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An application of lattices to logic

by

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This paper is a continuation of our papers "*Algebraic treatment of the notion of satisfiability*" [6] and "*On existential theorems in non-classical functional calculi*" [7] cited hereafter as [AT] and [ET] respectively.

The method used in this paper is indeed the same as in [AT] and [ET], but the subjects of research are sentential calculi with quantifiers. Analogously to [AT] we shall first examine a non-specified sentential calculus with quantifiers $\overline{\mathcal{S}}$, determined in an obvious way by a sentential calculus $\mathcal{S}$. Later on we shall study some special sentential calculi with quantifiers $\overline{\mathcal{S}}_\varkappa$, $\overline{\mathcal{S}}_\lambda$, $\overline{\mathcal{S}}_\chi$, $\overline{\mathcal{S}}_\mu$, determined by the classical calculus $\mathcal{S}_\varkappa$, the Lewis calculus $\mathcal{S}_\lambda$, the Heyting calculus $\mathcal{S}_\chi$, and the minimal calculus $\mathcal{S}_\mu$, respectively. Since the positive sentential calculus with quantifiers is equivalent (in a certain sense) to $\overline{\mathcal{S}}_\chi$, we shall not examine it separately.

We shall formulate a necessary and sufficient condition for a given formula α to be a theorem of one of these calculi with quantifiers. This condition will be formulated in the language of the lattice theory. We shall show later that, in the cases of $\overline{\mathcal{S}}_\lambda$, $\overline{\mathcal{S}}_\chi$ and $\overline{\mathcal{S}}_\mu$, this condition can also be expressed in a topological form.

As an application we shall obtain — by means of the same method as in [ET] — some theorems on the elimination of the quantifier $\sum\limits_a$ in systems $\overline{\mathcal{S}}_\lambda$, $\overline{\mathcal{S}}_\chi$ and $\overline{\mathcal{S}}_\mu$ (see 5.5, 6.5, 7.5). These theorems are analogous to Theorems (λ), (χ), (μ) in [ET]. However, in contrast to [ET], theorems on elimination of quantifiers in sentential calculi with quantifiers do not imply directly the decidability of formulas of the form $\Xi\beta$ where, roughly speaking, Ξ is a sequence of quantifiers, and β is a formula without quantifiers (see [ET], Theorems (λ'), (χ')). The problem of decidability remains open. An algebraical interpretation of the theorems on elimination of quantifiers will be given in theorems 6.7 and 7.7.

As the second application we shall prove (see 5.4, 6.4, 7.4) that in systems $\overline{\mathcal{S}}_\chi$, $\overline{\mathcal{S}}_\lambda$, $\overline{\mathcal{S}}_\mu$ there exist infinite sequences of closed non-equivalent formulas in contrast to the classical calculus $\overline{\mathcal{S}}_\varkappa$, which is complete, *i. e.* for each closed formula α in $\bar{\mathcal{S}}_\varkappa$ either α or the negation of α is a theorem. The construction of a sequence of closed non-equivalent formulas in $\bar{\mathcal{S}}_\chi$ and $\bar{\mathcal{S}}_\lambda$ has a very simple topological interpretation.

The third application (§ 8) is the proof of a connection between $\bar{\mathcal{S}}_\chi$ and $\bar{\mathcal{S}}_\lambda$ (see the analogous theorems 15.1 and 15.2 in [AT]).

There is a slight difference between the method used in [AT] and the one applied in this paper. In order to ensure that all the infinite operations are feasible we operated in [AT] exclusively with complete lattices. In this paper we consider a wider class of lattices, called $\bar{\mathcal{S}}$-algebras. The question whether it suffices to consider only the complete lattices remains open. We do not know whether, in case of $\bar{\mathcal{S}}_\lambda$ (of $\bar{\mathcal{S}}_\chi$, of $\bar{\mathcal{S}}_\mu$), it suffices to consider the complete lattices of all (of all open) subsets of topological spaces.

This paper can be studied independently of [AT] and [ET]. In order to avoid the repetition of definitions, the definitions of an $\mathcal{S}$-algebra, of Lindenbaum algebras, and of the systems $\mathcal{S}_\varkappa$, $\mathcal{S}_\lambda$, $\mathcal{S}_\chi$, $\mathcal{S}_\pi$, $\mathcal{S}_\mu$ is merely outlined. The reader can find the exact definitions in [AT].

§ 1. Sentential calculi with quantifiers. Let $\mathcal{S}$ be a fixed consistent system of a sentential calculus [1]) containing as primitive symbols the sentential variables $a_1, a_2, \ldots$, parentheses and the following primitive constants:

(a) the disjunction sign $+$, the conjunction sign $\cdot$, the implication sign $\rightarrow$;

(b) some other binary sentential operators $o_1, \ldots, o_r$;

(c) some unary sentential operators $o^1, \ldots, o^s$.

The set of operators mentioned in (b) or (c) may be empty. The rules of inference in $\mathcal{S}$ are: the rule of substitution for sentential variables, modus ponens, and the rule of replacement of equivalent parts. We suppose that all the theorems of the positive sentential calculus are theorems of $\mathcal{S}$. However it is possible that some other formulas are also theorems of $\mathcal{S}$.

The letter I_0 will always denote the set of all positive integers.

The system $\mathcal{S}$ determines in an obvious way a system $\bar{\mathcal{S}}$ of sentential calculus with quantifiers $\sum\limits_{a_i}$ and $\prod\limits_{a_i}$.

The primitive symbols of $\bar{\mathcal{S}}$ are the parentheses, the sentential variables a_i (where $i \in I_0$), the constants mentioned in (a), (b), (c), and the quantifiers $\sum\limits_{a_i}, \prod\limits_{a_i}$ $(i \in I_0)$. The set $\bar{S}$ of formulas in $\bar{\mathcal{S}}$ is the smallest set such that

1) $a_i \in \bar{S}$ where $i \in I_0$;

[1]) The system $\mathcal{S}$ is exactly described in [AT], § 1.

2) if $\alpha, \beta \in \bar{S}$, then $(\alpha+\beta) \in \bar{S}$, $(\alpha\cdot\beta) \in \bar{S}$, $(\alpha\rightarrow\beta) \in \bar{S}$, $(\alpha o_k \beta) \in \bar{S}$ $(k=1,\ldots,r)$, $o^k\alpha \in \bar{S}$ $(k=1,\ldots,s)$, $\sum\limits_{a_i} \alpha \in \bar{S}$, $\prod\limits_{a_i} \alpha \in \bar{S}$.

We write $(\alpha\equiv\beta)$ instead of $\big((\alpha\rightarrow\beta)\cdot(\beta\rightarrow\alpha)\big)$.

We assume that the notion of free and bound occurrence of a sentential variable a_i is familiar.

The axioms of $\bar{\mathcal{S}}$ are all substitutions of all theorems of $\mathcal{S}$.

The set $\bar{T}$ of all theorems in $\bar{\mathcal{S}}$ is the least set of formulas such that

(i) $\bar{T}$ contains all axioms of $\bar{\mathcal{S}}$;

(ii) if $\alpha \in \bar{T}$, and if β is obtained from α by the admissible replacement of all free occurrences of a_i $(i \in I_0)$ by a formula $\gamma \in \bar{S}$

$$\beta = \alpha\binom{\gamma}{a_i},$$

then $\beta \in \bar{T}$;

(iii) if $\alpha \in \bar{T}$ and $(\alpha\rightarrow\beta) \in \bar{T}$, then $\beta \in \bar{T}$;

(iv) if $\alpha \in \bar{T}$ and $(\gamma\equiv\delta) \in \bar{T}$, and if γ is a part of α, then the formula β obtained from α by the replacement of the part γ by δ, is also in $\bar{T}$;

(v) if $(\alpha\rightarrow\beta) \in \bar{T}$ and there is no free occurrence of a_i in α (β), then $(\alpha\rightarrow\prod\limits_{a_i}\beta) \in \bar{T}$ (then $(\sum\limits_{a_i}\alpha\rightarrow\beta) \in \bar{T}$);

(vi) if $(\alpha\rightarrow\prod\limits_{a_i}\beta) \in \bar{T}$ (if $(\sum\limits_{a_i}\alpha\rightarrow\beta) \in \bar{T}$), then $(\alpha\rightarrow\beta) \in \bar{T}$.

If $\alpha \in \bar{T}$, we shall also write $\vdash \alpha$.

§ 2. $\bar{\mathcal{S}}$-algebras. The system $\mathcal{S}$ determines also a kind of abstract algebras (called *$\mathcal{S}$-algebras* [2])) with algebraic operations corresponding to the logical sentential operators $+, \cdot, \rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s$. The $\mathcal{S}$-algebras, which are the matrices of the system $\mathcal{S}$, are relatively pseudo-complemented lattices (with the sum (join) $a+b$ and the product (meet) $a\cdot b$) having the unit element e which is the distinguished element corresponding to the logical value of truth. If an $\mathcal{S}$-algebra is a complete lattice, it is called an *$\mathcal{S}^*$-algebra*.

Let A be an $\mathcal{S}$-algebra. Every formula $\alpha \in \bar{S}$ can be interpreted as an algebraic function (denoted by $(A)\Phi_\alpha$) by treating

(a) all sentential variables a_i as variables running over A;

(b) each of the logical signs mentioned in § 1 (a), (b), (c) as the corresponding algebraic operation in A;

(c) the quantifiers $\sum\limits_{a_i}$ and $\prod\limits_{a_i}$ as the signs of the infinite sums $(A)\sum\limits_{a_i \in A}$ and products $(A)\prod\limits_{a_i \in A}$ respectively [3]).

[2]) See [AT], § 3.

[3]) If A is a lattice and $a_u \in A$ for each $u \in U$, then $(A)\sum\limits_{u \in U} a_u$ (then $(A)\prod\limits_{u \in U} a_u$) is the sum (the product) of all the elements a_u in the lattice A, whenever this sum (product) exists. See [AT], p. 68.

Let $\{x_i\}$ be an infinite sequence of elements of A. If, for a given formula $\alpha \in \bar{S}$ and for the substitution

$$\text{(s)} \qquad a_i = x_i \qquad (i \in I_0)$$

there exist all the sums and products corresponding to the logical quantifiers in the algebraic interpretation $(A)\Phi_\alpha$ of α, then the sequence $\{x_i\}$ is said to be a *valuation of* α.

Obviously, it is sufficient to determine the substitution (s) only for those a_i whose free occurrences appear in α. However, for the simple formulation of the definition of valuation of α it is convenient to assume that the substitution (s) is determined for all a_i $(i \in I_0)$.

The value of the function $(A)\Phi_\alpha$ for the substitution (s) will then be denoted by $(A)\Phi_\alpha(\{x_i\})$.

If the sequence $\{x_i\}$ is a valuation of every formula $\alpha \in \bar{S}$, then $\{x_i\}$ is said to be a *valuation of* $\bar{S}$.

If every sequence $\{x_i\}$ of elements of an $\mathcal{S}$-algebra A is a valuation of $\bar{S}$, then A will be called an *$\bar{\mathcal{S}}$-algebra.*

Obviously the element $(A)\Phi_\alpha(\{x_i\})$ depends only on the elements x_i where i is a positive integer such that α contains a free occurrence of a_i.

The following equations can be considered as the inductive definition of $(A)\Phi_\alpha(\{x_i\})$.

2.1. *If* $\alpha, \beta \in \bar{S}$ *and* o *is a binary operation* (see § 1 (a), (b)), *then*

$$(A)\Phi_{\alpha o \beta}(\{x_i\}) = (A)\Phi_\alpha(\{x_i\})\, o\, (A)\Phi_\beta(\{x_i\}).$$

If o *is a unary operation* (see § 1 (c)), *then*

$$(A)\Phi_{o\alpha}(\{x_i\}) = o\,(A)\Phi_\alpha(\{x_i\}).$$

Analogously

$$(A)\Phi_{\sum\limits_{a_k}\alpha}(\{x_i\}) = (A)\sum_{x'_k \in A}(A)\Phi_\alpha(\{x'_i\}), \qquad (A)\Phi_{\prod\limits_{a_k}\alpha}(\{x_i\}) = (A)\prod_{x'_k \in A}(A)\Phi_\alpha(\{x'_i\}),$$

where $\{x'_i\}$ *is any sequence of elements of* A *such that* $x'_i = x_i$ *if* $i \neq k$, *and* x'_k *is arbitrary.*

If $(A)\Phi_\alpha(\{x_i\}) = a \in A$ for each valuation $\{x_i\}$ of α, we write $(A)\Phi_\alpha = a$.

2.2. *If* $\vdash \alpha$, *then* $(A)\Phi_\alpha = e$ *for every* $\mathcal{S}$*-algebra* A.

The easy proof by induction with respect to the length of the proof of α is omitted.

An $\bar{\mathcal{S}}$-algebra A is said to be *functionally* $\bar{\mathcal{S}}$*-free* if, for every $\alpha \in \bar{S}$, the condition $(A)\Phi_\alpha = e$ implies that $(B)\Phi_\alpha = e$ for every $\bar{\mathcal{S}}$-algebra B.

§ 3. The Lindenbaum algebra $\bar{L}$. For every $\alpha \in \bar{S}$, let $|\alpha|$ denote the class of all $\beta \in \bar{S}$ such that $\vdash(\alpha \equiv \beta)$. Let $\bar{L}$ be the set of all cosets $|\alpha|$ where $\alpha \in \bar{S}$. We define the algebraic operation in $\bar{L}$ as follows (see [AT], § 4):

$$|\alpha|\, o\, |\beta| = |\alpha\, o\, \beta|$$

if o is one of the binary operations mentioned in § 1 (a), (b), and

$$o|\alpha| = |o\alpha|$$

if o is one of the unary operations mentioned in § 1 (c). The element $|\alpha|$, where $\vdash \alpha$, will be denoted by e.

3.1. *$\bar{L}$ is an $\bar{\mathcal{S}}$-algebra. In particular $\bar{L}$ is relatively pseudocomplemented lattice with the operation $+$ (join), $\cdot$ (meet), $\rightarrow$ (relative pseudo-complement), with the unit e and with the zero element $0 = |\prod\limits_{a_1} a_1|$. The lattice inclusion $|\alpha| \subset |\beta|$ holds if and only if $\vdash(\alpha \rightarrow \beta)$.*

$|\alpha| = e$ if and only if $\vdash \alpha$.

For every $\alpha \in \bar{S}$, $k \in I_0$ and $\beta \in \bar{S}$, let $\alpha\binom{\beta}{a_k}$ be the formula obtained from α by the substitution of the formula β for each free occurrence of a_k. We assume that the necessary changes in the bound occurrences of variables of α have been performed before the operation of substitution has been applied. Obviously the element $\left|\alpha\binom{\beta}{a_k}\right|$ of $\bar{L}$ is uniquely determined.

More generally, let $\{\beta_i\}$ be a sequence of formulas of $\bar{S}$. Then, for every $\alpha \in \bar{S}$, let $\alpha\left(\left\{\begin{matrix}\beta_i\\ a_i\end{matrix}\right\}\right)$ be the formula obtained from α by the substitution of formulas β_i for the variables a_i $(i \in I_0)$ respectively.

3.2. *Let $\alpha \in \bar{S}$, $\beta_i \in \bar{S}$ for $i \in I_0$. Then*

$$(*) \qquad \left|\left(\sum_{a_k}\alpha\right)\left(\left\{\begin{matrix}\beta_i\\ a_i\end{matrix}\right\}\right)\right| = (\bar{L})\sum_{\beta'_k \in \bar{S}}\left|\alpha\left(\left\{\begin{matrix}\beta'_i\\ a_i\end{matrix}\right\}\right)\right|$$

and

$$(**) \qquad \left|\left(\prod_{a_k}\alpha\right)\left(\left\{\begin{matrix}\beta_i\\ a_i\end{matrix}\right\}\right)\right| = (\bar{L})\prod_{\beta'_k \in \bar{S}}\left|\alpha\left(\left\{\begin{matrix}\beta'_i\\ a_i\end{matrix}\right\}\right)\right|$$

where $\{\beta'_i\}$ is any sequence such that

$$\beta'_i = \beta_i \quad \textit{if} \quad i \neq k,$$

and β'_k is an arbitrary formula of $\bar{S}$.

The proof is similar to that of 4.3 in [AT], therefore it is omitted.

3.3. *Let $\alpha\in\bar{S}$ and $\beta_i\in\bar{S}$ for $i\in I_0$. Then*

(*) $$\left|\left(\sum_{a_k}\alpha\right)\left(\begin{Bmatrix}\beta_i\\ a_i\end{Bmatrix}\right)\right| = (\bar{L})\sum_{\beta'_k\in V}\left|\alpha\left(\begin{Bmatrix}\beta'_i\\ a_i\end{Bmatrix}\right)\right|$$

and

(**) $$\left|\left(\prod_{a_k}\alpha\right)\left(\begin{Bmatrix}\beta_i\\ a_i\end{Bmatrix}\right)\right| = (\bar{L})\prod_{\beta'_k\in V}\left|\alpha\left(\begin{Bmatrix}\beta'_i\\ a'_i\end{Bmatrix}\right)\right|$$

where V is the set of all sentential variables and $\{\beta'_i\}$ is any sequence such that $\beta'_i=\beta_i$ for $i\neq k$ and $\beta'_k\in V$.

The proof of 3.3 is similar to that of 3.2.

3.4. *For every $\alpha\in\bar{S}$ and for every sequence $\beta_i\in\bar{S}$,*

$$(\bar{L})\Phi_\alpha(\{|\beta_i|\}) = \left|\alpha\left(\begin{Bmatrix}\beta_i\\ a_i\end{Bmatrix}\right)\right|.$$

The proof is by induction with respect to the length of α. In the case of $\alpha=a_k$ the theorem is obvious. Suppose that $\alpha=\gamma o\delta$, where o is one of the binary operators mentioned in § 1 (a), (b). If Theorem 3.4 holds for γ and δ, then it holds for α, since

$$(\bar{L})\Phi_\alpha(\{|\beta_i|\}) = (\bar{L})\Phi_\gamma(\{|\beta_i|\})\, o\, (\bar{L})\Phi_\delta(\{|\beta_i|\})$$
$$= \left|\gamma\left(\begin{Bmatrix}\beta_i\\ a_i\end{Bmatrix}\right)\right| o \left|\delta\left(\begin{Bmatrix}\beta_i\\ a_i\end{Bmatrix}\right)\right| = \left|\alpha\left(\begin{Bmatrix}\beta_i\\ a_i\end{Bmatrix}\right)\right|.$$

In the case of $\alpha=o\gamma$, where o is one of the unary operators mentioned in § 1 (c), the proof is similar.

Suppose that $\alpha=\sum_{a_k}\gamma$ and that Theorem 3.4 holds for γ. Then

$$(\bar{L})\Phi_\alpha(\{|\beta_i|\}) = (\bar{L})\sum_{\beta'_k\in\bar{L}}(\bar{L})\Phi_\gamma\{|\beta'_i|\} = (\bar{L})\sum_{\beta'_k\in\bar{L}}\left|\gamma\left(\begin{Bmatrix}\beta'_i\\ a_i\end{Bmatrix}\right)\right|$$
$$= \left|\left(\sum_{a_k}\gamma\right)\left(\begin{Bmatrix}\beta_i\\ a_i\end{Bmatrix}\right)\right| = \left|\alpha\left(\begin{Bmatrix}\beta_i\\ a_i\end{Bmatrix}\right)\right|,$$

where $\beta'_i=\beta_i$ for $i\neq k$ and $\beta'_k\in\bar{L}$.

In the case of $\alpha=\prod_{a_k}\gamma$ the proof is similar.

It follows immediately from 3.4 that

3.5. *$\bar{L}$ is an $\bar{\mathcal{S}}$-algebra.*

3.6. *For every $\alpha\in\bar{S}$*

$$(\bar{L})\Phi_\alpha(\{|a_i|\}) = |\alpha|.$$

It follows from 3.6 that

3.7. *If non $\vdash\alpha$, then $(\bar{L})\Phi_\alpha(\{|a_i|\})\neq e$.*

We infer from 3.7 and 2.2 that

3.8. *The algebra $\bar{L}$ is functionally $\bar{\mathcal{S}}$-free.*

Combining 2.2 and 3.6 we obtain

3.9. *The following conditions are equivalent for every formula $\alpha\in\bar{S}$:*
(i) $\vdash\alpha$;
(ii) *$(A)\Phi_\alpha=e$ for every $\bar{\mathcal{S}}$-algebra A;*
(iii) *$(\bar{L})\Phi_\alpha=e$.*

§ 4. The classical calculus. Consider the case where $\mathcal{S}$ is the classical sentential calculus (see [AT], § 9) denoted here as $\mathcal{S}_\varkappa$. The sentential calculus with quantifiers, determined by $\mathcal{S}=\mathcal{S}_\varkappa$ by the method described in § 1, is the classical sentential calculus with quantifiers (see [1] and [2]), denoted by $\bar{\mathcal{S}}_\varkappa$.

The set of all formulas in $\bar{\mathcal{S}}_\varkappa$ will be denoted by $\bar{S}_\varkappa$.

$\bar{L}_\varkappa$ will denote the Lindenbaum algebra constructed by the method described in § 3.

The $\bar{\mathcal{S}}_\varkappa$-algebras are Boolean algebras and conversely. Every Boolean algebra is also an $\bar{\mathcal{S}}_\varkappa$-algebra. The letter B will always denote a Boolean algebra. In particular B_0 will always denote the two-element Boolean algebra.

The following theorems are known (see [1] and [2]), but their proofs seem to be new.

4.1. *For each $\alpha\in\bar{S}_\varkappa$, $\vdash\alpha$ if and only if $(B_0)\Phi_\alpha=e$.*

The necessity follows immediately from 2.2. To prove the sufficiency, let us suppose that non $\vdash\alpha$. Let $\mathfrak{p}$ be a prime ideal of $\bar{L}_\varkappa$ such that [4])

1) $|\alpha|\in\mathfrak{p}$,

2) $\mathfrak{p}$ preserves all the sums and products mentioned in 3.2 (*) and (**).

The quotient algebra $\bar{L}_\varkappa/\mathfrak{p}$ is the two-element Boolean algebra B_0. For every $\beta\in\bar{S}_\varkappa$ we have

$$(B_0)\Phi_\beta(\{[|a_i|]\}) = [|\beta|],$$

where $[b]$ denotes the element of $B_0=\bar{L}_\varkappa/\mathfrak{p}$ determined by an element $b\in\bar{L}_\varkappa$. The proof of this equation is by induction with respect to the length of α.

In particular

$$(B_0)\Phi_\alpha(\{[|a_i|]\}) = [|\alpha|] = 0, \qquad \text{q. e. d.}$$

4.2. *If a_k, $\sum_{a_k}$ and $\prod_{a_k}$ do not appear in α, then*

$$\vdash\left(\prod_{a_i}\alpha \equiv \left(\alpha\left(\begin{matrix}\beta\\ a_i\end{matrix}\right)\cdot\alpha\left(\begin{matrix}\gamma\\ a_i\end{matrix}\right)\right)\right)$$

[4]) Such a prime ideal exists. See *e. g.* [5], Lemma (iv), p. 197.

and

$$\vdash \left(\sum_{a_i} \alpha \equiv \left(\alpha\binom{\beta}{a_i} + \alpha\binom{\gamma}{a_i}\right)\right)$$

where $\beta = (a_k \cdot (-a_k))$ *and* $\gamma = (a_k + (-a_k))$.

This follows immediately from 4.1. Indeed, the algebraical functions corresponding to both sides of each of the above equations are identically equal (the algebra is supposed to be B_0) since $(B_0)\Phi_\beta = 0$ and $(B_0)\Phi_\gamma = e$.

4.3. *Let B be an $\overline{\mathcal{S}}_\varkappa$-algebra. Then for every $\alpha \in \overline{S}_\varkappa$*

$$(B)\prod_{x_k \in B}(B)\Phi_\alpha(\{x_i\}) = (B)\Phi_\alpha(\{x_i'\}) \cdot (B)\Phi_\alpha(\{x_i''\})$$

and

$$(B)\sum_{x_k \in B}(B)\Phi_\alpha(\{x_i\}) = (B)\Phi_\alpha(\{x_i'\}) + (B)\Phi_\alpha(\{x_i''\})$$

where

$$x_i' = x_i = x_i'' \text{ for } i \neq k, \text{ and } x_k' = 0 \in B,\ x_k'' = e \in B.$$

This results from 4.2 since the elements on both sides of the equations in 4.3 are the values of the algebraical functions corresponding to the formulas on both sides of the equivalences in 4.2. The algebraical functions corresponding to equivalent formulas are always identical.

4.4. *Every Boolean algebra B is a functionally $\overline{\mathcal{S}}_\varkappa$-free algebra.*

By 2.2 and 4.1 it suffices to prove that if $(B)\Phi_\alpha = e$ for a formula $\alpha \in \overline{S}_\varkappa$, then $(B_0)\Phi_\alpha = e$. It follows from 4.3 that if $x_i = 0$ or e for $i \in I_0$, then $(B_0)\Phi_\alpha = (B)\Phi_\alpha$ since all sums and products $\sum_{x \in B_0}$ and $\prod_{x \in B_0}$ can be replaced by $\sum_{x \in B}$ and $\prod_{x \in B}$ respectively (B_0 is interpreted as the subalgebra of B composed only of the zero element and the unit element). This remark completes the proof.

§ 5. The modal calculus. Now consider the case where $\mathcal{S}$ is the sentential calculus of Lewis (see *e. g.* [AT], § 10), denoted here by $\mathcal{S}_\lambda$. Besides the signs $+$, $\cdot$, $\rightarrow$, the system $\mathcal{S}_\lambda$ contains also the two unary sentential operators: the negation sign $-$, and the possibility sign $\mathbf{C}$. We shall write $\mathbf{I}$ instead of $-\mathbf{C}-$.

The sentential calculus with quantifiers, obtained by the method described in § 1 where $\mathcal{S} = \mathcal{S}_\lambda$, will be denoted by $\overline{\mathcal{S}}_\lambda$.

The $\mathcal{S}_\lambda$-algebras are closure algebras and conversely (see *e. g.* [AT], § 10). The letter C will exclusively denote closure algebras. A closure algebra C is said to be *topological* if it is formed of some subsets of a topological space $\mathcal{X}$, finite sums and products in C are the usual set-theoretical operations, and the operation $\mathbf{C}$ in C is that induced by the clo-

sure operation in $\mathcal{X}$. If $Z \subset \mathcal{X}$, then $\mathbf{I}Z$ is the interior of Z. The closure algebra of all subsets of a topological space $\mathcal{X}$ will be denoted by $\boldsymbol{C}(\mathcal{X})$.

$\overline{S}_\lambda$ will denote the set of all formulas in $\overline{\mathcal{S}}_\lambda$, and $\overline{L}_\lambda$ will denote the Lindenbaum algebra of $\overline{S}_\lambda$ (see § 2 where $\overline{\mathcal{S}} = \overline{\mathcal{S}}_\lambda$).

The $\overline{\mathcal{S}}_\lambda$-algebra $\overline{L}_\lambda$ is isomorphic to a topological closure algebra (see [AT], Theorem 10.1) C_λ of some subsets of a space $\mathcal{X}_\lambda$, and this isomorphism h preserves all the infinite sums and products corresponding to the logical quantifiers, *i. e.* all the sums and products mentioned in 3.2 (*) and (**), the corresponding infinite operations in C_λ are the usual set-theoretical ones.

5.1. *The following conditions are equivalent for every $\alpha \in \overline{S}_\lambda$:*

(i) $\vdash \alpha$;

(ii) $(C)\Phi_\alpha = e \in C$ *for every $\overline{S}_\lambda$-algebra C;*

(iii) $(C)\Phi_\alpha = \mathcal{X}$ *for every topological $\overline{\mathcal{S}}_\lambda$-algebra C formed of subsets of a space $\mathcal{X}$;*

(iv) $(C_\lambda)\Phi_\alpha = \mathcal{X}_\lambda$ *(i. e., by isomorphim, $(\overline{L}_\lambda)\Phi_\alpha = e \in \overline{L}_\lambda$).*

The implication (i)→(ii) follows from 2.2. The implications (ii)→(iii) and (iii)→(iv) are trivial. The implication (iv)→(i) follows from 3.7.

The equivalence (i)≡(iv) can also be formulated as follows:

5.2. *The topological closure algebra C_λ is functionally $\overline{\mathcal{S}}_\lambda$-free.*

The problem whether there exists a functionally $\overline{\mathcal{S}}_\lambda$-free closure algebra $\boldsymbol{C}(\mathcal{X})$, where $\mathcal{X}$ is a topological space, is unsolved.

5.3. *If $\mathcal{X}$ is a T_1-space dense in itself, then the closure algebra $\boldsymbol{C}(\mathcal{X})$ is not functionally $\overline{\mathcal{S}}_\lambda$-free* [5]).

Consider the formula

(a) $$\alpha_0 = \sum_{a_1}\left((Ca_1) \cdot (C - a_1)\right).$$

If X is a T_1-space, then $(\boldsymbol{C}(\mathcal{X}))\Phi_{\alpha_0}$ is identically equal to the derived set (*i. e.* the set of all cluster points) of the space $\mathcal{X}$ since it is the sum of boundaries of all subsets of $\mathcal{X}$. Hence if $\mathcal{X}$ is dense in itself, then $(\boldsymbol{C}(\mathcal{X}))\Phi_{\alpha_0} = \mathcal{X} =$ the unit of $\boldsymbol{C}(\mathcal{X})$. On the other hand, if $\mathcal{X}$ contains an isolated point, then $(\boldsymbol{C}(\mathcal{X}))\Phi_{\alpha_0} \neq \mathcal{X}$.

In particular, we infer that non $\vdash \alpha_0$.

It follows from 4.1 that, in the classical calculus $\overline{\mathcal{S}}_\varkappa$, for each closed formula $\alpha \in \overline{S}_\varkappa$, either $\vdash \alpha$ or $\vdash(-\alpha)$. This means, roughly speaking, that the classical calculus $\overline{\mathcal{S}}_\varkappa$ is indeed the two-valued logic. In particular, there is no infinite sequence of closed non-equivalent formulas $\alpha_n \in \overline{S}_\varkappa$ (more exactly if $n > 2$, there is no n-element sequence of closed

[5]) The idea of the proof of 5.3 was suggested to us by a remark due to A. Grzegorczyk.

non-equivalent formulas of $\mathcal{S}_\varkappa$). The Lewis calculus $\bar{\mathcal{S}}_\lambda$ does not possess this property, it is indeed an infinitely valued logic.

5.4. *There is an infinite sequence $\{\alpha_n\}$ of closed non-equivalent formulas of the system $\bar{\mathcal{S}}_\lambda$.*

Let α_0 be the formula defined by (a), and let

$$\alpha_n = \sum_{a_{n+1}} \Big(C(\alpha_{n-1}\cdot a_{n+1})\cdot C\big(\alpha_{n-1}\cdot(-a_{n+1})\big)\Big) \quad \text{for} \quad n=1,2,\ldots$$

Let $\mathcal{X}$ be a T_1-space. Then the functions $\big(\mathbf{C}(\mathcal{X})\big)\Phi_{\alpha_n}$ are constant since α_n contains no free variable. The set $Z_n=\big(\mathbf{C}(\mathcal{X})\big)\Phi_{\alpha_n}$ is the derived set of the closed set Z_{n-1}, since it is the sum of all boundaries (relative to the space Z_{n-1}) of all subsets of Z_{n-1}. Consequently, Z_n is the $(n+1)$th derived set of the space $\mathcal{X}$ [6]).

There are T_1 spaces such that all derived sets Z_n $(n=0,1,2,\ldots)$ are different, $Z_n\neq Z_m$ for $n\neq m$. Since the algebraical functions of equivalent formulas are always identically equal, we infer that no two of the formulas α_n are equivalent.

Let x_0 be a fixed element such that $x_0 \notin \mathcal{X}_\lambda$, and let $\mathcal{X}_\lambda^0=\mathcal{X}_\lambda+(x_0)$. We shall consider the set $\mathcal{X}_\lambda^0$ as a topological space with the following definition of the closure operation in $\mathcal{X}_\lambda^0$: $\mathbf{C}^0(0)=0$, and $\mathbf{C}^0(X)=\mathbf{C}^0(X\mathcal{X}_\lambda)+(x_0)$ if $0\neq X\subset\mathcal{X}_\lambda^0$ ($\mathbf{C}$ denotes the closure operation in $\mathcal{X}_\lambda$). Let $\mathbf{I}$ and $\mathbf{I}^0$ denote the interior operations in the space $\mathcal{X}_\lambda$ and $\mathcal{X}_\lambda^0$ respectively, *i. e.* $\mathbf{I}(X)=X-\mathbf{C}(\mathcal{X}_\lambda-X)$, and analogously for $\mathbf{I}^0$.

Let C_λ^0 be the closure algebra of all sets X and $X+(x_0)$ where $X\in C_\lambda$, the closure operation in C_λ^0 being $\mathbf{C}^0$.

5.5. *If the formula $\sum_{a_k}\mathbf{I}\alpha\in\bar{S}_\lambda$ is a theorem of $\bar{\mathcal{S}}_\lambda$, $\vdash\sum_{a_k}\mathbf{I}\alpha$, then there exists such a formula $\beta\in\bar{S}_\lambda$ that $\vdash\alpha\binom{\beta}{a_k}$.*

Let h be the isomorphism of $\bar{L}_\lambda$ onto C_λ mentioned on p. 91.

One can prove by induction with respect to the length of the formula that C_λ^0 is an $\bar{\mathcal{S}}_\lambda$-algebra and that

(b) $$\mathcal{X}_\lambda\cdot\big((C_\lambda^0)\Phi_\beta(\{X_i\})\big)=(C_\lambda)\Phi_\beta(\{X_i\mathcal{X}_\lambda\})$$

for every $\beta\in\bar{S}_\lambda$ and for every sequence $X_i\in C_\lambda^0$.

Suppose that $\vdash\gamma$, $\gamma=\sum_{a_k}\mathbf{I}\alpha$. By 5.1 and 2.1

(c) $$\mathcal{X}_\lambda^0=(C_\lambda^0)\Phi_\gamma\big(\{h(|a_i|)\}\big)=\sum_{X_k\in C_\lambda^0}\mathbf{I}\big((C_\lambda^0)\Phi_\alpha(\{X_i\})\big)$$

where $X_i=h(|a_i|)$ for $i\neq k$, and $X_k\in C_\lambda^0$ is arbitrary.

[6]) The definition of the nth derived set of a set $Z\subset X$ is by induction: the $(n+1)$th derived set of Z is the derived set of the nth derived set of Z.

All the sets under the sign $\sum$ in (c) are open. Since there is only one open set which contains the element x_0, viz. the set $\mathcal{X}_\lambda^0$, we infer that

$$\mathbf{I}\big((C_\lambda^0)\Phi_\alpha(\{X_i\})\big)=\mathcal{X}_\lambda^0$$

for a sequence $\{X_i\}$ where $X_i=h(|a_i|)$ for $i\neq k$, and where X_k is an element of C_λ^0. Consequently, for the sequence $\{X_i\}$,

$$(C_\lambda^0)\Phi_\alpha(\{X_i\})=\mathcal{X}_\lambda^0,$$

i. e., by (b)

(d) $$(C_\lambda)\Phi_\alpha(\{X_i\mathcal{X}_\lambda\})=\mathcal{X}_\lambda^0\mathcal{X}_\lambda=\mathcal{X}_\lambda.$$

We have $X_i\mathcal{X}_\lambda=h(|a_i|)$ for $i\neq k$, and $X_k\mathcal{X}_\lambda=h(|\beta|)$ where β is a formula $\in\bar{S}_\lambda$ since all elements of C_λ^0 are of this form. Let $\beta_i=a_i$ for $i\neq k$ and $\beta_k=\beta$. By (d)

$$(\bar{L}_\lambda)\Phi_\alpha(\{|\beta_i|\})=e\in\bar{L}_\lambda$$

since h is an isomorphism. Consequently, by 3.4

$$\left|\alpha\binom{\beta}{a_k}\right|=(\bar{L}_\lambda)\Phi_\alpha(\{|\beta_i|\})=e\in\bar{L}_\lambda,$$

i. e.

$$\vdash\alpha\binom{\beta}{a_k}, \qquad \text{q. e. d.}$$

By the same method we obtain

5.6. *If $\alpha,\beta\in\bar{S}_\lambda$, and $\vdash(\mathbf{I}\alpha+\mathbf{I}\beta)$, then either $\vdash\alpha$ or $\vdash\beta$.*

§ 6. The intuitionistic calculus. Consider now the case where $\mathcal{S}$ is the Heyting sentential calculus (see [AT], § 11) denoted here by $\mathcal{S}_\chi$. Besides the signs $+$, $\cdot$, $\to$ the system $\mathcal{S}_\chi$ contains the negation sign $\urcorner$. The sentential calculus obtained from $\mathcal{S}_\chi$ by the method described in § 1 will be denoted by $\bar{\mathcal{S}}_\chi$. Of course $\bar{\mathcal{S}}_\chi$ is the sentential intuitionistic calculus with quantifiers. The set of all formulas of $\bar{\mathcal{S}}_\chi$ will be denoted by $\bar{S}_\chi$, and the Lindenbaum algebra of $\bar{\mathcal{S}}_\chi$ will be denoted by $\bar{L}_\chi$.

The $\mathcal{S}_\chi$-algebras are Heyting algebras and conversely (see [AT], § 11). The letter H will exclusively denote Heyting algebras. A Heyting algebra is said to be *topological*, if it is formed of some open subsets of a topological space $\mathcal{X}$, the finite sums and products in H are the usual set-theoretical operations, the operation $a\to b$ is defined as the interior of the set $(X-a)+b$, and $\urcorner a$ is the interior of the set $X-a$. The Heyting algebra of all open subsets of a topological space $\mathcal{X}$ will be denoted by $\boldsymbol{H}(\mathcal{X})$.

The $\bar{\mathcal{S}}_\chi$-algebra $\bar{L}_\chi$ is isomorphic to a topological Heyting algebra (see [AT], Theorem 11.2) H_χ of some open subsets of a space $\mathcal{X}_\chi$, and this isomorphism h preserves all the infinite sums and products corresponding to the logical quantifiers (*i. e.* all the sums and products mentioned in 3.2 (*) and (**)), the corresponding infinite operation $\sum$ in H_χ is the set-theoretical union, and the corresponding operation $\prod$ in H_χ is the interior of the set-theoretical intersection.

6.1. *The following conditions are equivalent for each formula* $\alpha \in \bar{S}_\chi$:

(i) $\vdash \alpha$;

(ii) $(H)\Phi_\alpha = e \in H$ *for every* $\bar{\mathcal{S}}_\chi$*-algebra* H;

(iii) $(H)\Phi_\alpha = \mathfrak{X}$ *for every topological* $\bar{\mathcal{S}}_\chi$*-algebra* H *formed of some open subsets of a space* $\mathfrak{X}$;

(iv) $(H_\chi)\Phi_\alpha = \mathfrak{X}_\chi$ (*i. e., by isomorphism,* $(\bar{L}_\chi)\Phi_\alpha = e \in \bar{L}_\chi$).

The proof is the same as the proof of the corresponding theorem 5.1.

The equivalence (i) ≡ (iv) can also be formulated as follows:

6.2. *The topological Heyting algebra* H_χ *is functionally* $\bar{\mathcal{S}}_\chi$*-free.*

The problem whether there exists a functionally $\bar{\mathcal{S}}_\chi$-free Heyting algebra $\boldsymbol{H}(\mathfrak{X})$, where $\mathfrak{X}$ is a topological space, is unsolved.

6.3. *If* $\mathfrak{X}$ *is a* T_1*-space dense in itself, then the Heyting algebra* $\boldsymbol{H}(\mathfrak{X})$ *is not functionally* $\bar{\mathcal{S}}_\chi$*-free.*

Let

(a) $$a_1 = \prod_{a_1} \big(a_1 + (\neg a_1)\big)$$

and let $\alpha = \neg \alpha_1$. If $\mathfrak{X}$ is a T_1-space, then $\big(\boldsymbol{H}(\mathfrak{X})\big)\Phi_{\alpha_1}$ is identically equal to the interior of the intersection of all open dense subsets of $\mathfrak{X}$, since an open set $G \subset X$ is dense in X if and only if it is of the form $A + (\neg A) = A + \mathbf{I}(X - A)$ where $A \in \boldsymbol{H}(\mathfrak{X})$, and conversely. Otherwise speaking, $\big(\boldsymbol{H}(\mathfrak{X})\big)\Phi_{\alpha_1}$ is the set of all isolated points of the space $\mathfrak{X}$. Consequently, $\big(\boldsymbol{H}(\mathfrak{X})\big)\Phi_\alpha$ is the interior of the derived set of the space $\mathfrak{X}$.

Therefore $\big(\boldsymbol{H}(\mathfrak{X})\big)\Phi_\alpha = \mathfrak{X}$ if $\mathfrak{X}$ is dense in itself. On the other hand, $\big(\boldsymbol{H}(\mathfrak{X})\big)\Phi_\alpha \neq \mathfrak{X}$ if $\mathfrak{X}$ is not dense in itself.

6.4. *There is an infinite sequence* $\{\alpha_n\}$ *of closed non-equivalent formulas of the system* $\bar{\mathcal{S}}_\chi$ [7]).

Let $\mathfrak{X}$ be a T_1-space. Notice that, for $A, B \in \boldsymbol{H}(\mathfrak{X})$, the set

$$A \to B = \mathbf{I}\big((\mathfrak{X} - A) + B\big)$$

is equal to the set

$$B + \mathbf{I}_{\mathfrak{X}-B}\big((\mathfrak{X} - B) - A\big)$$

where, for $Z \subset Y$, the set $\mathbf{I}_Y(Z)$ is the interior of the set Z with respect to the subspace $Y \subset \mathfrak{X}$, *i. e.*

$$\mathbf{I}_Y(Z) = Y - Y \cdot \mathbf{C}(Y - Z).$$

[7]) The analogous theorem for the system $\mathcal{S}_\chi$ was proved by J. C. C. McKinsey and A. Tarski [4].

In fact, $B + \mathbf{I}_{\mathfrak{X}-B}\big((\mathfrak{X} - B) - A\big)$ is the greatest set G open in $\mathfrak{X}$ such that $G(A - B) = 0$, *i. e.* such that $GA \subset B$. The same property characterizes (see [AT], p. 68, footnote [11])) the set $G = A \to B$.

Consequently, if $A, B \in \boldsymbol{H}(\mathfrak{X})$, then the set

(b) $$A + (A \to B)$$

is the complement of the boundary of $A(\mathfrak{X} - B)$ taken with respect to the space $\mathfrak{X} - B$, *i. e.* the set $A + (A \to B)$ is the sum of the set B and a set open and dense in the space $\mathfrak{X} - B$. Conversely, each set G which is the sum of B and a set open and dense in $\mathfrak{X} - B$ can be written in the form (b), viz.

$$G = G + (G \to B).$$

In fact, G is open in $\mathfrak{X}$, $G - B$ is open and dense in $\mathfrak{X} - B$. Therefore $\mathbf{I}_{\mathfrak{X}-B}\big((\mathfrak{X} - B) - G\big) = 0$, *i. e.* $G \to B = B$. Consequently

$$G + (G \to B) = G + B = G.$$

By (b) the set

(c) $$\prod_{A \in \boldsymbol{H}(\mathfrak{X})} \big(A + (A \to B)\big)$$

is the sum of B and the set of all isolated points of $\mathfrak{X} - B$. The set (c) is thus open in $\mathfrak{X}$.

Let

$$B_1 = \prod_{A \in \boldsymbol{H}(\mathfrak{X})} \big(A + (A \to 0)\big) = \prod_{A \in \boldsymbol{H}(\mathfrak{X})} \big(A + (\neg A)\big),$$

and by induction

$$B_{n+1} = \prod_{A \in \boldsymbol{H}(\mathfrak{X})} \big(A + (A \to B_n)\big) \quad \text{for} \quad n = 1, 2, \ldots$$

The set B_1 is the set of all isolated points of $\mathfrak{X}$, *i. e.* B_1 is the complement of the first derived set of $\mathfrak{X}$. By (c), the set B_2 is the sum of B_1 and of the set of all isolated points of the $\mathfrak{X} - B_1$, *i. e.* B_2 is the complement of the second derived set of the space $\mathfrak{X}$. More generally, the set B_n $(n = 1, 2, \ldots)$ is the complement of the nth derived set of the space $\mathfrak{X}$.

Let X be a T_1-space such that all the derived sets of $\mathfrak{X}$ of the order $1, 2, \ldots$ are different. Then

$$B_1 \subset B_2 \subset B_3 \subset \ldots$$

but

$$B_n \neq B_m \quad \text{if} \quad n \neq m.$$

Let

$$\alpha_1 = \prod_{a_1} \big(a_1 + (\neg a_1)\big),$$

and, by induction,

$$\alpha_{n+1} = \prod_{a_{n+1}} \big(a_{n+1} + (a_{n+1} \to \alpha_n)\big), \qquad n = 1, 2, \ldots$$

It follows from 2.1 that $(\boldsymbol{H}(\mathfrak{X}))\Phi_{a_n}$ is identically equal to $B_n \subset \mathfrak{X}$, $n=1,2,\ldots$ The closed formulas a_n $(n=1,2,\ldots)$ are not equivalent, since the functions $(H)\Phi_a$ of equivalent formulas are always identically equal, and in the case of the space $\mathfrak{X}$ mentioned above

$$(\boldsymbol{H}(\mathfrak{X}))\Phi_{a_n} \neq (\boldsymbol{H}(\mathfrak{X}))\Phi_{a_m} \quad \text{for} \quad n \neq m.$$

6.5. *If the formula* $\sum_{a_k} a \in \overline{S}_\chi$ *is a theorem of* $\overline{\mathcal{S}}_\chi$, $\vdash \sum_{a_k} a$, *then there exists such a formula* $\beta \in \overline{S}_\chi$ *that* $\vdash a\binom{\beta}{a_k}$.

The proof of this theorem is the same as that of theorem 5.5. It suffices to omit the sign $\mathbf{I}$ and to replace the spaces $\mathfrak{X}_\lambda$, $\mathfrak{X}_\lambda^0$, the closure algebras C_λ and C_λ^0 and the Lindenbaum algebra $\overline{L}_\lambda$ by the spaces $\mathfrak{X}_\chi$, $\mathfrak{X}_\chi^0$, the Heyting algebras H_χ, H_χ^0 and the Lindenbaum algebra $\overline{L}_\chi$ respectively. The space $\mathfrak{X}_\chi^0$ is constructed in the same way as the space $\mathfrak{X}_\lambda^0$, viz. $\mathfrak{X}_\chi^0 = \mathfrak{X}_\chi + (x_0)$, where x_0 is an element, $x_0 \notin \mathfrak{X}_\chi$. The open sets in $\mathfrak{X}_\chi^0$ are the sets G and $\mathfrak{X}_\chi^0$, where G is open in $\mathfrak{X}_\chi$. The Heyting algebra H_χ^0 is formed of the set $\mathfrak{X}_\chi^0$ and of all sets belonging to H_χ.

By the same method we obtain

6.6. *If* $a, \beta \in \overline{S}_\chi$, and $\vdash (a+\beta)$, *then either* $\vdash a$ *or* $\vdash \beta$.

Theorems 6.5 and 6.6 can also be formulated as follows:

6.7. *The set* $\mathfrak{p} = \overline{L}_\chi - (e)$ *is an ideal of* $\overline{L}_\chi$. *The ideal* $\mathfrak{p}$ *is the only maximal ideal of* $\overline{L}_\chi$. *It is enumerably additive in the following sense: if all components of an infinite sum corresponding to the logical quantifier* $\sum$ *are in* $\mathfrak{p}$, *then the sum is also in* $\mathfrak{p}$.

Let $\overline{\mathcal{S}}_\pi$ denote the system (with quantifiers) obtained from the system $\mathcal{S} = \mathcal{S}_\pi$ by the method described in § 1, where $\mathcal{S}_\pi$ is the positive sentential calculus (see [AT], § 12).

6.8. *The system* $\overline{\mathcal{S}}_\chi$ *is equivalent to the system* $\overline{\mathcal{S}}_\pi$ *with the rule of definition and with the following definition of the negation*:

(*) $$\neg a = a \to \prod_{a_1} a_1.$$

It follows from 6.8 that the separate examination of $\overline{\mathcal{S}}_\pi$ is superfluous.

To prove 6.8 let us notice that the formulas

(d) $$\Big(\big(a_i \to \prod_{a_1} a_1\big) \to (a_i \to a_k)\Big)$$

(e) $$\Big(\big(a_i \to \big(a_k \to \prod_{a_1} a_1\big)\big) \to \big(a_k \to \big(a_i \to \prod_{a_1} a_1\big)\big)\Big)$$

are theorems in $\overline{\mathcal{S}}_\pi$. Hence, using the definition of negation, we infer that the following formulas are theorems in $\overline{\mathcal{S}}_\pi$:

(d′) $$\big(\neg a_i \to (a_i \to a_k)\big),$$

(e′) $$\Big(\big(a_i \to (\neg a_k)\big) \to \big(a_k \to (\neg a_i)\big)\Big).$$

If we add those formulas to the set of axioms of $\mathcal{S}_\pi$, we obtain a set of axioms of $\mathcal{S}_\chi$. Hence, if $a \in \overline{S}_\chi$ and a is a theorem of $\overline{\mathcal{S}}_\chi$, then a is also a theorem of $\overline{\mathcal{S}}_\pi$.

Conversely, if a formula $a \in \overline{S}_\pi$ is a theorem of $\overline{\mathcal{S}}_\pi$, then a is also a theorem of $\overline{\mathcal{S}}_\chi$. This remark is obvious if a does not contain the sign $\neg$. Suppose that a contains the sign $\neg$. Let a^* be the formula obtained from a by the elimination of the sign $\neg$ on account of the definition (*). Of course a^* is a theorem of $\overline{\mathcal{S}}_\pi$. Thus a^* is a theorem of $\overline{\mathcal{S}}_\chi$. It is easy to see that the formula

$$\neg a_i \equiv \big(a_i \to \prod_{a_1} a_1\big)$$

is a theorem of $\overline{\mathcal{S}}_\chi$. We obtain a from a^* by replacing some parts of the form $\beta \to \prod_{a_1} a_1$ by $\neg \beta$. This implies that a is also a theorem of $\overline{\mathcal{S}}$.

§ 7. The minimal calculus.

Consider now the case where $\mathcal{S}$ is the minimal sentential calculus (see [AT], § 13) denoted here by $\mathcal{S}_\mu$. This system contains the operators $+$, $\cdot$, $\to$, and the negation sign $\sim$. The sentential calculus with quantifiers (obtained from $\mathcal{S}_\mu$ by the method described in § 1) will be denoted by $\overline{\mathcal{S}}_\mu$. The set of all formulas of $\overline{\mathcal{S}}_\mu$ and the Lindenbaum algebra of $\overline{\mathcal{S}}_\mu$ will be denoted by $\overline{S}_\mu$ and $\overline{L}_\mu$ respectively.

$\mathcal{S}_\mu$-algebras are algebras $\langle M; e; +, \cdot, \to, \sim \rangle$ such that $\langle M; e; +, \cdot, \to \rangle$ is a relatively pseudocomplemented lattice with the unit e, and $\sim a = a \to \sim e$ where $\sim e$ is an arbitrary but fixed element of M (see [AT], § 13). The letter M will exclusively denote $\mathcal{S}_\mu$-algebras. An $\mathcal{S}_\mu$-algebra is said to be *topological* if it is formed of some open subsets of a topological space $\mathfrak{X}$, the definition of $a+b$, $a \cdot b$, $a \to b$ being the same as in the case of topological Heyting algebras (see § 6). $\boldsymbol{M}(\mathfrak{X}, G)$ denotes the $\mathcal{S}_\mu^*$-algebra of all open subsets of a topological space $\mathfrak{X}$ such that $\sim e = \sim \mathfrak{X} = G \in \boldsymbol{M}(\mathfrak{X})$.

The $\overline{\mathcal{S}}_\mu$-algebra $\overline{L}_\mu$ is isomorphic (see [AT], Theorem 13.2) to a topological $\overline{\mathcal{S}}_\mu$-algebra M_μ of some open subsets of a topological space $\mathfrak{X}_\mu$, and this isomorphism preserves all the infinite sums and products corresponding to the logical quantifiers (see 3.2 (*) and (**)), the corresponding infinite operations $\sum$ and $\prod$ in M_μ being the same as in the case of H_μ (see p. 94).

7.1. *The following conditions are equivalent for each formula* $\alpha \in \overline{S}_\mu$:

(i) $\vdash \alpha$;

(ii) $(M)\Phi_\alpha = e \in M$ *for every* $\overline{S}_\mu$*-algebra* M;

(iii) $(M)\Phi_\alpha = \mathfrak{X}$ *for every topological* $\overline{S}_\mu$*-algebra* M *formed of open subsets of a space* $\mathfrak{X}$;

(iv) $(M_\mu)\Phi_\alpha = \mathfrak{X}_\mu$ (*i. e., by isomorphism,* $(\overline{L}_\mu)\Phi_\alpha = e \in \overline{L}_\mu$).

7.2. *The topological* $\overline{S}_\mu$*-algebra* M_μ *is functionally* $\overline{S}_\mu$*-free.*

The proof is the same as in § 6.

The problem whether there exists a functionally $\overline{S}_\mu$-free algebra $\boldsymbol{M}(\mathfrak{X}, G)$, where $\mathfrak{X}$ is a topological space, is unsolved.

7.3. *If* X *is a* T_1*-space dense in itself, and* G *is an open subset of* X, *then* $\boldsymbol{M}(\mathfrak{X}, G)$ *is not functionally* $\overline{S}_\mu$*-free.*

The formula

$$\alpha = \Big(\prod_{a_2}\Big(a_2 + \big(a_2 \to \prod_{a_1} a_1\big)\Big)\Big) \to \prod_{a_1} a_1$$

has the following property: $\big(\boldsymbol{M}(\mathfrak{X}, G)\big)\Phi_\alpha = \mathfrak{X}$ if and only if $\mathfrak{X}$ is dense in itself. The proof of 7.3 is the same as that of 6.3.

7.4. *There is an infinite sequence* $\{\alpha_n\}$ *of closed non-equivalent formulas of the system* $\overline{S}_\mu$.

In fact, the formulas α_n $(n = 1, 2, \ldots)$ satisfying theorem 6.4 are not equivalent in $\overline{S}_\mu$ since they are not equivalent in the stronger system $\overline{S}_\chi$.

7.5. *If the formula* $\sum_{a_k} \alpha \in \overline{S}_\mu$ *is a theorem of* $\overline{S}_\mu$, *then there exists a formula* $\beta \in \overline{S}_\mu$ *such that the substitution* $\alpha\binom{\beta}{a_k}$ *is a theorem of* $\overline{S}_\mu$.

The proof is the same as that of 6.5.

By the same method we obtain

7.6. *If* $\alpha, \beta \in \overline{S}_\mu$ *and* $\vdash (\alpha + \beta)$, *then either* $\vdash \alpha$ *or* $\vdash \beta$.

Theorems 7.5 and 7.6 can also be formulated as follows:

7.7. *The set* $\mathfrak{p} = \overline{L}_\mu - (e)$ *is an ideal of* $\overline{L}_\mu$. *The ideal* $\mathfrak{p}$ *is the only maximal ideal of* $\overline{L}_\mu$. *It is enumerably additive in the following sense: if all components of an infinite sum corresponding to the logical quantifier* $\sum$ *are in* $\mathfrak{p}$, *then the sum is also in* $\mathfrak{p}$.

7.8. *The negation* $\neg$ *defined by the formula*

$$\neg\alpha = \alpha \to \prod_{a_1} a_1 \quad \textit{for} \quad \alpha \in \overline{S}_\mu$$

is the intuitionistic negation in $\overline{S}_\mu$. *The negation* $\neg$ *is not equivalent (in the system* $\overline{S}_\mu$) *to the minimal negation* $\sim$.

The first remark has been proved in 6.8. The second remark follows from the fact that the formula

$$\beta = (\sim a_2) \to \big(a_2 \to \prod_{a_1} a_1\big)$$

is not a theorem of $\overline{S}_\mu$. In fact, let M be a complete pseudocomplemented lattice and let $\sim e = e$. The last equation determines uniquely the operation $\sim a$ in M, therefore M is an $\overline{S}_\mu$-algebra. If $x_i = e \in M$ for $i = 1, 2, \ldots$, then

$$(M)\Phi_\alpha(\{x_i\}) = (\sim e) \to (e \to 0) = e \to 0 = 0,$$

which completes the proof (see 2.2).

§ 8. The connection between the intuitionistic and the modal logic.

Let ψ be the transformation of $\overline{S}_\chi$ into $\overline{S}_\lambda$ defined by induction as follows (see [AT], § 15):

(i) $\psi(a_1) = \mathbf{I}(a_1)$,

(ii) $\psi(\alpha + \beta) = \psi(\alpha) + \psi(\beta)$,

(iii) $\psi(\alpha \cdot \beta) = \psi(\alpha) \cdot \psi(\beta)$,

(iv) $\psi(\alpha \to \beta) = \mathbf{I}\big(-\psi(\alpha) + \psi(\beta)\big)$,

(v) $\psi(\neg\alpha) = \mathbf{I}\big(-\psi(\alpha)\big)$,

(vi) $\psi\big(\sum_{a_k} \alpha\big) = \sum_{a_k} \psi(\alpha)$,

(vii) $\psi\big(\prod_{a_k} \alpha\big) = \mathbf{I}\prod_{a_k} \psi(\alpha)$.

8.1. *Let* C *be a closure algebra and let* $\boldsymbol{H}(C)$ *be the Heyting algebra of all open elements of* C. *If the sequence* $\{x_i\}$ *of elements of* C *is a valuation of* $\psi(\alpha)$ *where* $\alpha \in \overline{S}_\chi$, *then the sequence* $\{\mathbf{I}x_i\}$ *is a valuation of* α *and*

$$(C)\Phi_{\psi(\alpha)}(\{x_i\}) = \big(\boldsymbol{H}(C)\big)\Phi_\alpha(\{\mathbf{I}x_i\}).$$

The proof is by induction with respect to the length of α.

8.2. *A formula* $\alpha \in \overline{S}_\chi$ *is a theorem of* $\overline{S}_\chi$ *if and only if* $\psi(\alpha)$ *is a theorem of* $\overline{S}_\lambda$.

Suppose that $\psi(\alpha)$ is a theorem of $\overline{S}_\lambda$. Let C be a closure algebra such that $\overline{L}_\chi = \boldsymbol{H}(C)$ is the algebra of all open elements of C (such a closure algebra exists by a theorem of Tarski and McKinsey [3][8]). By 8.1, 3.6 and 2.2 we have

$$|\alpha| = (\overline{L}_\chi)\Phi_\alpha(\{|a_i|\}) = (C)\Phi_{\psi(\alpha)}(\{|a_i|\}) = e,$$

which proves that α is a theorem in $\overline{S}_\chi$ (see 3.1).

[8]) The Brouwerian algebras examined in that paper are dual to Heyting algebras.

Suppose now that $\psi(\alpha)$ is not a theorem of $\bar{\mathcal{S}}_\lambda$. Let H be the Heyting algebra of all open elements of $\bar{L}_\lambda$, $H=\boldsymbol{H}(\bar{L}_\lambda)$. Then by the same argumentation as above

$$(H)\Phi_\alpha(\{\mathbf{I}|a_i|\})=(\bar{L}_\lambda)\Phi_{\psi(\alpha)}(\{|a_i|\})=|\alpha|\neq e,$$

which proves that α is not a theorem of $\bar{\mathcal{S}}_\chi$ (see 2.2).

References

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Some applications of formalized consistency proofs

by

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Introduction

A well-known result due to Gödel [5] states: If (F) is one of the usual systems of arithmetic (for sufficient conditions on (F) see [9], p. 285) the formula which is ordinarily taken as the arithmetization of the *consistency* of (F) cannot be proved in (F) itself. Thus the consistency proof for Z_μ^1 ([9], p. 293) due to Ackermann [2] uses the principle of ordinal induction up to the first ε-number, which cannot be formalized in Z_μ, and the consistency proof by means of a truth definition ([9], p. 339) uses a predicate which cannot be formalized in Z_μ either. It is now natural to ask whether the "ideas" of these consistency proofs may be formalized in Z_μ: the result of such a step would then be a consistency proof for a subsystem (F) of Z_μ; (F) would be demonstrably weaker than Z_μ since a formula of Z_μ which expresses the consistency of (F) would be provable in Z_μ but not in (F). We shall denote such a formula by $\mathrm{Con}(F)$; it is to be understood that the formula chosen for expressing the consistency of (F) satisfies conditions sufficient to ensure the application of Gödel's second undecidability theorem.

In this way we are led to systems which are obtained from Z_μ by suppressing all those proofs of Z_μ which are too "complex"; several definitions of complexity will be used, the principal ones being the maximum number of bound variables occurring in any formula of the proof, and the number of distinct critical ε-matrices (Grundtypen, [9], p. 93), if the Hilbert ε-symbol is used instead of quantifiers. These measures of complexity are suggested by the two consistency proofs mentioned above. We may note in passing that, for our present purpose, the consistency proof by means of a truth definition is more appropriate because it can be immediately applied to any extension of Z_μ by means of transfinite induction, and other principles of proof which satisfy the rule of infinite induction ([11], p. 124).

Our first application of these results concerns the elimination of the induction scheme of Z_μ by means of a finite set of axioms (which are themselves formulae of Z_μ). It was established in [13] and [16] that