Suppose now that \( \varphi(a) \) is not a theorem of \( \mathcal{S}_a \). Let \( H \) be the Heyting algebra of all open elements of \( E, H = H(L_0) \). Then by the same argumentation as above

\[
(H) \varphi_a((I[a_1])) = (L_0) \varphi_{a_0}((I[a_1])) = [a] \neq \varepsilon,
\]

which proves that \( a \) is not a theorem of \( \mathcal{S}_a \) (see 2.2).

References


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Some applications of formalized consistency proofs

by

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Introduction

A well-known result due to Gödel [5] states: If \( (P) \) is one of the usual systems of arithmetic (for sufficient conditions on \( (P) \) see [9], p. 285) the formula which is ordinarily taken as the arithmetization of the consistency of \( (P) \) cannot be proved in \( (P) \) itself. Thus the consistency proof for \( Z^*_a \) ([9], p. 293) due to Ackermann [2] uses the principle of ordinal induction up to the first \( \varepsilon \)-number, which cannot be formalized in \( Z_a \), and the consistency proof by means of a truth definition ([9], p. 339) uses a predicate which cannot be formalized in \( Z_a \) either. It is now natural to ask whether the \( \varepsilon \)-idea of these consistency proofs may be formalized in \( Z_a \); the result of such a step would then be a consistency proof for a subsystem \( (P) \) of \( Z_a \) of the consistency of \( (P) \) would be demonstrably weaker than \( Z_a \) since a formula of \( Z_a \) which expresses the consistency of \( (P) \) would be provable in \( Z_a \) but not in \( (P) \). We shall denote such a formula by \( \text{Con}(P) \); it is to be understood that the formula chosen for expressing the consistency of \( (P) \) satisfies conditions sufficient to ensure the application of Gödel's second undecidability theorem.

In this way we are led to systems which are obtained from \( Z_a \) by suppressing all those proofs of \( Z_a \) which are too "complex"; several definitions of complexity will be used, the principal ones being the maximum number of bound variables occurring in any formula of the proof, and the number of distinct critical \( \varepsilon \)-matrices (Grundtypen, [9]), p. 83, if the Hilbert \( \varepsilon \)-symbol is used instead of quantifiers. These measures of complexity are suggested by the two consistency proofs mentioned above. We may note in passing that, for our present purpose, the consistency proof by means of a truth definition is more appropriate because it can be immediately applied to any extension of \( Z_a \) by means of transfinite induction, and other principles of proof which satisfy the rule of infinite induction ([11], p. 124).

Our first application of these results concerns the elimination of the induction scheme of \( Z_a \) by means of a finite set of axioms (which are themselves formulae of \( Z_a \)). It was established in [13] and [16] that
there is no finite set of theorems of $Z_{\kappa}$ from which all the theorems of $Z_{\kappa}$ may be deduced by means of the predicate calculus of first order. We now show that there is no finite set of formulae of $Z_{\kappa}$ consistent over the predicate calculus, from which all theorems of $Z_{\kappa}$ may be deduced by means of the predicate calculus. This is a fortiori the result of [10]: there is a finite set of formulae such that a formula $A$ of $Z_{\kappa}$ may be deduced from them if and only if $A$ is a theorem of $Z_{\kappa}$; but these formulae contain (auxiliary) predicate symbols which do not belong to $Z_{\kappa}$. Much of what we do is implicit in [13], but it seems convenient to set out the results anew.

Our second application concerns Gödel's result on the length of proofs, stated in his short note [4]. Using two (different) definitions of length — or, perhaps, better: complexity — of proofs we find: suppose two systems of arithmetic (F), (F') are such that any proof of (F) is also a proof of (F') and Con(F) is a theorem of (F'); then, for any function $\Phi$, not necessarily computable, there are infinitely many formulae $\varphi_n$ of (F) whose shortest proof in (F') has length $\varphi_n$ and whose shortest proof in (F') has length $\Lambda_\varphi$ and $\Lambda_\varphi > \Phi(\varphi_n)$. Mostowski's variant of Gödel's result, given in [14], is restricted to computable functions $\Phi$, because of his different definition of „length of proof"; we may note that on his definition there are less than $n$ proofs of length $\varphi_n$ while on both our definitions there are infinitely many proofs of length $\varphi_n$.

In the third application we return to the well-worn topic of axiom systems (F) of the predicate calculus with a finite number of axioms. It is known (Skołem-Gödel-Bernays) that if Con(F) is a theorem of $Z_{\kappa}$ then a model for the axioms of (F) may be established in $Z_{\kappa}$; in other words, there are predicates and functions of $Z_{\kappa}$ such that the axioms of (F) turn into theorems of $Z_{\kappa}$ if these predicates and functions are substituted for the non-logical constants of (F) (cf. Theorem 6 of [6]). We now obtain a converse to it: if a model for (F) can be established in $Z_{\kappa}$ then Con(F) is a theorem of $Z_{\kappa}$.

1. The consistency proofs

Definition. The system $Z^{(\delta)}$. A proof of $Z_{\kappa}$ which contains $\varphi_n$ distinct critical $\varepsilon$-matrices, is a proof of $Z^{(\delta)}$.

Observe that no restriction is here imposed on the range of these $\varepsilon$-matrices; however, on inspection of the rules of proof of $Z_{\kappa}$, there is a relation between the highest rank and the number of critical $\varepsilon$-matrices: e.g., if a quantifier-free formula of $Z_{\kappa}$ cannot be proved by means of formulae of rank $\varphi_n$, it cannot be proved with $\varphi_n$ critical matrixes, and $\varphi_n$ is unbounded. Note further that, for sufficiently large $n$, $Z^{(\delta)}_{\kappa}$ satisfies the conditions under which Gödel's second undecidability theorem has been established.

Lemma A. For each integer $n$, Con($Z^{(\delta)}_{\kappa}$) can be proved in $Z_{\kappa}$.

This, in effect, established in [2]: there, an (informal) consistency proof is given which establishes Con($Z^{(\delta)}_{\kappa}$) by means of ordinal induction up to $\varphi_n$. By [9], p. 366, for each $\kappa$, ordinal induction up to $\varphi_{\kappa+1}$ can be formalized in $Z_{\kappa}$.

This lemma was mentioned on p. 123 of [11] for a subsystem of $Z_{\kappa}$ analogous to $Z^{(\delta)}_{\kappa}$.

Definition. The system $Z^{(\eta)}$. A proof of $Z_{\kappa}$ (cf. p. 49) whose formulae contain $\varphi_n$ bound variables, is a proof of $Z^{(\eta)}_{\kappa}$.

Observe that here no restriction is imposed on the number of distinct $\varepsilon$-matrices.

Lemma B. For each integer $n$, Con($Z^{(\eta)}_{\kappa}$) can be proved in $Z_{\kappa}$.

This can be proved by modifying the truth definition of [9], p. 330-338.

It is assumed that “natural” definitions of the following syntactical terms and predicates have been chosen:

$\varphi_n^{(0)}(\alpha), \ldots, \varphi_n^{(n)}(\alpha)$ as in [9], p. 235,

$\varphi_n^{(m)}(\alpha, a, n)$ is the number of the formulae got from $A$ (with number $\alpha$) by replacing the variable $\varphi_1, \varphi_2, \ldots, \varphi_n$ by $\varphi_n^{(0)}(\alpha)$ and $\varphi_j, j > m$, by 0.

It is not assumed that all $\varphi_1, \varphi_2, \ldots, \varphi_n$ occur in $A$. All the variables we use in the proofs will be $\varphi_1, \varphi_2, \varphi_3$. Trivially, if $A$ is a closed formula, $\varphi_n^{(m)}(\alpha, a, n) = \varphi_n^{(m)}(\alpha, a, 0)$ ($\varphi_1, \ldots, \varphi_n$ are free variables).

$P(a, b)$ if and only if $a$ is a numerical proof of the formula $b$ (i.e., a proof in the elementary calculus, no variables). We recall that the consistency of numerical arithmetic can be proved in $Z_{\kappa}$.

If $a$ and $b$ are numbers of $A$ and $B$, then $\iota(a, b)$ is the number of $A|B$.

$U(a)$ if and only if $a$ is the number of a formula of the form $(\exists x) B(x)$, and then $\iota(a, g, y)$ is the number of $B(g)$. Similarly, $Q(a)$ if and only if $a$ is the number of a formula of the form $(\exists x) B(x)$, and then $\iota(a, g, y)$ is the number of $B(g)$.
For each k, a truth definition $T_k(b)$ can be given by means of a formula of $Z$, satisfying the following conditions (compare [9], p. 334):

$$T_k[\varphi(m,a,n)] \text{ if and only if } \varphi(y,q(m,a,n))$$

or $(Ez)(Ey)(x < \varphi(m,a,n) \land y < \varphi(m,a,n))$ and $T_k[\varphi(m,a,n)]$ or $(Ez)[(x,y) \in T_k[\varphi(m,a,n)]]$.

$$T_{k+1}[\varphi(m,a,n)] \text{ if and only if } T_k[\varphi(m,a,n)]$$

or $U[\varphi(m,a,n)] \land T_k[\varphi(m,a,n)]$ or $Q[\varphi(m,a,n)] \land T_k[\varphi(m,a,n)]$.

It can be verified in the usual manner that $T_k(b)$ is a normal truth definition in the sense of [7], and hence Con$(Z'_k)$ may be proved in $Z$. Note that $T_k(b)$ is a truth definition for the system $Z'_{k+1}$ only and not for $Z$; in particular, $s$ is not a free variable.

Observe that the consistency proof of Lemma B is capable of various extensions: e.g., if $Z$ is an extension of $Z$ by some principle of transfinite induction, we get a consistency proof of $Z''_0$ in $Z$.

II. Systems of arithmetic based on the predicate calculus

$P_n$ denotes the system consisting of the predicate calculus of first order with the closed formula $\forall$ of $Z$ as its only axiom. Prov$_n(m,n)$ is a primitive recursive formula of $Z$ such that for numerals $O^n$, $O^n$ can be proved in $Z$ if and only if $n$ is the number of a proof in $P_n$ of the formula with the number $m$; $\epsilon(O^n)$ is the number of the negation of the formula with number $n$.

We use $\uparrow_m$, $\uparrow A$ to mean: $A$ can be proved in $P_n$, $Z$.

**Theorem 1.** If $P_n$ is consistent, there is a theorem of $Z$ which cannot be proved in $P_n$.

By means of the Gödel substitution function we obtain a term $q$ of $Z$ whose value is the number of the following formula (which we call $\Omega$), namely

$$(\exists z)(Ey)(\varphi(x,z) \land y < \varphi(x,q))$$

$\Omega$ denotes $(Ey)(y < O^0 \land \varphi[y,q])$, and $\Omega^0$ denotes the disjunction $\varphi[0',q] \land \varphi[0',q] \lor \ldots \lor \varphi[O'^n,q]$. 

**Lemma 1.** The following argument is easily formalized in $Z$: if $\Omega$ can be proved in $P_n$, i.e., $\Omega \rightarrow \Omega$ can be proved in the predicate calculus, then, by Herbrand's theorem, there is a proof of $\Omega \rightarrow \Omega$ in the predicate calculus, in which no formula contains more quantifiers than the formula $\Omega \rightarrow \Omega$ itself, say $t$. This proof is also a proof in $Z'$.

By Lemma B,

$$\equiv_z([[Ez][\varphi(x,q) \rightarrow T_k[\varphi(x,q)]] \rightarrow T_k[\varphi(x,q)]]$$

Also

$$\equiv_z[[\varphi(x,q) \rightarrow T_k[\varphi(x,q)]]$$

But, since $T_k(a)$ is normal:

$$\equiv_z[T_k(a) \rightarrow \Omega], \equiv_z[T_k(a) \rightarrow \Omega]$$

Hence

$$\equiv_z[[\varphi(x,q) \rightarrow \Omega] \rightarrow \Omega]$$

**Lemma 2.** Either there is a theorem of $Z$ which cannot be proved in $P_n$, or there is a numeral $O^0$ such that $\equiv_z \Omega^0$.

(1) $\equiv_z \Omega$, i.e., $\equiv_z \Omega$ is false, and then, by lemma 1, we have a theorem of $Z$ which is not a theorem of $P_n$, or

(2) $\equiv_z \Omega$ can be proved in $P_n$ by a proof with number $n$, say.

In case (2) $\equiv_z \varphi[O^n, q]$, and thus:

either

(2.1) $\equiv_z \varphi[O^n, q]$ is false though $\equiv_z \varphi[O^n, q]$.

or

(2.2) $\equiv_z \varphi[O^n, q]$ and, since $\equiv_z \Omega$, also $\equiv_z \Omega$.

In case (2.2), since $\equiv_z \Omega$, $\Omega^0$ either

(2.21) $\equiv_z \Omega^0$ is false though $\equiv_z \Omega^0$ and $\equiv_z \Omega^0$.

or

(2.22) $\equiv_z \Omega^0$.

Since in each case whose number contains a digit 1, we have a theorem of $Z$ which cannot be proved in $P_n$, the lemma follows.

**Lemma 3.** If $\equiv_z \Omega^0$, then either $P_n$ is inconsistent or there is a theorem of $Z$ which cannot be proved in $P_n$.

Since $\Omega^0$ is a numerical formula, either $\equiv_z \Omega^0$ or $\equiv_z \Omega^0$. In the former case, $\equiv_z \Omega$ (when $P_n$ is inconsistent) or $\equiv_z \Omega$ is not a theorem of $P_n$, though, by lemma 1, it is a theorem of $Z$. In the latter case, either $\equiv_z \Omega^0$ cannot be proved in $P_n$ though $\equiv_z \Omega^0$, or both $\equiv_z \Omega^0$ and $\equiv_z \Omega^0$ when $P_n$ is inconsistent.

The theorem follows from the last two lemmata.
Remark 1. A shorter but less informative proof of Theorem 1 is the following. Let \( k \) be the number of quantifiers of \((a, b)\) (a prenex normal form) of \( \text{Con}(F, a) \). If \( \vdash \text{Con}(F, a) \), then there is a proof in the predicate calculus of \( \vdash \). By Herbrand's theorem, which can be established in \( \vdash \), if \( \vdash \) can be proved in the predicate calculus, it can be proved by a proof each formula of which contains \( k \) quantifiers. Thus \( \vdash \) can be proved in \( Z^0 \). Hence, by the normal truth definition of \( Z^0 \), \( \vdash \text{Con}(F, a) \rightarrow \vdash \). Therefore, the following formula is a theorem of \( Z \):

\[
(i) \quad \vdash \text{Con}(F, a).
\]

Now, if \( F \) contained all theorems of \( Z \), i.e., if \( F \) were an extension of \( Z \), the deductibility conditions of \( [9] \), p. 285 which ensure the application of Gödel's second undecidability theorem, would apply to \( F \), since we are using the "natural" proof predicate for \( F \); further, the formula \( (i) \) and hence \( \vdash \text{Con}(F, a) \) would be theorems of \( F \), and hence \( F \) would be inconsistent.

By analyzing the proofs in \( Z \) of the formulae 1, 2, 3 on p. 285, 286 of \([9]\), one could exhibit, as in our proof of Theorem 1, for any given consistent \( F \), a theorem of \( Z \) which cannot be proved in \( F \).

Remark 2. A proof of Theorem 1 can also be obtained by using Skolem's models for arithmetic as in \([16]\). Thus, given \( F \) and \( F \), either some theorems of \( Z \) are not derivable in \( F \), then nothing is left to prove. Or all theorems of \( Z \) are provable in \( F \). Then we can carry out for \( F \), Ryll-Nardzewski's construction in \([16]\), and get a case of the induction schema which is derivable in \( Z \) but not derivable in \( F \) if \( F \) is consistent. Hence, if all theorems of \( Z \) are provable in \( F \), \( F \) must be inconsistent. The proof of Theorem 1 is complete.

Remark 3. The theorem shows once again — if such a lesson were needed — the inadequacy of the "intuitive" (uncritical) approach to truth definitions: \( \vdash \) may well be an "intuitively false" formula, and, therefore, "intuitively" — any formula should be provable from \( \vdash \) via a truth definition, yet even not all theorems of \( Z \) can be proved from \( \vdash \) (by means of the predicate calculus). (We find it difficult to work up a paradox since the flaw is too obvious.)

III. Length of proof

The obvious idea underlying the present section is this: suppose a definition of length of proof of a system \( (F) \) is given such that, for each \( n \), \( \text{Con}(F^n) \) can be proved in \( (F) \) where \( F^n \) is the system obtained from \( F \) by retaining only those proofs of \( (F) \) whose length does not exceed \( m \); suppose further that Gödel's second undecidability theorem applies to \( (F^n) \); then the length of the proof of \( \text{Con}(F^n) \) in \( (F) \) exceeds \( m \). If \( (F) \) is a system in which \( \text{Con}(F) \) itself can be proved, and if \( \text{Con}(F^n) \) can be proved from \( F \) by proofs of bounded length (for all \( n \)) we have the following result:

Given any function \( \Phi \), there are infinitely many formulae \( A_n \) of \( (F) \) which can be proved in \( (F) \) with shortest proof of length \( l(n) \), and can be proved in \( (F) \) by a proof of length \( l(n) > \Phi(l(n)) \).

To see this, let a proof of \( \text{Con}(F) \) in \( (F) \) have length \( c \); the proofs of \( \text{Con}(F) \rightarrow \text{Con}(F^n) \) have length \( < d \); then we may take for \( A_n \) the formula \( \text{Con}(F^{d+c+n}) \) where \( n > \max \{ \Phi(i) \} \), \( i < c + d \).

Note that the introduction of \( \Phi \) here is unnatural; it is done only to permit easier comparison with the results of \([4]\) and \([14]\), where such a function \( \Phi \) is used. In fact the natural formulation is this: the shortest proofs of \( A_n \) in \( (F) \) are of bounded length, while the length of the shortest proof of \( A_n \) in \( (F) \) exceeds \( n \).

We shall now show that the definitions of length given in the first section apply here.

Theorem 2. Suppose the length of a proof of \( Z \) is measured by the number of distinct e-matrices which it contains, and suppose \( (F) \) is an extension of \( Z \), in which \( \text{Con}(Z) \) can be proved. Then, for each \( n \), \( \text{Con}(Z^n) \) can be proved in \( (F) \) by proofs of bounded length, but its shortest proof in \( Z^n \) is longer than \( n \).

For, \( \text{Con}(Z) \) means that there is no proof of \( Z \), which leads to \( 0 \), and \( \text{Con}(Z^n) \) means that there is no such proof of length \( < n \). Hence, \( \text{Con}(Z) \rightarrow \text{Con}(Z^n) \) is proved in \( F \) by the use of a single critical e-matrix. Since \( (F) \) is an extension of \( Z \), the length of the shortest proof in \( (F) \) of \( \text{Con}(Z^n) \) does not exceed the length of the shortest proof of \( \text{Con}(Z) \) by more than 1. Hence the shortest proofs of \( \text{Con}(Z^n) \) are of bounded length. By Lemma A, \( \text{Con}(Z^n) \) can be proved in \( Z^n \), but only by proofs of length exceeding \( n \).

Theorem 3. Suppose the length of a proof of \( Z \) is measured by the maximum number of bound variables that occur in any one of its formulae, and suppose \( (F) \) is an extension of \( Z \) in which \( \text{Con}(Z) \) can be proved. Then, for each \( n \), \( \text{Con}(Z^n) \) can be proved in \( (F) \) by proofs of bounded length, but its shortest proof in \( Z \) is longer than \( n \).

Proof as of Theorem 2, but by using Lemma B in place of Lemma A.

Remark 1. We observe in passing that there is a crucial difference between our definitions of "length of proof" and the one used in Mostowski's book \([14]\), where the number of a proof in a Gödel-numbered of proofs is taken as its length. On this definition there are only a finite number of proofs of length \( < n \), and, for each \( n \), their consistency is
expressed by a numerical formula (without variables), while both $Z^0$ and $Z^0$ contain infinitely many proofs (for each $n$). It may be observed that, if Mostowski’s definition is used, $\Phi$ may not be arbitrary, e.g. not $\Phi(n) = 0$ if the formula proved in $(F')$ by the proof with number $n$ cannot be proved in $(F)$.

$\Phi(n)$ = length of its shortest proof in $(F)$ if the formula which is proved in $(F')$ by the proof with number $n$ can be proved in $(F)$.

Obviously, we were led to the systems $Z^0$, $Z^0$ since it so happens that their consistency is easily proved in $Z$. We do not wish to suggest that the definiteness of length of proof which are used in the present section, are particularly natural. Perhaps a more natural one measures the length by the number of lines used in a proof: this is covered by Theorem 2 since a proof with $n$ lines has $< n$ critical matrices.

Remark 2. If $L(n)$ is the length of the proof with number $n$, as measured by the number of distinct $e$-matrices, and the proof of $\text{Con}(Z_0)$ in $(F')$ is of length $n_0$, then we can also prove Theorem 2 if we replace the formulae $\text{Con}(Z^0)$ by the formulae

$$(n)[L(n) < \Phi(n_0) + i \iff \text{Prov}_{(n', n)}],$$

which have the Gödel numbers $n_i (i = 1, 2, \ldots, n_0)$.

An alternative proof of Theorem 3 can be obtained similarly.

IV. Finite axiom systems of the predicate calculus

**Theorem 4.** If a finite axiom system $F$ has a model in $Z$ then $\text{Con}(F)$ can be proved in $Z$.

Let the model of $F$ be the theorem $F'$ of $Z$ which, by definition of a “model”, is obtained by substituting non-logical constants of $Z$ for the non-logical constants of $F$.

If $\rightarrow \text{Con}(F)$ there would be a proof of $\rightarrow F'$ in the predicate calculus, hence a proof of $\rightarrow F'$ in $Z^0$ where $m$ is the number of quantifiers of $F$. Hence, we have in $Z$:

$$\rightarrow \text{Con}(F) \rightarrow \text{Prov}(F')$$

(where $F'$ is the number of $F'$).

Since $\text{Prov}(F')$ is a normal truth definition,

$$\rightarrow \text{Prov}(F') \rightarrow \text{Prov}(F).$$

Thus, in $Z$,

$$F' \rightarrow \text{Con}(F).$$

Remark to Theorem 4. It is clear that instead of $Z$, we could have used the extensions of arithmetic mentioned after lemma B since

the whole proof merely depends on the use of a normal truth definition. However, it is not clear that the result applies if instead of $Z$ we use a system of analysis, e.g. $\Pi$ of Supplement IV to [9].

**Application of Theorem 4.** There can be no model in $Z_0$ for the system $(S)$ of [12], based on Bernays’ construction of a set theory with a finite number of axioms. (This result is in sharp contrast to the primitive recursive model in [1] of general set theory with an axiom schema, where, it can be shown, each axiom has a model in $Z_0$.) First, $\text{Con}(S) \rightarrow \text{Con}(Z_0)$ can be proved in $Z_0$ simply by following out the usual development of arithmetic in set theory. Next, if we had a model of $(S)$ in $Z_0$ we should have a proof of $\text{Con}(S)$ in $Z_0$ by Theorem 4, and hence a proof of $\text{Con}(Z_0)$, which is excluded by [9]. Thus we have decided a question raised in [8], p. 400.

Digression. By following out the steps of [12], §11, we obtain a proof of $\text{Con}(Z_0) \rightarrow \text{Con}(S)$ in $Z_0$ so that $\text{Con}(Z_0) \leftrightarrow \text{Con}(S)$ in $Z_0$.

**Theorem 5.** $\Pi \rightarrow \text{Con}(Z_0) \rightarrow \text{Con}(S)$.

By slightly reformulating $(S)$, we can eliminate all existential quantifiers from axioms of $(S)$ (see [12], p. 48). Let $\Pi$ or $\Pi(x_1, \ldots, x_m)$ be the conjunction of the axioms of $(S)$. If $\rightarrow \text{Con}(S)$, then the negation of $\Pi$, which is equivalent to a formula

$$\left(\exists x_1 \ldots \exists x_m\right) \rightarrow B(x_1, \ldots, x_m),$$

is provable in the predicate calculus. By the extended first $e$-theorem ([8], p. 32), we can also prove in the predicate calculus, for some constant $n_0$, a disjunction of the formulae:

$$\rightarrow B(i_1, \ldots, i_m), \quad 1 < i < n_0.$$

But, by [12], the formulae $B(i_1, \ldots, i_m)$ all have provable models in $Z_0$. Hence, we may derive $\rightarrow \text{Con}(Z_0)$ from $\rightarrow \text{Con}(S)$ in $Z_0$.

**Application of Theorem 5.** Let $ZF$ and $NB$ be respectively the set theories of Zermelo-Fraenkel and von Neumann-Bernays (see, e.g., [7], p. 271). It is known that if $\text{Con}(ZF)$ is added to $Z_0$, there is a model of $ZF$ in $Z_0$. Combining these models of axioms of $ZF$ with models of additional axioms of $NB$ which are similar to those of axioms of $(S)$, we may prove in $Z_0$ similarly as with $Z_0$ and $(S)$, the theorem: $\text{Con}(ZF) \rightarrow \text{Con}(NB)$.

References


On manifolds and $r$-spaces

by

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We shall say that a point $p$ belonging to a space $K$ is an $r$-point of $K$ if each neighbourhood of $p$ contains a neighbourhood $U$ of $p$ such that, for each $q \in U$, $Fr(U)$ is a deformation retract of $U - (q)$. (See [4]; $Fr(U)$ denotes the boundary of $U$, i.e. the set $\overline{U} \setminus (K - U)$.) The neighbourhood with the property just mentioned will be called a canonical neighbourhood. (It is worth noting that a canonical neighbourhood $U$ of a point $p$ is also a canonical neighbourhood for each point $q \in U$.)

The space $K$ is said to be an $r$-space if it is compact, metric, separable, finite dimensional, and if each point of $K$ is its $r$-point.

It turns out that $r$-spaces have a very similar structure to that of topological manifolds. In particular, many of the classical theorems about the manifolds (such as for instance theorems on the invariance of domain, internal characterization of separating sets and so on) hold also in $r$-spaces. Moreover, among the spaces of dimension $\leq 3$ and among the polytopes of dimension $\leq 3$ connected $r$-spaces are identical with the manifolds. The notion of $r$-space gives therefore a new topological characterization of 2-manifolds. The basic problem whether there exists an $r$-space not homeomorphic to a manifold remains open.

This paper should be considered as a continuation of the researches of K. Borsuk on the "spheroidal spaces" ([5], [3]). In particular, the proofs of some lemmas in §1 are suitably adapted proofs of the corresponding lemmas in [5] and [3]. These proofs are based on some auxiliary lemmas, which are modified lemmas from the papers mentioned. For the convenience of the reader all these auxiliary lemmas are gathered in the supplement at the end of the paper. (§1, §II etc. denote the lemmas from the supplement).

Terminology and notation. All set-theoretical topological notions used here are defined in [9]. Manifolds, pseudomanifolds, polytopes are meant in the sense defined in [2]. The homology theory here used

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