$, and every point $(x,1)\in(X_0-\bigcup_{i=1}^{k}G_i)\times\langle 0,1\rangle\subset X_1$ with the point $(y_0,i)=y_0$. In this manner we obtain from the set $X_0\cup X_1\cup Y_1\cup\ldots\cup Y_k$ a space X of dimension $\leqslant m+1$, containing each of the sets $X_0,X_1,Y_1,\ldots,Y_k$ [2]). Let us observe that each of the functions $f_1,f_2,\ldots,f_k$ is extendable over X. In fact, setting

$$\begin{aligned} f_i'(x,t)&=f_i(x) && \text{for} && (x,t)\in X_0\times\langle 0,1\rangle,\\ f_i'(y,i)&=y && \text{for} && (y,i)\in Y_0\times(i)=Y_i,\\ f_i'(y,j)&=y_0 && \text{for} && (y,j)\in Y_0\times(j)=Y_j \quad \text{for} \quad i\neq j,\end{aligned}$$

we obtain a continuous extension f_i' of f_i over X with respect to Y_0.

It follows that for every function $g\in\mathfrak{K}_{m+1}(f_1,f_2,\ldots,f_k)$ there exists an extension $g'\in Y_0^X$. Setting

$$\psi_i(y)=g'(y,i) \quad \text{for every} \quad y\in Y_0,$$

we obtain a function $\psi_i\in Y_0^{X_0}$. But the function $g(x)=g'(x,0)$ is homotopic to the function $g'(x,1)$ and one sees immediately that the last function is a join of the functions $\psi_if_i'(x,1)$ identical with the functions ψ_if_i respectively. Evidently, the point $g'(y_0)$ and the sets $G_1,G_2,\ldots,G_k$ realize the separation of the functions ψ_if_i. Hence the theorem is proved.

[2]) The construction of the space X constitutes a slight modification of the construction of the so called *mapping cylinder*, due to J. H. C. Whitehead [12], p. 259. Cf. also [6], p. 43.

It is known [4] that if $f,g\in Y_0^{X_0}$ and f/g, then there exists a function $\psi\in Y_0^{X_0}$ such that g is homotopic with ψf. If χ denotes the endomorphism of the group $H^n(Y_0,\mathfrak{A})$ induced by ψ, then $\chi_g^{(\mathfrak{A},n)}=\chi[\chi_f^{(\mathfrak{A},n)}]$. By virtue of corollary 2 in section 9 we have

COROLLARY 1. *If the functions* $f_1,f_2,\ldots,f_k\in Y_0^{X_0}$ *are separable and* $m\geqslant\dim X_0$, $m+1\geqslant\dim Y_0$, *then for every function* $g\in\mathfrak{K}_{m+1}(f_1,f_2,\ldots,f_k)$ *there exist endomorphisms* $\chi_1,\chi_2,\ldots,\chi_k$ *of the group* $H^n(Y_0,\mathfrak{A})$ *such that* $\chi_g^{(\mathfrak{A},n)}=\sum_{i=1}^{k}\chi_i[\chi_{f_i}^{(\mathfrak{A},n)}]$ *for every* $n=0,1,\ldots$ *and for every Abelian group* $\mathfrak{A}$.

If Y_0 is an orientable n-dimensional pseudomanifold (*i. e.*, a simple n-circuit), then for every function $\psi\in Y_0^{X_0}$ there exists an integer (the degree of ψ) m such that for every n-dimensional true cycle γ of Y_0 we have $\psi(\gamma)\sim m\cdot\gamma$ in Y_0. Hence we obtain the following

COROLLARY 2. *Let* Y_0 *be an orientable* n*-dimensional manifold,* X_0 *a compact space and* m — *an integer such that* $n\leqslant m+1$ *and* $\dim X_0\leqslant m$. *If the functions* $f_1,f_2,\ldots,f_k\in Y_0^{X_0}$ *are separable, then for every function* $g\in\mathfrak{K}_{m+1}(f_1,f_2,\ldots,f_k)$ *there exist some integers* $m_1,m_2,\ldots,m_k$ *such that for every* n*-dimensional true cycle* γ *of* X_0 *we have*

$$g(\gamma)\sim\sum_{i=1}^{k}m_if_i(\gamma) \quad \text{in} \quad Y_0.$$

Applying the theorem of section 8 we obtain

COROLLARY 3. *If* $f_1,f_2,\ldots,f_k\in Y_0^{X_0}$ *where* $\dim X_0<2n$ *and* Y_0 *is the Euclidean* n*-sphere, and if* m *is an integer satisfying the inequality* $m\geqslant\max[\dim X_0+1,\dim Y_0]$, *then for every function* $g\in\mathfrak{K}_m(f_1,f_2,\ldots,f_k)$ *there exist integers* $m_1,m_2,\ldots,m_k$ *such that for every true cycle* γ *of* X_0 *it is*

$$g(\gamma)\sim\sum_{i=1}^{k}m_if_i(\gamma) \quad \text{in} \quad Y_0.$$

Since $\mathfrak{K}(f_1,f_2,\ldots,f_k)\subset\mathfrak{K}_{m+1}(f_1,f_2,\ldots,f_k)$ for every system of functions $f_1,f_2,\ldots,f_k\in Y_0^{X_0}$ and for every integer m (finite or not) we get from the corollaries 1, 2 and 3 the following corollaries:

COROLLARY 1'. *If* $g\in Y_0^{X_0}$ *depends on the separable functions* $f_1,f_2,\ldots,f_k\in Y_0^{X_0}$, *then there exist endomorphisms* $\chi_1,\chi_2,\ldots,\chi_k$ *of the group* $H^n(Y_0,\mathfrak{A})$ *such that*

$$\chi_g^{(\mathfrak{A},n)}=\sum_{i=1}^{k}\chi_i[\chi_i^{(\mathfrak{A},n)}].$$

COROLLARY 2'. *If Y_0 is an orientable n-dimensional pseudo-manifold and if $g \in Y_0^{X_0}$ depends on the separable functions $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$, then there exist integers $m_1, m_2, \ldots, m_k$ such that for every n-dimensional true cycle γ of X_0 we have*

$$g(\gamma) \sim \sum_{i=1}^{k} m_i f_i(\gamma) \quad in \quad Y_0.$$

COROLLARY 3'. *If $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$ where $\dim X_0 < 2n$ and if Y_0 is the Euclidean n-sphere, then for every function $g \in \mathfrak{R}(f_1, f_2, \ldots, f_k)$ there exist integers $m_1, m_2, \ldots, m_k$ such that for every true cycle γ of X_0 we have*

$$g(\gamma) \sim \sum_{i=1}^{k} m_i f_i(\gamma) \quad in \quad Y_0.$$

12. For the mappings into Euclidean spheres we have the following theorem, stronger than the theorem of section 11:

THEOREM. *If $f_1, f_2, \ldots, f_k \in Y_0^{X_0}$ where $\dim X_0 < 2n$ and Y_0 is the Euclidean n-sphere, and if m is an integer such that $\dim X_0 < m < 2n$, then $f \in \mathfrak{R}_m(f_1, f_2, \ldots, f_k)$ if and only if f is a join of certain multiples $g_1, g_2, \ldots, g_k$ of $f_1, f_2, \ldots, f_k$.*

Proof. By virtue of theorem of section 11 it suffices to show that every join of the functions $g_1, g_2, \ldots, g_k$ belongs to $\mathfrak{R}_m(f_1, f_2, \ldots, f_k)$. Since $g_1, g_2, \ldots, g_k$ are separable and $\mathfrak{R}_{2n-1}(g_1, g_2, \ldots, g_k) \subset \mathfrak{R}_m(f_1, f_2, \ldots, f_k)$, it suffices to show that every join f of the functions $f_1, f_2, \ldots, f_k$ belongs to $\mathfrak{R}_{2n-1}(f_1, f_2, \ldots, f_k)$. Obviously we may assume that there exist a point $y_0 \in Y_0$ and open subsets $G_1, G_2, \ldots, G_k$ of X_0 realizing the separation of $f_1, f_2, \ldots, f_k$ and such that

$$f(x) = f_i(x) \quad \text{for every} \quad x \in G_i,$$
$$f(x) = y_0 \quad \text{for every} \quad x \in X_0 - \bigcup_{i=1}^{k} G_i.$$

Let X be a compact space such that $X_0 \subset X$, that $\dim(X - X_0) \leqslant 2n-1$ and that for every $i = 1, 2, \ldots, k$ there exists an extension $f_i' \in Y_0^X$ of f_i over X. Setting

$$\varphi(x) = (f_1'(x), f_2'(x), \ldots, f_k'(x)) \quad \text{for every} \quad x \in X,$$

we obtain a continuous function φ mapping X onto the Cartesian product $T = S_1 \times S_2 \times \ldots \times S_k$, where $S_\nu = Y_0$ for $\nu = 1, 2, \ldots, k$. Let us set $a_\nu = y_0$ and let b_ν be a point of S_ν different from a_ν. Consider, as in the proof of lemma 2 (section 5), the set A of all points $(x_1, x_2, \ldots, x_k) \in T$ such that at most one of the coordinates x_ν differs from a_ν, and the set B of all points $(x_1, x_2, \ldots, x_k) \in T$ such that for two indices $\mu \neq \nu$ we have $x_\mu = b_\mu$ and $x_\nu = b_\nu$. By lemma 2, there exists a retraction by deformation $r(p,t)$ of the set $T - B$ to the set $A \subset T - B$. But $\dim T = kn$, $\dim B = (k-2)n$ and $\dim X \leqslant 2n-1$, hence lemma 4 implies that there exists a function $\varphi' \in T^X$ homotopic to φ and satisfying the conditions

$$\varphi'(X) \cap B = 0,$$
$$\text{if} \quad \varphi(x) \in A \quad \text{then} \quad \varphi'(x) = \varphi(x).$$

Since $\varphi(x) \in A$ for every $x \in X_0$, it follows by the last condition that $\varphi' \in T^X$ constitutes an extension of the function φ considered only in X_0. It follows that setting

$$\varphi''(x) = r[\varphi'(x), 1] \quad \text{for every} \quad x \in X$$

we obtain an extension $\varphi'' \in A^X \subset T^X$ of the function φ (considered only in X_0) and this extension is homotopic to φ. Let

$$\varphi''(x) = (f_1''(x), f_2''(x), \ldots, f_k''(x)) \quad \text{for every} \quad x \in X$$

and let G_i'' denote the subset of X consisting of all points $x \in X$ such that $f_i''(x) \neq y_0 = a_i$. It is evident that the sets $G_1'', G_2'', \ldots, G_k''$ are open and disjoint and that $G_i \subset G_i''$. Setting

$$f''(x) = f_i''(x) \quad \text{for every} \quad x \in G_i'', \quad i = 1, 2, \ldots, k,$$
$$f''(x) = y_0 \quad \text{for every} \quad x \in X - \bigcup_{i=1}^{k} G_i''$$

we obtain a join $f'' \in Y_0^X$ of the functions $f_1'', f_2'', \ldots, f_k''$. Since $\varphi''(x) = \varphi(x)$ for $x \in X_0$ and $G_i \subset G_i''$, the function f'' is an extension over X of the function f. Thus we have shown that f has an extension over X and consequently $f \in \mathfrak{R}_{2n-1}(f_1, f_2, \ldots, f_k)$.

13. In the n-dimensional case of Hopf, *i. e.* in the case where $\dim X_0 \leqslant n$ and Y_0 is the Euclidean n-sphere, the classes of homotopic functions belonging to $Y_0^{X_0}$ constitute the so called *Hopf group* of X_0 (it is a special case of the cohomotopy group studied by Spanier [6]). In the Hopf group the sum of two classes of homotopy $[f_1], [f_2] \subset Y_0^{X_0}$ is defined as a class $[f_3]$ such that for every true cycle γ lying in X_0 we have $f_3(\gamma) \sim f_1(\gamma) + f_2(\gamma)$. By virtue of the theorem of section 12 and of corollary 3' of section 11 we obtain the following

COROLLARY. *In the n-dimensional case of Hopf the set of classes of homotopy contained in the set $\mathfrak{R}_m(f_1, f_2, \ldots, f_k)$, where $n < m < 2n$, coincides with the subgroup of the Hopf group of X_0 generated by the classes of homotopy $[f_1], [f_2], \ldots, [f_k]$.*

14. A subset Φ of $Y_0^{X_0}$ will be said to be a *system of generators* of $Y_0^{X_0}$, if $\mathfrak{R}(\Phi) = Y_0^{X_0}$. Since the set of all components of $Y_0^{X_0}$ is at most enumerable, there exists a finite or enumerable system of generators of $Y_0^{X_0}$ for every X_0 and Y_0. If there exists a finite system of generators, we denote by $I(X_0, Y_0)$ the minimal number of elements in such a system. If, however, a finite system of generators does not exist, then we set $I(X_0, Y_0) = \aleph_0$.

It is easy to give examples of X_0 and Y_0 such that $I(X_0, Y_0)$ has an arbitrarily given integer value ν with $0 \leqslant \nu \leqslant \aleph_0$.

Similarly, if m is an integer $\geqslant 0$, then a subset Φ of $Y_0^{X_0}$ with $\mathfrak{R}_m(\Phi) = Y_0^{X_0}$ will be said to be a *system of generators at the dimension m* of $Y_0^{X_0}$. By $I_m(X_0, Y_0)$ we shall denote the minimal number of elements in such a system. It is evident that $I_m(X_0, Y_0)$ is finite or equal $\aleph_0$ and that

$$I(X_0, Y_0) \geqslant I_{m+k}(X_0, Y_0) \geqslant I_m(X_0, Y_0)$$

for every integers $k, m \geqslant 0$.

In the n-dimensional case of Hopf and under the hypothesis that X_0 is a polytope the Hopf group of X_0 has a finite system of generators. It follows by the corollary of section 13 that in this case the number $I_m(X_0, Y_0)$, where $n < m < 2n$, is finite (equal to the minimal number of the generators of the Hopf group) [3]).

Problem. *Is it true that for every polytope X_0 the number $I(X_0, Y_0)$ is necessarily finite?*

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[3]) P. J. Hilton has been so kind as to call my attention to a mistake on page 254 of my preliminary report [5]. Namely, it is written there that in the case of Hopf the number $I(X_0, Y_0)$ is finite. Instead of this it should be written (and it is so in the present note) that in the n-dimensional case of Hopf the number $I_m(X_0, Y)$ is finite if $n < m < 2n$.

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