On the extending of models (I) *

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Introduction. In mathematics it is often necessary to extend a given model $\mathfrak{M}=\langle A, R_1, \dots, R_n\rangle$ of a certain axiomatic $\mathcal{A}$ to model $\mathfrak{M}^*=\langle A^*, R_1^*, \dots, R_n^*\rangle$ in such a way that, besides the axiomatics $\mathcal{A}$ in $\mathfrak{M}^*$, also each of the sentences of a given set $\mathcal{C}$ be valid. In that case sentences belonging to $\mathcal{A}$ are called *primary conditions of extension*, sentences belonging to $\mathcal{C}$ — *secondary conditions of extension* and, given these two sets, we search for the necessary and sufficient conditions to be satisfied by the model in order that extension may exist.

This kind of problem is called the *problem of extension with secondary conditions*. The simplest example is here the problem of extension of a commutative ring to a field, where $\mathcal{A}$ is the axiomatic of commutative rings and $\mathcal{C}$ consists of one sentence which states the existence of a con-

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verse element for each element differing from 0. As it is known, a sufficient and necessary condition for the existence of extension is in that case the fact that the extended ring contains no divisors of zero.

In this paper we present the general problem of the extension of models with elementary primary and secondary conditions; it is formulated in more general terms than above (see § 3), since in secondary conditions it admits the occurrence of new relations, not found in primary conditions. A general solution of the problem of extensions is given in theorem 1.

If we revert to the problem of extension of a ring to a field and write symbolically the condition of the absence of divisors of zero

$$x\cdot y=0\to(x=0\vee y=0),$$

we see that this condition is written in the form of an open formula, namely, a formula in which there are no quantifiers whatever [1]). As follows from theorem 1, the general rule is that in every defined case of the problem of extension with primary and secondary conditions the necessary and sufficient conditions of extension may be found in the form of a certain set of open formulas.

Paragraphs 1-4 of the present paper are of a preliminary character. They contain notions and lemmas for proving theorem 1, as well as a classification of the problems of extension. Paragraph 5 includes the proof of theorem 1 followed by its general application. Paragraphs 13-16 contain details pertaining to the application of theorem 1 to various examples of extension known from algebra. It is of interest to note that such examples may be found among well known results. Besides the problem of the extension of a ring to a field, also the problem of the extension of a semigroup to a group has been solved (by Malcev [7]) in accordance with theorem 1, *i. e.* by giving conditions without quantifiers.

§ 1. Elementary formulas and consequences.

We shall take into consideration three elementary theories which differ as regards the primitive signs occurring in them.

The first is the theory with the signs $r_1, r_2, \dots$ of relations and the signs $f_1, f_2, \dots$ of functions, in the second theory, apart from these signs, the signs $q_1, q_2, \dots$ of relations occur, and in the third, in addition to the above mentioned signs, there will be a certain number of individual

[1]) The lack of divisors of zero may of course be written with quantifiers in the form of $\prod_x \prod_y (x\cdot y=0\to x=0\vee y=0)$. On account of the rule of omitting and adding general quantifiers these sentences are deductively equivalent.

signs g_t, where the index t runs over a set T. Besides the identity sign "$=$" will occur in all three theories.

The rules of constructing well-formed formulas will be the same in all three theories. We thus assume that $r_1, r_2, \ldots, q_1, q_2, \ldots$, respectively, are signs of $\nu_1, \nu_2, \ldots, \mu_1, \mu_2, \ldots$-ary relations and $f_1, f_2, \ldots$, are signs of $\varkappa_1, \varkappa_2, \ldots$-ary functions. According to well known rules we build well-formed formulae from primitive signs of the theory, of propositional connectives $\rightarrow$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction), $\equiv$ (equivalence), $'$ (negation), of individual variables $x_1, x_2, \ldots$, of quantifiers $\prod\limits_{x}$ and $\sum\limits_{x_j}$, and of brackets.

We denote by $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ the sets of well-formed formulas of the first, second and third theory. We obviously have

(1.1) $$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3.$$

Let us note that to $\mathcal{E}_1$ belong not only sentences, *i. e.* formulas without free variables but also formulas in which free variables occur, and even those in which there occur no bound variables. We distinguish therefore in each of the sets $\mathcal{E}_i$ a subset $\mathcal{S}_i$ of sentences of that theory, hence of such formulas in which there occur no free variables, and a subset $\mathcal{O}_i$ of open formulas of that theory, *i. e.* formulas in which there occur no bound variables. Let us observe the fact that the sets $\mathcal{S}_i$ and $\mathcal{O}_i$ need not be disjoint; in particular, in the third theory the common part of these sets will contain all those sentences in which there occur no variables but only the signs of individual constants g_t.

In each of the three theories we shall apply the rules of inference known from the so-called *functional calculus with identity* (engere Prädikatenkalkül mit Identität [2])). For a set $\mathfrak{X} \subset \mathcal{E}_i$ we shall denote by $Cn_i(\mathfrak{X})$ the set of all formulas belonging to $\mathcal{E}_i$ derivable from the formulas belonging to $\mathfrak{X}$ by means of these rules. Thus we obtain a certain operation on subsets of set $\mathcal{E}_i$, called the *operation of consequence*. Let us recall its fundamental properties.

(1.2) *If* $\mathfrak{X} \subset \mathcal{E}_i$, *then* $\mathfrak{X} \subset Cn_i(\mathfrak{X}) = Cn_i\big(Cn_i(\mathfrak{X})\big) \subset \mathcal{E}_i$.

(1.3) *If* $\mathfrak{X} \subset \mathcal{E}_i$, *then* $Cn_i(\mathfrak{X}) = \bigcup\limits_{\substack{\mathfrak{Y} \subset \mathfrak{X} \\ \mathfrak{Y} \text{ finite}}} Cn_i(\mathfrak{Y})$.

(1.4) *If* $\alpha \in \mathcal{S}_i$, $\beta \in \mathcal{E}_i$, $\mathfrak{X} \subset \mathcal{E}_i$, *then* $\beta \in Cn_i\big(\mathfrak{X} \cup (\alpha)\big)$ *if and only if* $\alpha \rightarrow \beta \in Cn_i(\mathfrak{X})$.

(1.5) *There exists such* $\alpha \in \mathcal{E}_i$, *that* $Cn_i\big((\alpha)\big) = \mathcal{E}_i$ [3]).

[2]) See Hilbert-Bernays [3].

[3]) Cf. Tarski [10].

The set $\mathfrak{X} \subset \mathcal{E}_i$ is called *consistent* in $\mathcal{E}_i$ if $Cn_i(\mathfrak{X}) \neq \mathcal{E}_i$. From (1.3) and (1.5) it follows that

(1.6) *If* $\mathfrak{X} \subset \mathcal{E}_i$ *is inconsistent in* $\mathcal{E}_i$, *there exists such a finite* $\mathfrak{Y} \subset \mathfrak{X}$, *that* $\mathfrak{Y}$ *is inconsistent in* $\mathcal{E}_i$.

From (1.4) and from the laws of the sentential calculus it follows that

(1.7) *If* $\alpha \in \mathcal{S}_i$, $\mathfrak{X} \subset \mathcal{E}_i$ *and* $\mathfrak{X} \cup (\alpha)$ *is an inconsistent set then* $\alpha' \in Cn_i(\mathfrak{X})$.

Let $\alpha = \alpha(g_{t_1}, \ldots, g_{t_n})$ belong to $\mathcal{S}_3$; we denote by $\alpha(x_{s_1}, \ldots, x_{s_n})$ a formula which differs from α only in that everywhere in α the constant signs g_{t_i} have been replaced by the variables x_{s_i}. It is easy to prove the following lemma:

(1.8) *If* $\alpha(g_{t_1}, \ldots, g_{t_n}) \in \mathcal{S}_3$, $\mathfrak{X} \subset \mathcal{E}_2$ *and* $\alpha(g_{t_1}, \ldots, g_{t_n}) \in Cn_3(\mathfrak{X})$ *then for arbitrary* $x_{s_1}, \ldots, x_{s_n}$ *not occurring in* $\alpha(g_{t_1}, \ldots, g_{t_n})$ *we have* $\alpha(x_{s_1}, \ldots, x_{s_n}) \in Cn_2(\mathfrak{X})$ [4]).

A pair composed of a set of formulas $\mathcal{E}_i$ and of the consequence Cn_i operating in that set we call a *deductive theory*. In the sequel we shall denote this theory by the symbol of the set of formulas: $\mathcal{E}_i$.

§ 2. Models for $\mathcal{E}_i$. By a *model* in theory $\mathcal{E}_i$ we understand any interpretation of formulas from $\mathcal{E}_i$, *i. e.* a non-empty set A with some relations, functions and possibly constants by which the primitive signs of formulas from $\mathcal{E}_i$ are interpreted. Since sets $\mathcal{E}_i$ differ as regards their primitive signs, the models in these theories are also different. Thus for the theory $\mathcal{E}_1$ we have models of the form

$$\mathfrak{M} = \langle A, R_1, R_2, \ldots, F_1, F_2, \ldots \rangle,$$

for the theory $\mathcal{E}_2$ of the form

$$\mathfrak{M} = \langle A, R_1, R_2, \ldots, F_1, F_2, \ldots, Q_1, Q_2, \ldots \rangle,$$

and for the theory $\mathcal{E}_3$ of the form

$$\mathfrak{M} = \langle A, R_1, R_2, \ldots, F_1, F_2, \ldots, Q_1, Q_2, \ldots, \{G_t\}_{t \in T} \rangle.$$

R_i and Q_i are relations on A which interpret the signs r_i and q_i, they must therefore be ν_i and μ_i-ary relations, respectively. F_i are functions on A with values in A, which interpret the signs f_i, hence F_i must be a function of $\varkappa_i$ variables. Finally, G_t are interpretations for the constants g_t, they are therefore elements (not necessarily different) of the set A.

It is clear what we understand by saying that formula α is *valid* in model $\mathfrak{M}$. It will be observed that for the sentence "α is valid in $\mathfrak{M}$"

[4]) Cf. Henkin [1], p. 161.

to be meaningful, interpretations in $\mathfrak{M}$ must be given for each of the primitive signs occurring in α; in particular, this sentence has no meaning if $\alpha \in \mathcal{E}_{i+1} - \mathcal{E}_i$ and if $\mathfrak{M}$ is a model for $\mathcal{E}_i$ $(i=1,2)$.

Let us also note that the formula α containing the free variable x_i is valid in model $\mathfrak{M}$ if and only if formula $\prod\limits_{x_i}\alpha$ is valid in $\mathfrak{M}$; in other words: free variables are treated by interpretations in the model in the same manner as variables bound with general quantifiers. This is in conformity with the known rule of omitting and adding the general quantifiers.

There is no interpretation shown for the sign of identity in a model, as we assume that this sign is always interpreted as identity, *i. e.* as a relation occurring between each pair of the elements a,a and no other pair.

If every formula of the set $\mathfrak{X}$ is valid in the model $\mathfrak{M}$ we say that $\mathfrak{M}$ is a *model* of $\mathfrak{X}$.

The following lemmas are well known:

(2.1) *If $\mathfrak{M}$ is a model of the set $\mathfrak{X} \subset \mathcal{E}_i$, then $\mathfrak{M}$ is a model of the set $Cn_i(\mathfrak{X})$.*

(2.2) *Each set of formulas which has a model is consistent.*

Let us also note the well known

THEOREM OF GÖDEL. *Each consistent set of formulas has a model.*

Let it be observed that Gödel's theorem is valid for all three theories $\mathcal{E}_i$, irrespective of how many primitive signs are accepted in $\mathcal{E}_i$ [5]); in particular, it is valid for $\mathcal{E}_3$ even if the set of the signs g_t therein contained is of great power. If we do not assume (as is often the case) the set of primitive signs to be denumerable, then the proof of Gödel's theorem is ineffective, *i. e.* this theorem cannot be proved without the aid of the axiom of choice or other similar means [6]).

§ 3. Submodels and extensions. Let $\mathfrak{M} = \langle A, R_1, R_2, \ldots, F_1, F_2, \ldots \rangle$ and $\mathfrak{M}^* = \langle A^*, R_1^*, R_2^*, \ldots, F_1^*, F_2^*, \ldots \rangle$ be two models for $\mathcal{E}_i$.

If $A \subset A^*$ and $R_1^*|A = R_1$, $R_2^*|A = R_1$, ..., $F_1^*|A = F_1$, $F_2^*|A = F_2, \ldots$ [7]) we say that $\mathfrak{M}$ is a *submodel* of $\mathfrak{M}^*$, and $\mathfrak{M}^*$ is an *extension of the first kind* of the model $\mathfrak{M}$.

Let $\mathfrak{M}$ and $\mathfrak{M}^*$ be the models mentioned above. If there exists a one-to-one function φ mapping A on A^* in such a manner that

$$R_i(a_1, \ldots, a_{\nu_i}) \text{ if and only if } R_i^*\big(\varphi(a_1), \ldots, \varphi(a_{\nu_i})\big) \qquad (i=1,2,\ldots),$$

$$\varphi\big(F_i(a_1, a_2, \ldots, a_{\varkappa_i})\big) = F_i^*\big(\varphi(a_1), \ldots, \varphi(a_{\varkappa_i})\big) \qquad (i=1,2,\ldots),$$

we say that the models $\mathfrak{M}$ and $\mathfrak{M}^*$ are *isomorphic.*

[5]) Cf. Henkin [1], p. 164, and Łoś [4].

[6]) See Łoś [4].

[7]) Symbol $R_i|A$ denotes the relation R_i limited to set A, similarly symbol $F_i|A$.

With the aid of the notion of isomorphism it is possible to generalize the notion of submodel and of extension as follows:

The model $\mathfrak{M}$ is a *submodel* (in a generalized sense) of the model $\mathfrak{M}^*$ if there exists a submodel (in the above sense) $\mathfrak{M}^0$ of the model $\mathfrak{M}^*$ isomorphic with $\mathfrak{M}$. If $\mathfrak{M}$ is a submodel (in a generalized sense) of the model $\mathfrak{M}^*$, we say that $\mathfrak{M}^*$ is an *extension of the first kind* of the model $\mathfrak{M}$.

The notions of isomorphism, submodel and extension of the first kind are analogously defined for models of the theories $\mathcal{E}_2$ and $\mathcal{E}_3$.

Now let $\mathfrak{M}$ be a model for $\mathcal{E}_i$ ($i=2$ or 3). By $\mathfrak{M}|j$ $(j<i)$ we denote the model for $\mathcal{E}_j$ obtained by eliminating from $\mathfrak{M}$ the interpretations of signs not occurring in $\mathcal{E}_j$. For example:

If $\mathfrak{M} = \langle A, R_1, R_2, \ldots, F_1, F_2, \ldots, Q_1, Q_2, \ldots, \{G_t\}_{t\in T}\rangle$ then

$$\mathfrak{M}|2 = \langle A, R_1, R_2, \ldots, F_1, F_2, \ldots, Q_1, Q_2, \ldots\rangle$$

and

$$\mathfrak{M}|1 = \mathfrak{M}|2|1 = \langle A, R_1, R_2, \ldots, F_1, F_2, \ldots\rangle.$$

The model $\mathfrak{M}^*$ for $\mathcal{E}_i$ is called *extension of the second kind* of the model $\mathfrak{M}$ for $\mathcal{E}_j$, $j<i$, if $\mathfrak{M}$ is a submodel of $\mathfrak{M}^*|j$.

Moreover if $\mathfrak{M}$ is isomorphic with $\mathfrak{M}^*|j$, we say that $\mathfrak{M}^*$ is a *weak extension of the second kind* of the model $\mathfrak{M}$.

Let us note that if $\mathfrak{M}$ is a submodel of $\mathfrak{M}^*$, the formula α being valid in $\mathfrak{M}^*$, then α need not be valid in $\mathfrak{M}$. The converse does not hold either: if α is valid in $\mathfrak{M}$, α need not be valid in $\mathfrak{M}^*$. However, we have the following theorem:

(3.1) *If $\mathfrak{M}^*$ is a model of set $\mathfrak{X} \subset \mathcal{O}_i$, $\mathfrak{M}$ a submodel of $\mathfrak{M}^*$, then $\mathfrak{M}$ is the model of the set $\mathfrak{X}$.*

§ 4. Descriptions of models for $\mathcal{E}_1$. The notion of description as given in this paragraph is essentially due to Robinson [8]). It may be defined for models of any theory, hence not only for $\mathcal{E}_1$ but also for $\mathcal{E}_2$ and $\mathcal{E}_3$ provided an extended theory with a sufficiently large number of constants is available for the theory in question. That extended theory for $\mathcal{E}_1$ will be $\mathcal{E}_3$ and we shall therefore limit ourselves to defining the description of models for $\mathcal{E}_1$.

Let $\mathfrak{M} = \langle A, R_1, R_2, \ldots, F_1, F_2, \ldots\rangle$ be the model for $\mathcal{E}_1$. The description of the model $\mathfrak{M}$ will be a certain set of sentences belonging to $\mathcal{E}_3$. Let us first put $T = A$; in other words, let us assume that the index of the constants g_t runs over A. Such an assumption is of course equivalent to assuming that there are as many constants g_t in $\mathcal{E}_3$ as elements in A (hence $\overline{\overline{T}} = \overline{\overline{A}}$) and that a one-to-one mapping of T on A is given. We are free to make such assumptions since, so far, we have not made any with respect to T.

[8]) Robinson [9], p. 74.

If we now consider the constant signs g_a in $\mathcal{E}_3$ as names of elements $a \in A$, then we shall find in $\mathcal{E}_3$ a set of sentences describing model $\mathfrak{M}$ as exactly as possible. These sentences are

(i) $(g_a = g_b)'$, for $a, b \in A$ and $a \neq b$ [9]);

(ii) $r_i(g_{a_1}, \ldots, g_{a_{\nu_i}})$, if $R_i(a_i, \ldots, a_{\nu_i})$ holds;

(iii) $r_i(g_{a_1}, \ldots, g_{a_{\nu_i}})'$, if $R_i(a_1, \ldots, a_{\nu_i})$ does not hold;

(iv) $f_i(g_{a_i}, \ldots, g_{a_{\varkappa_i}}) = g_a$, if $F_i(a_1, \ldots, a_{\varkappa_i}) = a$.

The set of all sentences from $\mathcal{E}_3$ of the form (i)-(iv) will be called the *description* of model $\mathfrak{M}$ and denoted by $\mathcal{D}(\mathfrak{M})$.

Let us remember that

(4.1) $\mathcal{D}(\mathfrak{M}) \subset \mathcal{S}_3 \frown \mathcal{O}_3$.

(4.2) *In sentences belonging to $\mathcal{D}(\mathfrak{M})$ the signs $q_1, q_2, \ldots$ do not occur.*

From the definition of description it follows immediately that

(4.3) *If $\mathcal{D}(\mathfrak{M}) \subset \mathfrak{X} \subset \mathcal{S}_3$ and $\mathfrak{M}^*$ is a model of $\mathfrak{X}$, then $\mathfrak{M}^*|1$ is an extension of the first kind and $\mathfrak{M}^*|2$ an extension of the second kind of the model $\mathfrak{M}$.*

We shall now prove:

(4.4) *If $\mathfrak{M}$ is a model of the set $\mathfrak{X} \subset \mathcal{E}_i$, then the set $\mathfrak{X} \smile \mathcal{D}(\mathfrak{M})$ is consistent in $\mathcal{E}_3$.*

In fact the weak extension of the second kind

$$\mathfrak{M}^* = \langle A, R_1, R_2, \ldots, F_1, F_2, \ldots, Q_1, Q_2, \ldots, \{G_a\}_{a \in A}\rangle$$

where $G_a = a$ and Q_i are arbitrary, is in that case the model of the set $\mathfrak{X} \smile \mathcal{D}(\mathfrak{M})$, hence (4.4) results from (2.2).

(4.5) *If $\mathfrak{M}$ is a model of the set $\mathcal{A} \subset \mathcal{E}_1$, and the set $\mathcal{A} \smile \mathcal{C} \smile \mathcal{D}(\mathfrak{M})$ where $\mathcal{C} \subset \mathcal{E}_2$ is inconsistent in $\mathcal{E}_3$, then there exists such $\alpha_1 \in \mathcal{O}_1$, that $\alpha_1 \in Cn_2(A \smile C)$ and α_1 is not valid in $\mathfrak{M}$.*

Proof. It follows from (1.6) that there exists such a finite set $\mathcal{D}_0 \subset \mathcal{D}(\mathfrak{M})$ that $\mathcal{A} \smile \mathcal{C} \smile \mathcal{D}_0$ is inconsistent in $\mathcal{E}_3$. Let us denote by $\alpha = \alpha(g_{a_1}, \ldots, g_{a_n})$ the conjunction of all sentences belonging to $\mathcal{D}_0$. $\mathcal{A} \smile \mathcal{C} \smile (\alpha)$ is obviously inconsistent in $\mathcal{E}_3$, hence it follows from (1.7) that $\alpha' = \alpha(g_{a_1}, \ldots, g_{a_n})'$ belongs to $Cn_3(\mathcal{A} \smile \mathcal{C})$, but $\mathcal{A} \smile \mathcal{C} \subset \mathcal{E}_2$, hence from (1.8) it follows that $\alpha_1 = \alpha(x_{s_1}, \ldots, x_{s_n})'$ belongs to $Cn_2(\mathcal{A} \smile \mathcal{C})$. From (4.1) and (4.2) it follows that $\alpha_1 \in \mathcal{O}_1$.

[9]) The symbol = has two meanings in this paper: as a primitive sign occurring in formulas of theories, and as an identity sign for elements. In this formula it is used in both meanings.

$\alpha = \alpha(g_{a_1}, \ldots, g_{s_n})$ is the conjunction of sentences belonging to $\mathcal{D}(\mathfrak{M})$ and from the definition of description $\mathcal{D}(\mathfrak{M})$ it follows that the elements $a_1, \ldots, a_n$ satisfy the formula $\alpha(x_{s_1}, \ldots, x_{s_n})$ in $\mathfrak{M}$, thus not satisfying $\alpha(x_{s_1}, \ldots, x_{s_n})'$, which is to say, that α_1 is not valid in $\mathfrak{M}$, q. e. d.

§ 5. Theorem on the existence of extensions. Let $\mathcal{A} \subset \mathcal{E}_1$, $\mathcal{C} \subset \mathcal{E}_2$, and let $\mathfrak{M}$ be a model of the set $\mathcal{A}$. Every extension of the second kind $\mathfrak{M}^*$ of the model $\mathfrak{M}$, which is a model of the set $\mathcal{A} \smile \mathcal{C}$ will be called the *extension of $\mathfrak{M}$ with primary conditions $\mathcal{A}$ and secondary conditions $\mathcal{C}$.*

THEOREM 1. *In order that an extension of the model $\mathfrak{M}$ with primary conditions $\mathcal{A}$ and secondary conditions $\mathcal{C}$ may exist, it is necessary and sufficient that $\mathfrak{M}$ be the model of the set $\mathcal{O}_1 \frown Cn_2(\mathcal{A} \smile \mathcal{C})$.*

Proof. Necessity. Let $\mathfrak{M}^*$ be an extension of $\mathfrak{M}$ with the conditions $\mathcal{A}$ and $\mathcal{C}$; then every formula which belongs to $\mathcal{A} \smile \mathcal{C}$ and consequently every formula which belongs to $Cn_2(\mathcal{A} \smile \mathcal{C})$ is valid in $\mathfrak{M}^*|1$. In view of $\mathfrak{M}$ being a submodel of $\mathfrak{M}^*|1$, it follows from (3.1) that $\mathfrak{M}$ is a model of $\mathcal{O}_1 \frown Cn_2(\mathcal{A} \smile \mathcal{C})$.

Sufficiency. It follows from (4.5) that if $\mathfrak{M}$ is a model of set $\mathcal{O}_1 \frown Cn_2(\mathcal{A} \smile \mathcal{C})$, the set $\mathcal{A} \smile \mathcal{C} \smile \mathcal{D}(\mathfrak{M})$ is consistent in $\mathcal{E}_3$. Gödel's theorem shows that there exists a model $\mathfrak{M}^0$ of that set. It follows from (4.5) that $\mathfrak{M}^0|2 = \mathfrak{M}^*$ is an extension of the second kind of the model $\mathfrak{M}$. Considering that $\mathcal{A} \smile \mathcal{C} \subset \mathcal{E}_2$, and that $\mathfrak{M}^0$ is a model of that set, hence $\mathfrak{M}^*$ is also a model of that set.

Remark 1. If $\mathcal{C} \subset \mathcal{E}_1$, then $\mathcal{C} \subset \mathcal{E}_2$, hence in this case theorem 1 is of course valid. In theorem 1 we state the existence of an extension of the second kind but by assuming $\mathcal{C} \subset \mathcal{E}_1$ we can put $\mathfrak{M}^* = \mathfrak{M}^0|1$ and thus we obtain an extension of the first kind with conditions $\mathcal{A}$ and $\mathcal{C}$; in that case a condition for the existence of extension is that $\mathfrak{M}$ be the model of set $\mathcal{O}_1 \frown Cn_1(\mathcal{A} \smile \mathcal{C})$ because, as may easily be verified, for $\mathfrak{X} \subset \mathcal{E}_1$ we have $\mathcal{O}_1 \frown Cn_1(\mathfrak{X}) = \mathcal{O}_1 \frown Cn_2(\mathfrak{X})$.

THEOREM 2. *If $\mathcal{C} \subset \mathcal{O}_2$ then for a weak extension of the model $\mathfrak{M}$ with conditions $\mathcal{A}$ and $\mathcal{C}$ to exist it is necessary and sufficient that $\mathfrak{M}$ be the model of set $\mathcal{O}_1 \frown Cn_2(\mathcal{A} \smile \mathcal{C})$.*

The proof follows immediately from theorem 1 and lemma (3.1).

Remark 2. It should be noted that theorem 2 is valid only for the reason that we pass from set $\mathcal{E}_1$ to $\mathcal{E}_2$ by adding certain signs of relations and not signs of functions. When we do so it frequently happens that $\mathcal{E}_2$ differs from $\mathcal{E}_1$ in as much as in formulas $\mathcal{E}_2$ occur signs of functions which are not found in formulas $\mathcal{E}_1$. If this is the case, we may apply theorem 1, since every sign of an s-ary function may be replaced by the sign of an $(s+1)$-ary relation, assuming this relation to be a function with respect to its last argument. These conditions are expressed in the form of certain formulas belonging to $\mathcal{E}_2$, but not to $\mathcal{O}_2$, which must be added to the secondary conditions of extension. Thus we may apply theorem 1, but theorem 2 is not applicable. Namely, if even the initial set of secondary conditions consisted of open formulas only, then, after replacing the signs in functions by the sign of relation, we obtain a set of conditions in the formulas of which quantifiers will occur and therefore the assumption of the theorem will not be satisfied.

§ 6. Characterization of $\mathcal{O}$-systems [10]. In this paragraph we shall make use of the following theorem, known as the theorem of completeness of the rules of elementary theories [11]:

THEOREM OF COMPLETENESS. *If $\mathfrak{X} \subset \mathcal{E}_1$, $\alpha \in \mathcal{E}_1$ and $\alpha \notin Cn_1(\mathfrak{X})$, then there exists a model of $\mathfrak{X}$ in which α is not valid.*

If $\mathfrak{X} \subset \mathcal{E}_1$ and $Cn_1(\mathfrak{X}) = \mathfrak{X}$, then we call $\mathfrak{X}$ a *system*.

If $\mathfrak{X}$ is a system and $\mathfrak{X} = Cn_1(\mathcal{O}_1 \cap \mathfrak{X})$, then $\mathfrak{X}$ is called *$\mathcal{O}$-system*.

From (2.1) and (3.1) it follows that

(6.1) *If $\mathfrak{X}$ is the $\mathcal{O}$-system, then every submodel of a model of the system $\mathfrak{X}$ is a model of the system $\mathfrak{X}$.*

We shall prove the following inversion of this theorem:

THEOREM 3. *If every submodel of a model of the system $\mathfrak{X}$ is a model of the system $\mathfrak{X}$, then $\mathfrak{X}$ is an $\mathcal{O}$-system.*

Proof. Let us assume that $\alpha \in \mathfrak{X}$ and $\alpha \notin Cn_1(\mathcal{O}_1 \cap \mathfrak{X})$. It results from the theorem of completeness that there exists a model $\mathfrak{M}$ of the set $\mathcal{O}_1 \cap \mathfrak{X}$, in which α is not valid. Putting $\mathcal{A} = \mathcal{O}_1 \cap \mathfrak{X}$ and $\mathcal{C} = \mathfrak{X}$ we obtain $\mathcal{O}_1 \cap Cn_1(\mathcal{A} \cup \mathcal{C}) = \mathcal{O}_1 \cap Cn_1(\mathfrak{X}) = \mathcal{O}_1 \cap \mathfrak{X} = \mathcal{A}$, hence the condition of theorem 1 is satisfied. From theorem 1 it follows that there exists an extension (of the first kind) $\mathfrak{M}^*$ of the model $\mathfrak{M}$ which is a model of the system $\mathfrak{X}$. Following the assumption of the theorem, $\mathfrak{M}$ as a submodel of $\mathfrak{M}^*$ is a model of the system $\mathfrak{X}$ and therefore α is satisfied in $\mathfrak{M}$, contrary to the assumption. We have shown that there exists no $\alpha \in \mathfrak{X}$ such that $\alpha \notin Cn_1(\mathcal{O}_1 \cap \mathfrak{X})$, hence $\mathfrak{X} \subset Cn_1(\mathcal{O}_1 \cap \mathfrak{X})$. Since — as is easy to see — $Cn_1(\mathcal{O}_1 \cap \mathfrak{X}) \subset Cn_1(\mathfrak{X}) = \mathfrak{X}$, therefore finally $\mathfrak{X} = Cn_1(\mathcal{O}_1 \cap \mathfrak{X})$, q. e. d.

§ 7. Elementarily definable classes. Let $\boldsymbol{A}$ be a certain class of models. The class $\boldsymbol{A}$ is called *elementarily definable* if there exists such a set $\mathcal{A} \subset \mathcal{E}_1$ that $\mathfrak{M} \in \boldsymbol{A}$ if and only if $\mathfrak{M}$ is a model of the set $\mathcal{A}$. The set $\mathcal{A}$ is then called a *set of axioms* of class $\boldsymbol{A}$.

[10]) The results contained in this paragraph are given in [4].

[11]) This theorem is often called Gödel's theorem on the completeness of the functional calculus.

When considering definable classes we always assume that in formulas of the set $\mathcal{E}_1$ occur only those primitive signs which are essential for the problem under consideration. Thus, for example, considering the class of semigroups, we assume that in $\mathcal{E}_1$ there is only one sign of a binary function corresponding to multiplication in semigroups, whereas neither the signs of relations nor any other signs of functions occur in formulas of $\mathcal{E}_1$. At the same time we assume, of course, that in the models in question there are interpretations only for signs occurring in formulas of $\mathcal{E}_1$. Sometimes it may not be irrelevant whether the same problem is considered by means of a greater or smaller number of primitive signs. For example, as we know, groups may be considered with the aid of one binary operation — multiplication, or two operations — multiplication and unary inversion. Only in the second case it is possible to write the axiom of groups in the form of open formulas. From this results the well known fact that in the first case the submodel of a group need not be a group (it may be a semigroup only, *e. g.* the set of natural numbers forms a submodel of the group of integers), and in the second case a submodel of a group is always a group.

If $\boldsymbol{A}$ is a given class of models, then by an *$\boldsymbol{A}$-submodel* of model $\mathfrak{M}$ we shall understand that submodel of the model $\mathfrak{M}$ which belongs to the class $\boldsymbol{A}$.

For a given set $\mathcal{A} \subset \mathcal{E}_1$ we shall denote by $\boldsymbol{M}(\mathcal{A})$ the definable class of axiomatic $\mathcal{A}$. Obviously, if $\mathcal{A} \subset \mathcal{B}$, then $\boldsymbol{M}(\mathcal{B}) \subset \boldsymbol{M}(\mathcal{A})$. It follows from Gödel's theorem that $\boldsymbol{M}(\mathcal{A}) = \boldsymbol{M}(\mathcal{B})$ is equivalent to $Cn_1(\mathcal{A}) = Cn_1(\mathcal{B})$ [12].

§ 8. General conclusions from theorem 3. From theorem 3 follows immediately

THEOREM 4. *If every submodel of a model belonging to a definable class $\boldsymbol{A}$ belongs to $\boldsymbol{A}$, then there exists an axiomatic $\mathcal{A}$ of $\boldsymbol{A}$ contained in $\mathcal{O}_1$.*

This is, in fact, theorem 3 in the terminology of definable classes. A more specific conclusion from theorem 3 is the following

THEOREM 5. *If $\boldsymbol{A} = \boldsymbol{M}(\mathcal{A})$ is a definable class, $\boldsymbol{B}$ is a definable subclass of $\boldsymbol{A}$ and every $\boldsymbol{A}$-submodel of a model which belongs to $\boldsymbol{B}$ again belongs to $\boldsymbol{B}$, then there exists such a set $\mathcal{B} \subset \mathcal{O}_1$ that $\boldsymbol{B} = \boldsymbol{M}(\mathcal{A} \cup \mathcal{B})$* [13].

Let us note here one more lemma resulting from theorem 5 and from the known properties of elementary theories:

[12]) According to Tarski [11] classes definable by means of a finite axiomatic are called *arithmetical classes*. I do not use this terminology as this notion seems to be only loosely connected with arithmetic.

[13]) This theorem may be also proved by repeating the proof of theorem 3. It is in fact a generalization of theorem 3.

(8.1) *If $A = M(\mathcal{A})$ is a definable class, α a sentence and every A-submodel of a model belonging to class $M(A \cup (\alpha))$ also belongs to that class, then to $Cn_1(\mathcal{A})$ belongs the equivalence $\alpha \equiv \beta$ where β is a general sentence, i. e. a sentence of the form $\prod_{x_{s_1}} \prod_{x_{s_2}} \dots \prod_{x_{s_n}} \gamma$, $\gamma \in \mathcal{O}_1$.*

§ 9. Sentences persistent with respect to a given definable class. A sentence α will be called *persistent* with respect to a class of models A if and only if whenever it is valid in a model $\mathfrak{M}$ belonging to A it is also valid in all the extensions of $\mathfrak{M}$ which belong to A.

From (8.1) we obtain the following theorem which describes the form of persistent sentences with respect to definable classes:

THEOREM 6. *If the sentence α is persistent with respect to the definable class $A = M(\mathcal{A})$, then to $Cn_2(\mathcal{A})$ belongs an equivalence $\alpha \equiv \beta$ where β is an existential sentence, i. e. a sentence of the form*

$$\sum_{x_{s_1}} \sum_{x_{s_2}} \dots \sum_{x_{s_n}} \gamma, \qquad \gamma \in \mathcal{O}_1.$$

As Robinson[14]) has shown, every sentence persistent with respect to the class of all Abelian groups G (it being of no consequence whether we treat groups as models of an axiomatic of one or two primitive functions) and satisfied in at least one Abelian group is satisfied in a certain finite group.

On account of the definability of the class of Abelian groups, Gödel's theorem and theorem 6, the theorem of Robinson, may be expressed as follows:

(9.1) *If α is an existential sentence, consistent with the axiomatic of Abelian groups[15]), then α is valid in a certain finite group*

or in an entirely equivalent form:

(9.2) *If the formula $\alpha \in \mathcal{O}_1$ is valid in all finite Abelian groups, it is valid in all Abelian groups.*

There are analogous theorems for the class of commutative rings[16]), but it is false for the class of arbitrary groups[17]).

§ 10. The class of submodels of a given definable class. For a given class of models A let us denote by $\mathrm{sm}(A)$ the class of all submodels of models of class A. As every model is, according to the

[14]) Robinson [8], p. 234.

[15]) This means that set $\mathcal{A} \cup (\alpha)$ is consistent when $\mathcal{A}$ is a set of axioms for Abelian groups. It is a guarantee of the existence of Abelian groups in which α is valid.

[16]) Robinson [8], p. 146.

[17]) I am greatly obliged to Professor B. H. Neuman for this information.

definition, its own submodel, hence $A \subset \mathrm{sm}(A)$. From the definition of operation sm it follows that

$$\mathrm{sm}(\mathrm{sm}(A)) = \mathrm{sm}(A). \tag{10}$$

If A is a definable class, then, according to theorem 4, $A = \mathrm{sm}(A)$ is equivalent to the existence of such a set $\mathcal{A} \subset \mathcal{O}_1$ that $A = M(\mathcal{A})$. On the other hand, in accordance with theorem 5, for an arbitrary definable class $A = M(\mathcal{A})$ and its arbitrary definable subclass B, $A \cap \mathrm{sm}(B) = B$ is equivalent to the existence of such a set $\mathcal{B} \subset \mathcal{O}_1$ that $B = M(\mathcal{A} \cup \mathcal{B})$.

We thus find that operation sm enables us to express some theorems in a very simple manner.

From theorem 1 and the above remark follows

THEOREM 7. *If $A = M(\mathcal{A})$ is a definable class, then $\mathrm{sm}(A)$ is a definable class too, and $\mathrm{sm}(A) = M(\mathcal{O}_1 \cap Cn_1(\mathcal{A}))$.*

In order to generalize this theorem we shall consider simultaneously classes of models of $\mathcal{E}_1$ and $\mathcal{E}_2$. For a class A of models for $\mathcal{E}_2$ we shall denote by $A|1$ the class of all models $\mathfrak{M}|1$, where $\mathfrak{M} \in A$.

THEOREM 8. *If $A = M(\mathcal{A})$, $\mathcal{A} \subset \mathcal{E}_2$ is a definable class of models, the class $\mathrm{sm}(A|1)$ is a definable class too, and $\mathrm{sm}(A|1) = M(\mathcal{O}_1 \cap Cn_2(\mathcal{A}))$.*

THEOREM 9. *If $A = M(\mathcal{A})$, $\mathcal{A} \subset \mathcal{E}_2$, and $A|1 \subset B = M(\mathcal{B})$, $\mathcal{B} \subset \mathcal{E}_1$, then the class $B \cap \mathrm{sm}(A|1)$ is a definable class too, and*

$$B \cap \mathrm{sm}(A|1) = M\left[\mathcal{B} \cup \left(\mathcal{O}_1 \cap Cn_2(\mathcal{A} \cup \mathcal{C})\right)\right].$$

§ 11. Definability of some non-elementarily defined classes. So far we have discussed only elementary formulas, in particular sentences, *i. e.* those in which only individual variables occur. Now we proceed to discuss a special type of non-elementary sentences, namely, sentences of the form

$$\sum_{q_1} \sum_{q_2} \dots \sum_{q_s} \beta \tag{$*$}$$

where $\beta \in S_2$, and where no signs q_j occur in β for $j > s$. The signs q_j occurring in α are treated in this case as variables which run over all relations, and the signs r_i and f_i as constant relations and functions.

Let $\mathfrak{M}$ be a model of a set $\mathcal{A} \subset \mathcal{E}_1$. It is obvious what we should understand by the *validity of a sentence* α of the form ($*$), in $\mathfrak{M}$. It simply means that on the set A of the model $\mathfrak{M} = \langle A, R_1, R_2, \dots, F_1, F_2, \dots \rangle$ it is possible to determine such relations $Q_1, \dots, Q_s$ that β is valid in the model $\mathfrak{M}^* = \langle A, R_1, R_2, \dots, F_1, F_2, \dots, Q_1, \dots, Q_s \rangle$.

In other words: α is *valid* in M means that there exists a weak extension of the second kind of the model $\mathfrak{M}$ in which β is valid.

Let us denote by $\boldsymbol{M}(\mathcal{A}, a)$ the class of all those models of the set $\mathcal{A}$ in which the sentence a is valid. From the above remark it follows that for a of the form (*):

$$\boldsymbol{M}(\mathcal{A}, a)=\boldsymbol{M}\big(\mathcal{A} \cup (\beta)\big)|1. \tag{11.1}$$

As will be shown below (§ 12) class $\boldsymbol{M}(\mathcal{A}, a)$ need not be elementarily definable. From theorem 2, theorem 9 and (11.1) follows

THEOREM 10. *If a is of the form (*) and β is a general sentence (i. e. of the form $\prod\limits_{x_{s_1}}\prod\limits_{x_{s_2}}\ldots\prod\limits_{x_{s_n}}\gamma$, $\gamma \in \mathcal{O}_2$), then $\boldsymbol{M}(\mathcal{A}, a)$ is elementarily definable and*

$$\boldsymbol{M}(\mathcal{A}, a)=\boldsymbol{M}\Big(\mathcal{A} \cup \big(\mathcal{O}_1 \frown Cn_2\big(\mathcal{A} \cup (\beta)\big)\big)\Big).$$

THEOREM 11. *If a is of the form (*), then class*

$$\mathrm{sm}\big(\boldsymbol{M}(\mathcal{A}, a)\big) \quad \textit{and} \quad \boldsymbol{M}(\mathcal{A}) \frown \mathrm{sm}\big(\boldsymbol{M}(\mathcal{A}, a)\big)$$

are elementarily definable and

$$\mathrm{sm}\big(\boldsymbol{M}(\mathcal{A}, a)\big)=\boldsymbol{M}\Big(\mathcal{O}_1 \frown Cn_2\big(\mathcal{A} \cup (\beta)\big)\Big),$$

$$\boldsymbol{M}(\mathcal{A}) \frown \mathrm{sm}\big(\boldsymbol{M}(\mathcal{A}, a)\big)=\boldsymbol{M}\Big(\mathcal{A} \cup \big(\mathcal{O}_1 \frown Cn_2\big(\mathcal{A} \cup (\beta)\big)\big)\Big).$$

§ 12. Example of a non-elementarily definable class $\boldsymbol{M}(\mathcal{A}, a)$. Let us interpret the signs of functions f_1, f_2 as the operations of addition and multiplication of natural numbers, the signs f_3, f_4 as numbers 0 and 1 (hence $\varkappa_1=\varkappa_2=2$, $\varkappa_3=\varkappa_4=0$). In this interpretation let $\mathcal{A}$ be the set of all valid formulas. Set $\mathcal{A}$ is, of course, a complete system.

It is well known that the $\boldsymbol{M}(\mathcal{A})$ comprises a great number of different models but only in one of them the (non-elementary) axiom of induction is valid.

Let a be the sentence

$$\sum_{q_1}\Big\{q_1(f_3) \wedge \prod_{x_1}\Big[q_1(x_1) \rightarrow q_1\big(f_1(x_1, f_4)\big)\Big] \wedge \sum_{x_1} q_1(x_1)'\Big\}.$$

Class $\boldsymbol{M}(\mathcal{A}, a)$ is not equal to $\boldsymbol{M}(\mathcal{A})$ as it is composed only of those models belonging to $\boldsymbol{M}(\mathcal{A})$ in which the axiom of induction is not valid. This class is elementarily not definable since $\mathcal{A}$ is a complete system, hence none of its proper subclasses is elementarily definable.

§ 13. Extension of a semigroup to a group. In particular applications we shall refrain from denoting relations and operations by the letters r_i, q_i and f_i; we shall denote them in the way accepted in the respective branches of mathematics.

Let us assume that in formulas of $\mathcal{E}_1$ there occurs only one binary operation "$\cdot$" — multiplication. Let $\mathcal{A}_1$ consist only of the associative law

$$(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3).$$

Every model of $\mathcal{A}_1$ is a semigroup. Let set $\mathcal{C}_1$ be composed of the formulas

$$\prod_{x_1}\prod_{x_2}\sum_{x_3} x_1 \cdot x_3 = x_2,$$

$$\prod_{x_1}\prod_{x_2}\sum_{x_3} x_3 \cdot x_1 = x_2.$$

Every model of $\mathcal{A}_1 \cup \mathcal{C}_1$ is a group.

The problem of extension with primary conditions $\mathcal{A}_1$ and secondary conditions $\mathcal{C}_1$ is the problem of extension of a semigroup to a group. From theorem 1 it follows that this problem may be solved by giving open formulas written with the help of multiplication only, the validity of which in a given semi-group is the necessary and sufficient condition of the existence of extension. In precisely such manner Malcev has solved this problem in the first of his papers [7], and in the second he proved that the set of open formulas forming this condition of extension must be infinite.

§ 14. Extension of a semigroup to a ring. Let us now assume that in $\mathcal{E}_2$ besides multiplication $\cdot$ occurs also addition $+$. Let $\mathcal{C}_2$ consist of the following formulas:

$$\prod_{x_1}\prod_{x_2}\sum_{x_3} x_1 + x_3 = x_2,$$

$$(x_1+x_2)+x_3 = x_1+(x_2+x_3),$$

$$x_1+x_2 = x_2+x_1,$$

$$x_1 \cdot (x_2+x_3) = x_1 \cdot x_2 + x_1 \cdot x_3,$$

$$(x_1+x_2) \cdot x_3 = x_1 \cdot x_3 + x_2 \cdot x_3.$$

The problem of extension with primary conditions $\mathcal{A}_1$ and secondary conditions $\mathcal{C}_2$ is the problem of embodying a semigroup of a ring. From theorem 1 it follows that this problem may be solved by giving sufficient and necessary conditions containing no quantifiers. As far as I know, this problem has not yet been solved in this or any other form.

If we put $\boldsymbol{P}=\boldsymbol{M}(\mathcal{A}_1+\mathcal{C}_2)$ the above problem may be formulated thus:

find a set of axioms for the class $\mathrm{sm}(\boldsymbol{P})|1$.

On the basis of theorem 8 a set of axioms for this class is the set $\mathcal{O}_1 \frown Cn_2(\mathcal{A})$, this being, of course, not the only set of axioms of that class.

In purely logical terms we may formulate our problem thus: find a set $\mathfrak{X} \subset \mathcal{E}_1$ (possibly the simplest) such that

$$Cn_1(\mathfrak{X}) = Cn_1\big(\mathcal{O}_1 \cap Cn_2(\mathcal{A}_1 \cap C_2)\big).$$

It is of interest to note that the algebraic problem of extension may be reduced to a logical one. In that case it also seems that this cannot be of any practical value. We shall present below a case in which a similar translation of a problem into terms of logic immediately yields its solution (see § 16).

§ 15. Extension of a group to an ordered group. We call a group G *ordered* if for its elements an ordering relation $<$ is defined, such that for arbitrary elements a,b,c,d of G: if $a<b$, then $cad<cbd$.

The class of groups which can be ordered may be defined in a non-elementary way as follows:

By γ we denote the conjunction of the formulas

$$q_1(x_1, x_2) \to q_1(x_2, x_1)',$$

$$q_1(x_3, x_4) \wedge q_1(x_4, x_5) \to q_1(x_3, x_5),$$

$$q_1(x_6, x_7) \vee q_1(x_7, x_6) \vee x_6 = x_7.$$

By β we denote the formula

$$\prod_{x_1} \dots \prod_{x_7} \gamma$$

and, lastly, by α we denote the formula $\sum\limits_{q_1} \beta$.

The class of groups which can be ordered is the class $\boldsymbol{U} = \boldsymbol{M}(\mathcal{A}_1 \cup \mathcal{C}_1, \alpha)$.

In view of the fact that β is a general sentence, it follows from theorem 10 that $\boldsymbol{U}$ is an elementarily definable class and that a set of its axioms may be obtained by adding a certain class $\mathfrak{X} \subset \mathcal{O}_1$ to the axioms of the groups $\mathcal{A}_1 \cup \mathcal{C}_1$. On the basis of the known theorems on ordered groups it is possible to construct the set $\mathfrak{X}$ which, together with $\mathcal{A}_1 \cup \mathcal{C}_1$, defines the class $\boldsymbol{U}$ (see [5]). It is also possible to prove that $\mathfrak{X}$ must be an infinite set.

§ 16. Reduction of a ternary semigroup. Let us suppose that in the formulas of $\mathcal{E}_1$ there occurs one symbol of operation on three elements (a,b,c), and that $\mathcal{A}_3$ consists of the condition of complete associativity of that operation

$$\big((a,b,c),d,e\big) = \big(a,(b,c,d),e\big) = \big(a,b,(c,d,e)\big).$$

Every model of the class $\boldsymbol{T} = \boldsymbol{M}(\mathcal{A}_3)$ is called a *ternary semigroup*.

A ternary semigroup is called *reducible* if there exists such an associative operation on two elements that

$$(a,b,c) = (a \cdot b) \cdot c.$$

During the last war Banach showed [18]) that not every ternary semigroup is reducible and he put forward the problem whether every ternary semigroup may be extended to a reducible one. This problem of Banach may be presented as follows.

Let us assume that in $\mathcal{E}_2$ there occurs, besides the ternary operation occurring already in $\mathcal{E}_1$, the binary operation of multiplication — "$\cdot$". Let $\mathcal{C}_3$ consist of two formulas

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \qquad (a,b,c) = (a \cdot b) \cdot c.$$

Will $\mathrm{sm}\big(\boldsymbol{M}(\mathcal{A}_3 \cup \mathcal{C}_3)|1\big) = \boldsymbol{T}$?

We may here apply theorem 8, on the basis of which

$$\mathrm{sm}\big(\boldsymbol{M}(\mathcal{A}_3 \cup \mathcal{C}_3)|1\big) = \boldsymbol{M}\big(\mathcal{O}_1 \cap Cn_2(\mathcal{A}_3 \cup \mathcal{C}_3)\big).$$

It is easily seen that formulas of $\mathcal{A}_3 \cup \mathcal{C}_3$ permit only an arbitrary displacement of brackets, which is already guaranteed by $\mathcal{A}_3$ for formulas belonging to $\mathcal{O}_1$.

Thus we have [19])

$$\mathcal{O}_1 \cap Cn_1(\mathcal{A}_3) = \mathcal{O}_1 \cap Cn_2(\mathcal{A}_3 \cup \mathcal{C}_3)$$

which results in

$$\mathrm{sm}\big(\boldsymbol{M}(\mathcal{A}_3 \cup \mathcal{C}_3)|1\big) = \boldsymbol{M}\big(\mathcal{O}_1 \cap Cn_1(\mathcal{A}_3)\big) = \boldsymbol{M}(\mathcal{A}_3) = \boldsymbol{T},$$

i. e. in a positive solution of Banach's problem.

§ 17. The problem of weak extensions. In the general case the problem of giving conditions for the existence of weak extensions of the second kind of a given model $\mathfrak{M}$ with primary conditions $\mathcal{A}$ and secondary conditions $\mathcal{C}$ remains open. The example presented in § 12 seems to indicate that no conditions asserting the validity of certain sentences of $\mathcal{E}_1$ in $\mathfrak{M}$ can be sufficient. This seems to be indicated also by the theorem given below without proof, which can by no means be regarded as a solution of the problem. I am giving it for the purpose of drawing attention to the fact that the solution should be looked for in the conditions imposed upon the system $Cn_3\big(\mathcal{A} \cup \mathcal{C} \cup \mathcal{D}(\mathfrak{M})\big)$ and not only in the systems contained in $\mathcal{E}_1$ and $\mathcal{E}_2$.

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[18]) I am greatly obliged to Professor A. Alexiewicz for this information.

[19]) This requires a very simple proof which may be carried out for instance by mapping the systems in question in a theory in which no brackets are used for writing the individual formulas $x_1 \cdot x_2 \cdot \ldots \cdot x_n$.

We shall call the system $\mathfrak{X}$ in E_3 *ω-complete* if whenever all sentences of the form $\alpha(g_t)$, $t \in T$, $\alpha \in \mathcal{E}_3$ belong to $\mathfrak{X}$, then also the sentence $\prod_{x_s} \alpha(x_s)$, where x_s does not occur in $\alpha(g_t)$, belongs to $\mathfrak{X}$.

Theorem. *If an ω-complete system containing $\mathcal{A} \cup \mathcal{C} \cup \mathcal{D}(\mathfrak{M})$ exists in $\mathcal{E}_3$, then there exists a weak extension of model $\mathfrak{M}$ with the conditions $\mathcal{A}$ and $\mathcal{C}$.*

The inverse theorem is valid too.

Finally let us observe that the theorems and constructions contained in this paper, and particularly the method applied, are not entirely the author's own. The method was introduced first by Malcev [6] and Robinson [8], [9]; some theorems, very similar to those included here, were recently published by Henkin [2].

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Added during proof. Recently a paper by B. H. Neuman has been published (*An embedding theorem for algebraic systems*, Proceedings of the London Mathematical Society 3 (4) (1954), p. 138-153) containing theorems very similar to those given here.

О сепарабельности топологических групп

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Вполне аддитивная мера $\mu(X)$, определённая на некотором теле $\boldsymbol{X}$ подмножеств множества X, называется *сепарабельной*, если существует такой счётный класс $\boldsymbol{M}$ измеримых множеств, что для всякого множества $X \in \boldsymbol{X}$ конечной меры и всякого действительного числа $\varepsilon > 0$ найдётся в $\boldsymbol{M}$ множество M, удовлетворяющее условию $\mu(X \dot{-} M) < \varepsilon$. $X \dot{-} M$ обозначает симметрическую разность множеств X и M, т. е. $(X - M) \cup (M - X)$.

Теорема. *Для того, чтобы локально-компактная топологическая группа G была сепарабельной, необходимо и достаточно, чтобы её мера Хаара была сепарабельной.*

Необходимость условия очевидна, так как всякая борелевская регулярная мера в сепарабельном пространстве сепарабельна (в качестве счётного класса $\boldsymbol{M}$ можно принять класс сумм конечного числа множеств из базы открытых множеств пространства).

Для доказательства достаточности воспользуемся следующими леммами.

Лемма 1. *Если мера Хаара локально-компактной группы G сепарабельна, то группа G удовлетворяет первой аксиоме счётности и следовательно метризуема.*

Доказательство. Как известно из теоремы Вейля [3] (см. также [1]), система множеств $\mathop{E}_{x}\{\mu(xE \dot{-} E) < \varepsilon\}$, где μ — мера Хаара, $x \in G$, а ε — число, удовлетворяющее неравенствам $0 < \varepsilon < 2\mu(E)$, образует полную систему окрестностей единицы группы. Легко заметить, что можно получить полную систему окрестностей единицы, ограничиваясь числами ε вида $1/n$, где n натуральное число, а если мера сепарабельна, можно сверх того ограничиться множествами E из счётного класса $\boldsymbol{M}$. Таким образом получается счётная полная система окрестностей единицы и группа удовлетворяет первой аксиоме счётности. Метризуемость группы следует отсюда по теореме Какутани [2].

Лемма 2. *Пусть X метрическое, локально-компактное пространство, μ — борелевская мера, положительная для открытых множеств, конечная для компактных, регулярная и такая, что открытые множества конечной меры можно приближать по мере компактными. Тогда, если мера μ сепарабельна, то пространство X сепарабельно.*