Remarquons que le théorème peut être faux sans l'hypothèse que G soit connexe. En effet, soit $G=\{0,1\}$ un groupe de deux éléments, chacun de mesure 1/2. Soit $A=B=\{O\}$. Alors, $A+B=\{O\}$.

THÉORÈME 2. *Soit G un groupe topologique compact quelconque. Soient $A,B\subset G$ mesurables avec $m(A)+m(B)>1$. Alors on a $A+B=G$.*

En effet, soit $x\in G$. Alors, $m(x-A)=m(A)$, donc $(x-A)\cap B\neq\varphi$ (l'ensemble vide), ce qui implique que $x\in A+B$, c. q. f. d.

Travaux cités

[1] B. Eckmann, *Über monothetische Gruppen*, Com. Math. Helv. 16 (1943), p. 249-263.

[2] P. R. Halmos and H. Sammelson, *On monothetic groups*, Proc. Nat. Acad. Sc. 28 (1942), p. 254-258.

[3] A. M. Macbeath, *On measure of sum sets II*, Proc. Cambr. Philos. Soc. 49 (1953), p. 40-43.

[4] H. Ostmann, *Verfeinerte Lösung der asymptotischen Dichtenaufgabe*, Journ. Reine u. Ang. Math. 87 (1950), p. 183-188.

[5] Д. А. Райков, *О сложении множеств в смысле Шнирельмана*, Мат. Сборн. 5 (1939), p. 425-438.

Reçu par la Rédaction le 31. 3. 1954

A property of plane homeomorphisms

by

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In a real Euclidean plane E let (x,y) denote the Cartesian coordinates of a point p, and let Θ denote the set of all homeomorphisms of E onto itself which are of the form

$$(x,y)\to(x',y')$$

where

$$x'=x,\qquad y'=\Phi(x,y)\qquad\text{or}\qquad x'=\Phi(x,y),\qquad y'=y.$$

It is supposed that either the first alternative holds for every point (x,y) of E or the second alternative holds for every point of E. Denote by Ξ the group formed by all finite superpositions of any of the transformations of Θ. S. Ulam[1]) has raised the question as to whether it is possible to approximate to any arbitrary homeomorphism of the plane onto itself by members of Ξ.

The solution of the problem depends upon the meaning to be assigned to the word "approximate". In § 1 of this paper it is shown that if the approximation is to be uniform then the answer is in the negative, that is to say, if for any two homeomorphisms $\mathfrak{H}_1,\mathfrak{H}_2$ of the plane E we write

$$\delta(\mathfrak{H}_1,\mathfrak{H}_2)=\underset{p\in E}{\text{up. bd.}}\ \varrho\big(\mathfrak{H}_1(p),H_2(p)\big),$$

where ϱ denotes the Euclidean distance, then a homeomorphism $\mathfrak{G}$ can be constructed such that for any member $\mathfrak{H}$ of Ξ, $\delta(\mathfrak{H},\mathfrak{G})>1$.

The example that is constructed here, depends essentially upon the fact that the plane is not compact. If we restrict ourselves to compact subsets the situation is different. In §§ 2 and 3 we prove that if S is a closed square with its sides parallel to the coordinate axes and if Θ' is the subclass of the members of Θ which leave each frontier point of S fixed and if Ξ' is the group generated by finite superpositions of mem-

[1]) S. Ulam, *Problème 60*, Fundamenta Mathematicae 24 (1935), p. 324.

bers of Θ'; then given any plane homeomorphism $\mathfrak{G}$ of S onto itself that leaves the frontier points of S fixed, there exists a sequence of homeomorphisms $\mathfrak{H}_n$ belonging to Ξ' such that

$$\delta'(\mathfrak{H}_n, \mathfrak{G}) = \operatorname*{up.bd.}_{p \in S} \varrho\big(\mathfrak{H}_n(p), \mathfrak{G}(p)\big) \to 0 \quad \text{as} \quad n \to \infty .$$

The result proved in § 3 is used in § 4 to show that if we interpret $\mathfrak{H}_n \to \mathfrak{G}$ as $n \to \infty$ to mean that $\mathfrak{H}_n(p) \to \mathfrak{G}(p)$ for every point p of E then to any given homeomorphism $\mathfrak{G}$ corresponds a sequence of homeomorphisms $\mathfrak{H}_n$, belonging to Ξ such that $\mathfrak{H}_n \to \mathfrak{G}$ as $n \to \infty$.

§ 1. Uniform approximation

Let A denote a simple arc with end-points a_1, a_2 in the plane E and let p be any point of the plane which does not belong to A. Let q denote a variable point on A. Consider the change in direction of the line pq, directed from p to q when q describes A from a_1 to a_2. If we fix on a direction through p from which to measure angles, say pl, take a particular sense of rotation as positive and assign a particular appropriate value to $\sphericalangle a_1 pl$, then there will be two points (or more) of A say q' and q'' such that for all q of A,

$$\sphericalangle q'pl \leqslant \sphericalangle qpl \leqslant \sphericalangle q''pl .$$

Write

$$(1) \qquad \sphericalangle q''pl - \sphericalangle q'pl = \alpha(A, p),$$

$$(2) \qquad \beta(A) = \operatorname*{up.bd.}_{p \in E-A} \alpha(A, p).$$

$\alpha(A,p)$ is independent of the sense of description of A and of the particular way in which the angles $\sphericalangle qpl$ are measured. It is always a positive or zero number. It is also finite (for if it were infinite A would wind round p infinitely often and since p is at a positive distance from A this would mean that A was not locally connected) but $\beta(A)$ may be infinite. However we shall use this function of A only when it is finite. We then have the following lemma:

LEMMA 1. *If $\beta(A)$ is finite and H is a homeomorphism of the class Θ then*

$$(3) \qquad \beta\big(\mathfrak{H}(A)\big) \leqslant 2\pi + \beta(A).$$

Suppose that $\mathfrak{H}$ is $(x,y) \to (x',y')$ where

$$x' = x, \qquad y' = \Phi(x,y).$$

If H is of the alternative form the argument is similar with x and y interchanged. Since when p varies over $E-A$, $\mathfrak{H}(p)$ varies over $E-\mathfrak{H}(A)$ it is sufficient to show that for any p of $E-A$

$$(4) \qquad \alpha\big(\mathfrak{H}(A), \mathfrak{H}(p)\big) \leqslant 2\pi + \alpha(A, p).$$

In proving (4) we assume (without loss of generality since $\alpha(A,p)$ varies continuously with A) that A is a polygonal arc with no segments parallel to the y-axis. Let A' be a minimal subarc of A for which

$$(5) \qquad \alpha\big(\mathfrak{H}(A'), \mathfrak{H}(p)\big) = \alpha\big(\mathfrak{H}(A), \mathfrak{H}(p)\big).$$

If A' degenerates to a single point the result is trivially true, otherwise let p be the point (x_1, y_1).

Firstly consider the case when A' does not intersect the line $x = x_1$. Then $\mathfrak{H}(A')$ does not intersect the line $x = x_1$. Thus, since this line contains the point $\mathfrak{H}(p)$,

$$(6) \qquad \alpha\big(\mathfrak{H}(A), \mathfrak{H}(p)\big) = \alpha\big(\mathfrak{H}(A'), \mathfrak{H}(p)\big) \leqslant \pi \leqslant 2\pi + \alpha(A, p).$$

Next consider the case when A' does intersect the line $x = x_1$. Let A' have end-points e, f and meet the line $x = x_1$ in points whose order on A' from e to f is $p_1, p_2, \ldots, p_n$ where e may be p_1 and f may be p_n. Since the subarc ep_1 of A' lies entirely (except for the point p_1) on one side of $x = x_1$, so does the subarc $\mathfrak{H}(ep_1)$ of $\mathfrak{H}(A')$. Similarly $\mathfrak{H}(fp_2)$ lies entirely on one side of $x = x_1$. Thus if the line $\mathfrak{H}(p)m$ is in a fixed direction through $\mathfrak{H}(p)$ we have,

$$(7) \qquad \alpha\big(\mathfrak{H}(A'), \mathfrak{H}(p)\big) \leqslant 2\pi + |\sphericalangle \mathfrak{H}(p_n)\mathfrak{H}(p)m - \sphericalangle \mathfrak{H}(p_1)\mathfrak{H}(p)m| .$$

Further

$$(8) \qquad |\sphericalangle p_n pl - \sphericalangle p_1 pl| = |\sphericalangle \mathfrak{H}(p_n)\mathfrak{H}(p)m - \sphericalangle \mathfrak{H}(p_1)\mathfrak{H}(p)m|$$

and

$$(9) \qquad \alpha(A, p) \geqslant \alpha(A', p) \geqslant |\sphericalangle p_n pl - \sphericalangle p_1 pl| .$$

Thus from (5), (7), (8), (9)

$$(10) \qquad \alpha\big(\mathfrak{H}(A), \mathfrak{H}(p)\big) \leqslant 2\pi + \alpha(A, p),$$

and this is the required inequality (4).

We can now prove the main result which we state as a theorem.

THEOREM 1. *There are homeomorphisms of the plane onto itself which are not the uniform limit of any sequence of members of Ξ.*

Let K_n be the arc whose equation in polar coordinates is

$$(11) \qquad r = (4 + e^{-\theta} - 5e^{-2n\pi})(1 - e^{-2n\pi})^{-1}, \qquad 0 \leqslant \theta \leqslant 2\pi; \qquad n = 1, 2, \ldots$$

This arc is part of a spiral which starts at $r=5$, $\theta=0$ winds round the origin n times with r monotonically decreasing and ends at the point $r=4$, $\theta=2n\pi$.

Denote by M_n the arc obtained from K_n by a translation parallel to the x-axis by an amount $10n$. The arcs $M_1, M_2, \ldots, M_n, \ldots$ are disjoint. There is a homeomorphism of the plane onto itself which maps the segment $L_n = \{(x,y);\ 10n-1 \leqslant x \leqslant 10n,\ y=0\}$ onto the arc M_n, $n=1,2,\ldots$ Denote this homeomorphism by $\mathfrak{G}$ and let $\mathfrak{H}$ be a member of Ξ which we may suppose is obtained by the superposition of the members $\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_s$ of Θ.

Then $\beta(L_n)=\pi$ and thus by lemma 1,

$$\beta\big(\mathfrak{H}(L_n)\big) \leqslant (2s+1)\pi, \qquad n=1,2,\ldots \tag{12}$$

Consider the segment L_{s+1}. Let q be the point with coordinates $x=10(s+1)$, $y=0$. Write $p=\mathfrak{G}^{-1}(q)$. Then

$$a\big(\mathfrak{G}(L_{s+1})\mathfrak{G}(p)\big) = a(M_{s+1}, q) = (2s+2)\pi. \tag{13}$$

Now if it were the case that $\delta(\mathfrak{H},\mathfrak{G}) \leqslant 1$, then $\mathfrak{H}(L_{s+1})$ would be an arc whose end-points would be distant at most 1 from the end-points of $\mathfrak{G}(L_{s+1}) = M_{s+1}$ and which is such that when p' describes L_{s+1} the points $\mathfrak{H}(p')$ and $\mathfrak{G}(p')$ are distant apart at most 1. But under these circumstances $\mathfrak{G}(p')$ winds round the point q $s+1$ times keeping at a distance of at least 4 from it. Thus $\mathfrak{H}(p')$ also winds round q $s+1$ times except that its end-points may be such that it fails to complete the $(s+1)$th rotation by an angle less than $2\sin^{-1}(1/4)$ [2]). Hence

$$a\big(\mathfrak{H}(L_{s+1}), \mathfrak{G}(p)\big) \geqslant (2s+2)\pi - 2\sin^{-1}(1/4) > (2s+1)\pi. \tag{14}$$

Since the inequalities (12) and (14) are in contradiction with one another, $\delta(\mathfrak{H},\mathfrak{G}) > 1$ and the theorem is proved.

§ 2. Homeomorphisms of certain subsets of a closed square onto themselves

Let S be a closed square with sides parallel to the coordinate axes. We shall consider approximations to homeomorphisms of S onto itself which leave each frontier point of S fixed. Let Θ' be the class of homeomorphisms $\mathfrak{G}$ of the form

$$x'=x, \quad y'=\Phi(x,y) \quad \text{or} \quad x'=\Phi(x,y), \quad y'=y \tag{15}$$

and which map S onto itself and leave each frontier point of S fixed.

[2]) The angle $\sin^{-1}(1/4)$ that is needed here lies between 0 and $\pi/2$.

In the present paragraph we consider homeomorphisms which are defined over certain polygonal arcs contained in S and we shall use these results to show that homeomorphisms of a closed square onto itself can be uniformly approximated to by means of members of Ξ', the group generated by Θ'.

We require a number of auxiliary lemmas. The following convention is used. If a segment L is such that, apart from its end-points, it belongs to the interior of a closed Jordan domain T then we shall say that L belongs to the interior of T. If X is any point set then we denote its interior by X^0 and its frontier by $\operatorname{Fr} X$.

LEMMA 2. *If a simple polygonal arc $aq_0q_1\ldots q_mb$, where $m\neq 0$, together with the segment ab bounds a simple closed Jordan domain T, then at least one of the segments $aq_1, q_0q_2, \ldots, q_{i-1}q_{i+1}, \ldots, q_{m-1}b$ belongs to the interior of T.*

Either q_0b belongs to T^0 or there is a vertex q_j, $j\neq 0$ such that aq_j belongs to T^0. In the first case we replace $aq_0q_1\ldots q_mba$ by $q_0q_1\ldots q_mbq_0$ and in the second case we replace $aq_0q_1\ldots q_mba$ by $aq_0q_1\ldots q_{j-1}q_ja$. In either case we obtain a polygonal curve which bounds a domain contained in T, has less segments than $aq_0q_1\ldots q_mba$ and is such that all its segments but one belong to $aq_0q_1\ldots q_mb$. If this new polygonal curve is not a triangle then we repeat the process a finite number of times until we obtain a triangle that bounds a domain contained in T and two of whose sides belong to $aq_0q_1\ldots q_mb$. Now the construction of the new polygon from $aq_0q_1\ldots q_mb$ is such that the only side that does not belong to $aq_0q_1\ldots q_mb$ is contained in T^0. Thus finally the third side of the triangle which we ultimately obtain is contained in T^0 and it is also one of the segements $aq_1, q_0q_2, \ldots, q_{i-1}q_{i+1}, \ldots, q_{m-1}b$.

The lemma is proved.

LEMMA 3. *Two polygonal arcs $ap_0p_1\ldots p_kb$ and $aq_0q_1\ldots q_mb$, $m\geqslant 0$, together form the boundary of a simple closed Jordan domain T. The arc $ap_0p_1\ldots p_kb$ together with ab bounds a convex set U and those of the vertices $q_0, q_1, \ldots, q_m$ which lie on the same side of ab as does U lie either on the segment ab or interior to U, (it is assumed that U has interior points). Also all the points p_j, q_i lie in the strip bounded by the two lines that are perpendicular to ab and pass through a and b respectively. Then either*

(i) *the arc $aq_0q_1\ldots q_mb$ together with the segment ab bounds a convex set V that is exterior to T, or*

(ii) *one of the segments $aq_1, q_0q_2, \ldots, q_{i-1}q_{i+1}, \ldots, q_{m-1}b$ is contained in T^0.*

Let B denote the arc $ap_0p_1\ldots p_kb$ and let K denote the convex cover of the arc $aq_0q_1\ldots q_mb$. Since all the points q_i lie inside the strip bounded by the two lines through a and b perpendicular to ab both a and b are frontier points of K. Thus $\mathrm{Fr}\,K$ consists of two arcs that join a to b. (The set K will have interior points because the arc $aq_0q_1\dots q_mb$ has at least three vertices where by a vertex is meant a point which is either an end-point or a point that lies on two segments that lie on distinct lines. We take the notation to imply that the segments aq_0, q_0q_1, for example, are not collinear.) Let these two arcs be A_1 and A_2. Now no points of B are interior points of K and, of all the vertices of K (which are some of the points $a, b, q_0, q_1, \dots, q_m$), only a and b belong to B. Thus K meets B either in the whole segment ab or in just the two points a and b. The first case can arise only when B is precisely the segment ab. In this case U would have no interior points which is contrary to the hypotheses of the lemma. We suppose then that B meets K in the two points a and b.

The two arcs A_1 and B bound a domain say W_1. Similarly we define W_2 to be the domain bounded by A_2 and B. Either every segment of A_1 belongs to W_2^0 or every segment of A_2 belongs to W_1^0. We choose the notation so that the first of these alternatives holds. If the whole of A_1 belongs to the arc $aq_0q_1\dots q_mb$ then it coincides with the whole of this arc and the lemma is proved since (i) is true. If however there is a segment of A_1 that does not belong to $aq_0q_1\dots q_mb$ then we can find two points of the set $a, q_0, q_1, \dots, q_m, b$ say r_1 and r_2 such that these points belong to A_1 and there are no other points of the arc $aq_0q_1\dots q_mb$ that belong to the segment r_1r_2. The points r_1 and r_2 are not necessarily vertices of K but they are vertices of the arc $aq_0q_1\dots q_mb$.

The points r_1 and r_2 are not consecutive vertices of $aq_0q_1\dots q_mb$ for if they were the whole of the segment r_1r_2 would belong to the arc $aq_0q_1\dots q_mb$. If they are separated in this sequence by only one other vertex then the segment r_1r_2 itself is of the form required in (ii). If however r_1 and r_2 are separated by more than one vertex in the order $aq_0q_1\dots q_mb$ then the subarc of $aq_0q_1\dots q_mb$ whose end-points are r_1 and r_2, bounds with the segment r_1r_2 a domain contained in T. We can apply lemma 2 to it and deduce the existence of a segment as required in (ii).

The lemma is proved.

We now prove a result that will be needed in the next paragraph. Suppose that the square S is cut by a number of segments parallel to the x-axis. Let these segments, which join the two sides of S that are parallel to the y-axis, be denoted by $A_1, A_2, \dots, A_n$. We also suppose that the segments A_1 and A_n are sides of S and that the notation is such that the ordinate of the line containing A_i is greater than or less than that containing A_j according as i is less than or greater than j. Denote the totality of these segments by K.

THEOREM 2. *If $\mathfrak{G}^{-1}$ is a homeomorphism defined over $K \cup \mathrm{Fr}\,S$ which leaves every point of the frontier of S fixed and maps each of the segments A_i onto a polygonal line contained in S, then there is a homeomorphism $\mathfrak{H}$ of the group Ξ' such that for any point p of K $\mathfrak{H}(\mathfrak{G}^{-1}(p)) = p$.*

Write a_i, b_i for the two end points of A_i and use the symbols A_i' for $\mathfrak{G}^{-1}(A_i)$. By the given conditions $\mathfrak{G}^{-1}(a_i) = a_i$, and $\mathfrak{G}^{-1}(b_i) = b_i$.

We need the following lemmas.

LEMMA 4. *If A_2' is a polygonal line with t segments, $t > 1$, then either*

(a) *it bounds with the segment a_2b_2 a convex set that is exterior to the set bounded by A_1', A_2', segment a_1a_2 and segment b_1b_2, or*

(b) *there is a member $\mathfrak{H}$ of Ξ' such that $\mathfrak{H}(A_2')$ is a polygonal line of at most $t-1$ segments and every point of each A_j', $j \neq 2$ is a fixed point under $\mathfrak{H}$.*

We apply lemma 3 with a_2 for a and b_2 for b, with $a_2a_1b_1b_2$ for $ap_0p_1\dots p_kb$ and A_2' for $aq_0q_1\dots q_mb$. By that lemma if (a) is not true then there are two consecutive segments of A_2' say cd and de such that the segment ce is contained in T^0, where T is the domain bounded by A_1', A_2', a_1a_2, b_1b_2, *i. e.* ce is, apart from its end-points, contained in the interior of T. The segment ce does not meet any A_j' with $j \neq 2$. We next select two points d' and d'' on opposite sides of the line ce such that the closed quadrilateral $cd'ed''$ is convex, contains d as an interior point, does not meet A_j', $j \neq 2$, and meets A_2' only in the segments cd, de. This last requirement can be satisfied because only the end-points of the segment ce belong to A_2' and thus the triangle cde meets A_2' only in the segments cd and de. Join d to a point of ce other than c or e, by a polygonal line lieing in the interior of $cd'ed''$, such that each segment of this line is parallel either to the x-axis or to the y-axis. Let this polygonal line be $dr_1r_2\dots r_s$, where r_s is the point of ce.

We shall now show that there is a member of Ξ' say $\mathfrak{J}_1$ such that

$$\mathfrak{J}_1(d) = r_1; \qquad \mathfrak{J}_1(cd) = cr_1; \qquad \mathfrak{J}_1(de) = r_1e, \tag{16}$$

and such that every point on the frontier of or exterior to the quadrilateral $cd'ed''$ is fixed under $\mathfrak{J}_1$.

Suppose for the moment that such a homeomorphism exists, then the lemma follows. For just as we have defined $\mathfrak{J}_1$ so we can define $\mathfrak{J}_i$ such that

$$\mathfrak{J}_i(r_{i-1}) = r_i, \qquad \mathfrak{J}_i(cr_{i-1}) = cr_i, \qquad \mathfrak{J}_i(r_{i-1}e) = r_ie, \qquad i = 2, \dots, s$$

and every point on the frontier of or exterior to the quadrilateral $cd'ed''$ is fixed under $\mathfrak{J}_i$. Then the homeomorphism $\mathfrak{J}_s\mathfrak{J}_{s-1}\dots\mathfrak{J}_1$ maps cd onto cr_s and de onto r_se. Since all the other points of A_2' are fixed it reduces the number of segments of A'_2 by at least one. Also every point of every A'_j, $j \neq 2$ is fixed and the lemma is established.

Thus we have only to construct the homeomorphism $\mathfrak{J}_1$. Suppose that dr_1 is parallel to the y-axis. In the other case we use the same argument with x and y interchanged. On a particular line $x = x'$, $\mathfrak{J}_1$ is defined as follows. All points exterior to or on the frontier of $cd'ed''$ are fixed. If the line $x = x'$ meets the pair of segments cd, de in one point say p, then it also meets the pair of segments cr_1, r_1e in one point q. Define $\mathfrak{J}_1(p)$ to be q and complete the definition of $\mathfrak{J}_1$ on $x = x'$ by linearity. If the line $x = x'$ meets cd, de in two points say p' and p'' and the ordinate of p' is greater than that of p'', then the line also meets the segments cr_1, r_1e in two points which we may call q' and q'' where the ordinate of q' is greater than that of q''. Define $\mathfrak{J}_1(p')$ to be q' and $\mathfrak{J}_1(p'')$ to be q'' and complete the definition of $\mathfrak{J}_1$ on the line by linearity. If the line $x = x'$ contains the whole of the segment cd or de then it also contains the point r_1. Define $\mathfrak{J}_1(d)$ to be r_1 and complete as before by linearity. If the abscissae of the two points d' and d'' are equal to one of or lie between the abscissae of c and e then the definition of $\mathfrak{J}_1$ is complete. Otherwise we still have to define it for those lines $x = x'$ that meet the quadrilateral $cd'ed''$ but do not meet the pair of segments cd, de. In this case there is a line $x = x''$ on which $\mathfrak{J}_1$ has been defined, and which passes through one or more of the points c, d, e, and which is such that the segment of the line $x = x''$ that is interior to the quadrilateral $cd'ed''$, say tu, forms with either d' or d'', say d'', a triangle that contains that part of the line $x = x'$ that is interior to the quadrilateral $cd'ed''$. Let the part of $x = x'$ that is contained in $cd'ed''$ be the segment vw. To define $\mathfrak{J}_1$ on vw, join d'' to a point say z on vw, produce to meet tu in z'. Let $\mathfrak{J}_1(z')$, which has been defined to be a point of tu, be the point z'_1. Join z'_1 to d'' cutting vw in z_1. Define $\mathfrak{J}_1(z)$ to be z_1.

Then $\mathfrak{J}_1$ has been completely defined, it is of the form

$$(x, y) \to (x', y') \quad \text{where} \quad x' = x, \quad y' = \Phi(x, y),$$

and has the properties stated in (16).

The proof of the lemma is complete.

Lemma 5. *A rectangle T is given with its sides parallel to the coordinate axes. A homeomorphism $\mathfrak{K}$ defined over the frontier of T is of the form*

$$(x, y) \to (x', y') \quad \textit{where} \quad x' = \Phi(x, y). \tag{17}$$

Further $\mathfrak{K}$ leaves fixed the points of the sides of T that are parallel to the y-axis. Then there is a homeomorphism which is of the form (17), *maps the whole of T onto itself and coincides with $\mathfrak{K}$ on* Fr T.

Suppose that T is the rectangle $a_1 \leqslant x \leqslant a_2$; $b_1 \leqslant y \leqslant b_2$. Consider the two rectangles

$$T_1: \quad a_1 \leqslant x \leqslant a_2, \quad \tfrac{1}{2}(b_1 + b_2) \leqslant y \leqslant b_2,$$

$$T_2: \quad a_1 \leqslant x \leqslant a_2, \quad b_1 \leqslant y \leqslant \tfrac{1}{2}(b_1 + b_2).$$

In T_1 define the homeomorphism $\mathfrak{H}$ by $(x, y) \to (x', y)$ where

$$x' = x + \big(\Phi(x, b_2) - x\big)\Big(y - \frac{1}{2}(b_1 + b_2)\Big) \Big/ \frac{1}{2}(b_2 - b_1).$$

In T_2 define $\mathfrak{H}$ by $(x, y) \to (x', y)$ where

$$x' = x + \big(\Phi(x, b_1) - x\big)\Big(\frac{1}{2}(b_1 + b_2) - y\Big) \Big/ \frac{1}{2}(b_2 - b_1).$$

It may be verified that $\mathfrak{H}$ is a homeomorphism with the properties stated in the lemma.

Lemma 6. *There is a member $\mathfrak{H}$ of Ξ' such that $\mathfrak{H}(A'_j) = A_j$, $j = 1, \dots, n$.*

Consider A'_2. By lemma 4, either A'_2 is an arc convex with respect to a_2b_2 and A'_2 lies on the same side of a_2b_2 as A'_1, or we can find $\mathfrak{J}_1$ of Ξ' which reduces the number of segments of A'_2 and leaves every point of every A'_j, $j \neq 2$, fixed. By successive repetitions of this argument it follows that we can find a member of Ξ' say $\mathfrak{F}_1$ such that either $\mathfrak{F}_1(A'_2)$ is the straight line segment a_2b_2 or it is convex with respect to a_2b_2 and lies on the same side of a_2b_2 as does A'_1. In either case $\mathfrak{F}_1(A'_2)$ together with the segments a_2a_3, a_3b_3, b_3b_2, bounds a convex set. We may now apply lemma 3 to A'_3 exactly as we applied it to A'_2 in the proof of lemma 4. We obtain $\mathfrak{F}_2$ of Ξ' such that every point of every A'_j, $j > 3$, and of A_1, $\mathfrak{F}_1(A'_2)$ is fixed, and such that $\mathfrak{F}_2(A'_3)$ is convex with respect to the segment a_3b_3 and lies on the same side of a_3b_3 as does A'_1, or alternatively $\mathfrak{F}_2(A'_3)$ is a straight line segment. Proceeding in this fashion we eventually arrive at a member $\mathfrak{F}$ of Ξ' such that each $\mathfrak{F}(A'_j)$ is either the segment a_jb_j or bounds with a_jb_j a convex set.

Now every line parallel to the y-axis that meets $\mathfrak{F}(A'_j)$ at all does so in exactly one point and also meets A_j in exactly one point. Suppose that such a line meets $\mathfrak{F}(A'_j)$ in x_j and A_j in y_j. Then the points $x_1x_2 \dots x_n$ occur on this line in the same order as $y_1y_2 \dots y_n$. We define a homeomorphism on this line by mapping x_j onto y_j and making the mapping linear between x_j and x_{j+1}. Such a mapping is defined for all the points of S. For points outside S we define each point to be its own image. Denote this mapping by $\mathfrak{L}$. Then $\mathfrak{L}\mathfrak{F}$ is a member of Ξ' with the property stated in the enunciation of the lemma.

We can now complete the proof of theorem 2. This theorem is nearly contained in lemma 6 but although that lemma provides a homeomorphism that maps A'_j onto A_j we do not know that it coincides with $\mathfrak{G}$ for each point p of $\mathfrak{G}^{-1}(p)$. To secure this result we proceed as follows.

The homeomorphism $\mathfrak{L}\mathfrak{F}\mathfrak{G}^{-1}$ maps each segment A_j onto itself and leaves each point of the frontier of S fixed. If lemma 5 is applied to the rectangle that is bounded by A_j, A_{j+1}, and the two segments a_ja_{j+1}, b_jb_{j+1} then there is a homeomorphism of the form (17) say $\mathfrak{M}_j$ that maps this rectangle onto itself and coincides with $\mathfrak{L}\mathfrak{F}\mathfrak{G}^{-1}$ on the frontier of this rectangle. Denote by $\mathfrak{M}$ the homeomorphism of the whole plane onto itself that coincides with $\mathfrak{M}_j$ in the rectangle in which it is defined and leaves every point exterior to S fixed. Then for any point p of $K\,\mathfrak{M}(p)$ is the point $\mathfrak{L}\mathfrak{F}\mathfrak{G}^{-1}(p)$. Thus $p=\mathfrak{M}^{-1}\mathfrak{L}\mathfrak{F}\mathfrak{G}^{-1}(p)$, that is to say the homeomorphism $\mathfrak{M}^{-1}\mathfrak{L}\mathfrak{F}$ which is a member of the group Ξ' has the property required in the theorem.

§ 3. Homeomorphisms of a closed square onto itself

As before let S denote a closed square with sides parallel to the axes. In this paragraph all the homeomorphisms concerned map S onto itself and leave the frontier points of S fixed. This class of homeomorphisms is denoted by Γ. Again we write Θ' for the class of homeomorphisms that belong both to Γ and to Θ, and we write Ξ' for the group generated by finite superpositions of members of Θ'.

THEOREM 3. *If $\mathfrak{G}$ is a homeomorphism belonging to the class Γ, then, given a positive number ε there exists a homeomorphism $\mathfrak{H}$ of Ξ' such that*

$$\delta'(\mathfrak{H},\mathfrak{G})=\underset{p\in S}{\text{up. bd.}}\ \varrho\big(\mathfrak{H}(p),\mathfrak{G}(p)\big)<\varepsilon .$$

Let l be the side length of S and choose a positive integer n so large that

$$4\cdot 2^{1/2}l<n\varepsilon .$$

Divide S into n^2 equal squares each of side length l/n and let the vertices of these squares be a_{ij}, $i=1,2,\ldots,n+1$, $j=1,2,\ldots,n+1$, where the vertices of S itself are $a_{11},a_{1,n+1},a_{n+1,n+1},a_{n+1,1}$. Denote the segment $a_{i1}a_{i,n+1}$ by A_i and the segment $a_{1j}a_{n+1,j}$ by B_j. Also write $\mathfrak{G}^{-1}(A_i)=A'_i$, $\mathfrak{G}^{-1}(a_{ij})=a'_{ij}$, $\mathfrak{G}^{-1}(B_j)=B'_j$; and let E_{ij} be the subarcs of A'_i with end points $a'_{ij},a'_{i,j+1}$ and F_{ij} be the subarcs of B'_j with end points $a'_{ij},a'_{i+1,j}$. Let the axes be such that the x-axis is parallel to each A_i and the y-axis is parallel to each B_j.

Define η to be a positive number, less than the least distance apart of any pair of nonintersecting arcs E_{ij} or F_{ij}.

Let C_{ij} be the closed circle whose centre is a'_{ij} and whose radius is $\eta/3$. No two of the circles C_{ij} intersect; let C denote their point-set union. The arc A'_j contains a subarc that is minimal with respect to the property of joining C_{ji} to $C_{j,i+1}$, and which we denote by L_{ji}, $j=1,2,\ldots,n$, $i=1,2,\ldots,n+1$. Similarly there is a minimal subarc of B'_i that joins C_{ji} to $C_{j+1,i}$, say M_{ji}, $j=1,2,\ldots,n$, $i=1,2,\ldots,n+1$.

Any two segments of the form A_j or B_i either do not meet or do so in one of the points a_{ji}. Thus any two of the arcs A'_j, B'_i either do not meet at all or if they do meet their point of intersection belongs to the set C^0. Thus all the arcs of the form L_{ij} and M_{kr} are disjoint from one another. Also of all the circles C_{ij} the arc L_{pq} meets only two namely the two circles C_{pq} and $C_{p,q+1}$, similarly M_{rs} meets only the two circles C_{rs} and $C_{r+1,s}$. For suppose that the arc L_{pq} met a circle other than the two stated above, say it met C_{tu}, then on the one hand L_{pq} is a subset of E_{pq} and is thus at a distance of at least η from any of the arcs E_{ij} or F_{ij} which do not actually meet E_{pq} and on the other hand a'_{tu} lies on two arcs of the form E_{ij} and on two arcs of the form F_{ij}. It follows that a'_{tu} lies on an arc of one of these two types that does not meet E_{pq}. Thus the point a'_{tu} is distant at least η from L_{pq} and L_{pq} does not meet C_{tu}. In fact L_{pq} is distant at least $2\eta/3$ from C_{tu}.

Let η_1 be a positive number less than the distance apart of any two arcs L_{ij} or M_{rs} and let η_0 be the smaller of the two numbers η and η_1. Let P_{ij} be a polygonal line joining the end-points of L_{ij} and lieing both inside S and within a distance of $\eta_0/3$ of L_{ij} and outside C^0_{ij} and $C^0_{i,j+1}$, $i=2,3,\ldots,n$, $j=1,2,\ldots,n$. Let the point of intersection of the arc L_{ij} with the circle C_{ij} be l'_{ij} nad the point of intersection of L_{ij} with $C_{i,j+1}$ be l''_{ij}. Let P'_{ij} be the polygonal line consisting of segment $a'_{ij}l'_{ij}$, polygonal line P_{ij}, and segment $l''_{ij}a'_{i,j+1}$. We also suppose that P'_{1j} is the segment $a_{1j}c_{1,j+1}$ and that $P'_{n+1,j}$ is the segment $a_{n+1,j}a_{n+1,j+1}$. Finally let P_i denote the union $\bigcup_{j=1}^{n}P'_{ij}$, $i=1,2,\ldots,n+1$.

Next define polygonal lines Q_j in the same way as P_j has been defined but using the minimal arcs M_{rs} in place of the minimal arcs L_{ij} [3]).

The polygonal lines P_i,Q_j approximate to A'_i and B'_j respectively and have the following properties:

(i) P_i intersects Q_j in exactly one point namely a'_{ij}.

(ii) Of the points a'_{ij}; P_i contains in order the points $a'_{i1},a'_{i2},\ldots,a'_{i,n+1}$.

(iii) Of the points a'_{ij}; Q_j contains in order the points $a'_{1j},a'_{2j},\ldots,a'_{n+1,j}$.

(iv) The four sides of the square S are P_1,P_{n+1},Q_1,Q_{n+1}.

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[3]) Q_{rs}, Q'_{rs} are defined first with respect to M_{rs}.

Next let $\mathfrak{K}$ be any homeomorphism of the class Γ such that

$$\mathfrak{K}(P_j)=A_j, \qquad \mathfrak{K}(Q_j)=B_j.$$

It is clear that such a homeomorphism exists. If z is a point of S then it belongs to a domain bounded by four arcs of which two are of the form E_{ij} and two are of the form F_{rs}. z may belong to more than one such domain. (We take these domains to be closed.) Suppose that z belongs to the domain bounded by $E_{ij}, F_{i,j+1}, E_{i+1,j}, F_{ij}$. If $2 \leqslant i,j \leqslant n-1$, then z also belongs to the domain bounded by the polygonal lines $P'_{i-1,j-1}, P'_{i-1,j}, P'_{i-1,j+1}, Q'_{i-1,j+1}, Q'_{i,j+1}, Q'_{i+1,j+1}, P'_{i+1,j+1}, P'_{i+1,j}, P'_{i+1,j-1}, Q'_{i+1,j-1}$, $Q'_{i,j-1}, Q'_{i-1,j-1}$. A similar statement is true when one or both of i or j is one of 1 or n. Thus we have

$$\varrho\big(\mathfrak{K}(z), \mathfrak{G}(z)\big) \leqslant 2\cdot 2^{1/2}(l/n) < \frac{1}{2}\varepsilon. \tag{18}$$

Hence it is sufficient to show that there is a member $\mathfrak{H}$ of Ξ' such that for every point z of S

$$\varrho\big(\mathfrak{K}(z), \mathfrak{H}(z)\big) \leqslant \frac{1}{2}\varepsilon. \tag{19}$$

By theorem 2 there is a member $\mathfrak{J}$ of Ξ' such that for any point p of any of the polygonal lines P_j $j=1,2,\ldots,n+1$, $\mathfrak{J}(p)=\mathfrak{K}(p)$. Consider the homeomorphism $\mathfrak{K}\mathfrak{J}^{-1}$. This mapping leaves fixed each point of the frontier of S and of each of the segments A_j. Denote by T_j that rectangle which has A_j and A_{j+1} as two opposite sides. T_j is contained in S and is mapped onto itself by $\mathfrak{K}\mathfrak{J}^{-1}$. Thus for any point p of S the ordinate of $\mathfrak{K}(p)$ differs from that of $\mathfrak{J}(p)$ by at most the width of one of the rectangles T_j, *i. e.* by at most l/n.

Now the homeomorphism $\mathfrak{K}\mathfrak{J}^{-1}$ not only maps T_j onto itself but also leaves each point of the frontier of T_j fixed, and (even though T_j is a rectangle and not a square) all the preceding argument is valid with T in place of S. Thus by an argument similar to that used in theorem 2 with x and y interchanged we can show that there is a homeomorphism $\mathfrak{F}_j$ which belongs to Ξ, maps each point of the frontier of T_j onto itself and is such that for any point p of T_j the abscissae of $\mathfrak{F}_j(p)$ and $\mathfrak{K}\mathfrak{J}^{-1}(p)$ differ by at most l/n.

Define $\mathfrak{F}$ to be the homeomorphism which coincides with $\mathfrak{F}_j$ in T_j $j=1,2,\ldots,n+1$, and leaves every point of E exterior to S fixed. Then $\mathfrak{F}$ belongs to Ξ' and for any point p of S we have:

(i) The ordinates of $\mathfrak{F}(p)$ and $\mathfrak{K}\mathfrak{J}^{-1}(p)$ differ by at most l/n because these points both belong to the same rectangle T.

(ii) The abscissae of $\mathfrak{F}(p)$ and $\mathfrak{K}\mathfrak{J}^{-1}(p)$ differ by at most l/n.

Hence

$$\varrho\big(\mathfrak{F}(p), \mathfrak{K}\mathfrak{J}^{-1}(p)\big) \leqslant 2^{1/2}(l/n) < \frac{1}{2}\varepsilon.$$

Now for any point q of S there is a point p such that $p=\mathfrak{J}(q)$, and thus for any point q of S

$$\varrho\big(\mathfrak{F}\mathfrak{J}(q), \mathfrak{K}(q)\big) = \varrho\big(\mathfrak{F}(p), \mathfrak{K}\mathfrak{J}^{-1}(p)\big) < \frac{1}{2}\varepsilon$$

and this is the inequality (19) as required with $\mathfrak{F}\mathfrak{J}$ as the homeomorphism $\mathfrak{H}$.

Theorem 3 is proved.

§ 4. Non-uniform approximations

In this paragraph we use theorem 3 to establish the result stated in the introduction.

THEOREM 4. *If $\mathfrak{G}$ is any given homeomorphism of the plane onto itself, then there exists a sequence of members of Ξ say $\{\mathfrak{H}_n\}$ such that for every point q of the plane, $\mathfrak{H}_n(q)$ tends to $\mathfrak{G}(q)$ as n tends to infinity.*

Let $C=C(p,R)$ denote the set of points of the plane whose distance from the fixed point p is not more than R, and let $\mathfrak{G}(C)$ be D. It is sufficient to show that given two positive numbers R and ε, there is a member of Ξ say $\mathfrak{H}$, such that the distance apart of the points $\mathfrak{H}(z)$ and $\mathfrak{G}(z)$ for all points z of $C(p,R)$ is less than ε.

Let S be a square so large that it contains both C and D in its interior. We show first that there is a homeomorphism $\mathfrak{J}$ of E onto itself that coincides with $\mathfrak{G}$ on C and leaves each point of the frontier of S fixed. Let S_1 be the set S^0-C and let S_2 be the set S^0-D. Both S_1 and S_2 are open, connected, and doubly connected sets. Thus there are conformal mappings say $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that $\mathfrak{M}_1(S_1)$ is an annulus A_1 and $\mathfrak{M}_2(S_2)$ is an annulus A_2. Since $\mathrm{Fr}\,S$, $\mathrm{Fr}\,C$, and $\mathrm{Fr}\,D$ are all Jordan curves we can extend $\mathfrak{M}_1$ and $\mathfrak{M}_2$ to be homeomorphic over the closures of S_1 and S_2. We use the same notation $\mathfrak{M}_1$ and $\mathfrak{M}_2$ for these two homeomorphisms of the closed sets. Let $\mathfrak{C}$ be a homeomorphism of the closure of A_2 onto the closure of A_1 such that for any point p of $\mathrm{Fr}\,S$ $\mathfrak{C}\mathfrak{M}_2(p)=\mathfrak{M}_1(p)$. Let the circles bounding A_1 be K and L and suppose that L is the image under $\mathfrak{M}_1$ of $\mathrm{Fr}\,S$.

We next define a homeomorphism of K onto itself say $\mathfrak{K}$, as follows. For q belonging to K write

$$\mathfrak{K}(q) = \mathfrak{C}\mathfrak{M}_2\mathfrak{G}\mathfrak{M}_1^{-1}(q).$$

Now we can extend $\mathfrak{K}$ so that we obtain a homeomorphism of the closure of A_1 onto itself say $\mathfrak{J}$ with the properties,

$$\text{if} \quad p \in K \quad \mathfrak{J}(p)=\mathfrak{K}(p), \qquad \text{if} \quad p \in L \quad \mathfrak{J}(p)=p.$$

Now define $\mathfrak{J}$ as follows. For $p \in C$, $\mathfrak{J}(p)=\mathfrak{G}(p)$; for $p \in \mathrm{Fr}\, S$ $\mathfrak{J}(p)=p$; for $p \in S^0 - C$ $\mathfrak{J}(p)=\mathfrak{M}_2^{-1}\mathfrak{C}^{-1}\mathfrak{J}\mathfrak{M}_1(p)$; for $p \notin S$ $\mathfrak{J}(p)=p$. Then $\mathfrak{J}$ is a homeomorphism of the required form.

By § 3 we can find a member $\mathfrak{H}$ of Ξ such that for every point z of S, $\mathfrak{H}$ and $\mathfrak{J}$ differ by at most ε. The homeomorphism $\mathfrak{H}$ has the required property and the theorem 4 is proved.

Reçu par la Rédaction le 5. 4. 1954

Über eine Abschwächung des Auswahlpostulates

von

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Es sei M eine Menge. Wir bezeichnen die Elemente von M mit $a,b,\dots$, hingegen die Teilmengen von M mit $A,B,\dots$ Die Potenzmenge von M (d. h. die Menge sämtlicher Teilmengen von M, einschließlich der leeren Menge 0) bezeichnen wir mit M^*.

Definition. *Wir sagen, eine Menge M habe die Eigenschaft* (E), *wenn es eine eindeutige Abbildung φ von M^* in sich gibt, so daß für jedes aus mindestens zwei Elementen bestehende $A \subseteq M$ gilt:*

$$0 \subset \varphi(A) \subset A$$ [1].

Unsere Bedingung (E) steht in einem engen Zusammenhang mit dem bekannten [2]) Auswahlpostulat. Wie man ohne weiteres sieht, ist (E) formal schwächer als das Auswahlpostulat. Man kann aber außerdem leicht zeigen, daß das Kontinuum C noch (E) erfüllt. Denn mittels der unendlichen Folge der rationalen Zahlen gelingt es, da diese relativ zu C dicht liegen, jedes aus mindestens zwei Zahlen bestehende $A \subseteq C$ in ein echtes Anfangsstück und ein echtes Endstück von A zu zerlegen. Versteht man dann unter $\varphi(A)$ dieses echte, nicht leere Anfangsstück von A, so folgt unmittelbar unsere Behauptung. Wir sehen, unser (E) ist nichts weiter als eine gewisse Abschwächung des Auswahlpostulates.

Satz 1. *Jede Menge M mit der Eigenschaft* (E) *läßt sich ordnen.*

Beweis. Wir setzen [3]) $\bar{\varphi}(A)=A-\varphi(A)$ für jedes $A \subseteq M$. Dann folgt auch

$$0 \subset \bar{\varphi}(A) \subset A$$

für jedes aus mindestens zwei Elementen bestehende $A \subseteq M$. Ferner folgt

[1]) D. h. also, $\varphi(A)$ ist eine echte, nicht leere Teilmenge von A. Die vorliegende Arbeit enthält die wesentlichen Ergebnisse der (unveröffentlichten) Dissertation (Köln 1952) des erstgenannten Verfassers, der durch diese vom letztgenannten Verfasser stammende Definition sowie durch den Satz 1 angeregt wurde.

[2]) Siehe [7], Abschnitt 2, S. 514.

[3]) Es ist im folgenden bequem, für die aus nur einem Element bestehenden A, also (kurz geschrieben) für die $a \in M$, einfach $\varphi(a)=a$ vorauszusetzen. Ferner setzen wir im folgenden für die Nullmenge $\varphi(0)=0$ voraus.