On the paradox of the sphere

by

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I have been interested in the problem whether it is possible to combine the results of three papers: W. Sierpiński [8], R. M. Robinson [4], and J. F. Adams [1]. It appears to be quite easy. Using a lemma from [8] we can considerably generalize the theorems of [4] and [1], and obtain by means of them a generalization of the main theorem of [8].

The aim of this paper is to formulate these generalizations and to prove three new theorems related to them (theorems (T_1), (T_2) and (T_3)).

All the theorems given in this paper, except the first — (S_1), are not effective (in their proofs the axiom of choice has been used).

$\simeq$ denotes congruence of sets.

$\underset{\mathrm{rot}}{\simeq}$ denotes congruence of sets lying on the sphere, by a rotation around its centre.

All that will be said concerns the 3-dimensional Euclidean space. By a sphere and a solid sphere we understand the sets defined by $x^2+y^2+z^2=1$ and $x^2+y^2+z^2\leqslant 1$ or $x^2+y^2+z^2<1$ respectively.

The lemma required from [8][1]) is the following:

(S_1) *There exist $2^{\aleph_0}$ independent rotations of the sphere around its centre (i. e. rotations generating a free group).*

The generalizations of the theorems of R. M. Robinson are the following:

(R_1) *Let $0<\mathfrak{m}\leqslant 2^{\aleph_0}$, $0<\mathfrak{n}\leqslant 2^{\aleph_0}$ and $\overline{\overline{M}}=\mathfrak{m}$, $\overline{\overline{N}}=\mathfrak{n}$; $\mathfrak{m}$ independent rotations $\{\varphi_\mu\}_{\mu\in M}$ of a sphere S around its centre, and $\mathfrak{m}$ relations $\{R_\mu\}_{\mu\in M}$, each having the set N as domain and range, are given.*

Then the following propositions are equivalent:

1° *We can decompose the sphere S into $\mathfrak{n}$ disjoint sets $\{A_\nu\}_{\nu\in N}$ in such a way that for this subdivision φ_μ is compatible*[2]) *with R_μ for each $\mu\in M$.*

[1]) P. 238, Lemme 1. The proof is effective. For other applications of this theorem see my paper [2].

[2]) The rotation φ is said to be *compatible with the relation R for the subdivision* $\{A_\nu\}_{\nu\in N}$ if the implication $[\varphi(A_{\nu_1})\cap A_{\nu_2}\neq 0]\to[\nu_1 R\nu_2]$ is satisfied for each $\nu_1,\nu_2\in N$.

2° *Every product of any finite number of factors of the form $R_\mu^{\pm 1}$ has a fixed point*[3]).

(R_2) *Let $\mathfrak{m}$, $\mathfrak{n}$, M, N, $\{\varphi_\nu\}_{\nu\in N}$ and S satisfy the hypothesis of (R_1), and let $\{P_\mu\}_{\mu\in M}$, $\{Q_\mu\}_{\mu\in M}$ be classes of non-empty subsets of N different from N*[4]).

Then the sphere S may be decomposed into $\mathfrak{n}$ disjoint pieces $\{A_\nu\}_{\nu\in N}$ satisfying the system of congruences

$$\sum_{\nu\in P_\mu}A_\nu\underset{\mathrm{rot}}{\simeq}\sum_{\nu\in Q_\mu}A_\nu,\qquad \mu\in M$$

if and only if none of the congruences generated[5]) *by these assert the congruence of two complementary portions of S (i. e. none are of the form $\sum_{\nu\in P}A_\nu\underset{\mathrm{rot}}{\simeq}\sum_{\nu\in N-P}A_\nu$). Furthermore if that decomposition is possible, it can be done in such a way that the μth congruence can be effectuated by the rotation φ_μ (i. e. $\varphi_\mu\big(\sum_{\nu\in P_\mu}A_\nu\big)=\sum_{\nu\in Q_\mu}A_\nu\big)$ for each $\mu\in M$.*

The generalization of the theorem of J. F. Adams (it concerns the congruence $\simeq$; (R_2) concerns $\underset{\mathrm{rot}}{\simeq}$) is the following:

(A_1)[6]) *Let $\mathfrak{m}$, $\mathfrak{n}$, M, N, $\{P_\mu\}_{\mu\in M}$, $\{Q_\mu\}_{\mu\in M}$ satisfy the hypothesis of (R_2). Then the sphere may be decomposed into $\mathfrak{n}$ disjoint pieces $\{A_\nu\}_{\nu\in N}$ satisfying the system of congruences*

$$\sum_{\nu\in P_\mu}A_\nu\simeq\sum_{\nu\in Q_\mu}A_\nu,\qquad \mu\in M.$$

The proofs of these generalizations are the same as the proofs of the original theorems in [4] and [1].

Now, applying (S_1) and (R_2), we easily obtain the following generalization of another result of [4] (§ 5):

[3]) The *product RR'* and the *converse R^{-1}* are defined by the equivalences:

$$[\nu_1RR'\nu_2]\equiv[\text{there exists such a } \nu\in N \text{ that } \nu_1R\nu \text{ and } \nu R'\nu_2],$$

$$[\nu_1R^{-1}\nu_2]\equiv[\nu_2R\nu_1],\qquad \text{for each}\quad \nu_1,\nu_2\in N.$$

The relation R is said to *have a fixed point* if for a certain $\nu\in N$ we have $\nu R\nu$.

[4]) We do not suppose that $\mu_1\neq\mu_2$ implies $P_{\mu_1}\neq P_{\mu_2}$ or $Q_{\mu_1}\neq Q_{\mu_2}$.

[5]) A congruence is said to be *generated* by a system of congruences σ if it belongs to σ or if it is obtainable by finite application to the congruences of σ of the rules of taking complements and of transitivity.

The rule of *taking complements* consists in making use of the implication

$$\text{if}\quad \sum_{\nu\in P}A_\nu\underset{\mathrm{rot}}{\simeq}\sum_{\nu\in Q}A_\nu\quad\text{then}\quad \sum_{\nu\in N-P}A_\nu\underset{\mathrm{rot}}{\simeq}\sum_{\nu\in N-Q}A_\nu.$$

The rule of *transitivity* consists in the application of the fact that if two congruences have one member in common, then the remaining members are congruent.

[6]) A corollary of this theorem will be given below — (A_2).

(R$_3$) *If $2<\mathfrak{n}\leqslant 2^{\aleph_0}$, then the sphere can be decomposed into $\mathfrak{n}$ disjoint congruent sets.*

(It is interesting to compare this theorem with the following two theorems. One, which is much easier but also not effective, is an immediate generalization of the well known construction of unmeasurable-L sets (Sierpiński [10], p. 64, or Ruziewicz [5]):

If $0<\mathfrak{n}\leqslant 2^{\aleph_0}$, then the straight line and the circumference of the circle can be decomposed into $\mathfrak{n}$ disjoint congruent sets, and the semiclosed interval $\langle 0,1)$ can be decomposed into $\mathfrak{n}$ disjoint parts each two of which are $\underset{2}{=}$ [7]).

The other proved by J. von Neumann [3] is much more difficult than the first and also not effective:

Each of the intervals $\langle 0,1\rangle$, $\langle 0,1)$, $(0,1)$ can be decomposed into $\aleph_0$ disjoint congruent sets [8]).)

The following generalization of (R$_3$) will be proved here:

(T$_1$) *There exists such a set E on a sphere that for each $\mathfrak{n}$, for which $2<\mathfrak{n}\leqslant 2^{\aleph_0}$, that sphere can be decomposed into $\mathfrak{n}$ disjoint sets congruent to E by a rotation around its centre.*

(It is interesting to compare this theorem with the following theorem of W. Sierpiński ([6] or [7], p. 100, Proposition C$_{48}$):

The plane can be decomposed into $2^{\aleph_0}$ disjoint sets congruent to a set E (which is a graphical representation of a function, i. e. it is defined by an equation of the form $y=f(x)$) and is the sum of $\aleph_0$ sets congruent to E.

No proof of this theorem without the use of the continuum hypothesis ($2^{\aleph_0}=\aleph_1$) is known—our theorem (T$_1$) is free of that assumption.)

For proving (T$_1$) we shall introduce certain symbols and a lemma.

For a given class of sets $\{A_\nu\}_{\nu\in N}$ any set of congruences of the form

$$(*)\qquad \sum_{\nu\in P} A_\nu \underset{\text{rot}}{\simeq} \sum_{\nu\in Q} A_\nu ,$$

where $P,Q\subset N$, will be called a *system of congruences.*

The congruence $(*)$ will be denoted shortly by

$$P\sim Q$$

and P and Q will be called its *nominating sets.*

If σ is a system of congruences we shall denote by

$C(\sigma)$ the system generated by σ by the rule of taking complements,

$T(\sigma)$ the system generated by σ by the rule of transitivity,

$H(\sigma)$ the system generated by σ.

LEMMA. *For each system of congruences σ we have*

$$H(\sigma)=T\big(C(\sigma)\big).$$

Proof. Evidently $T\big(C(\sigma)\big)\subset H(\sigma)$; thus it is enough to verify, that the set $T\big(C(\sigma)\big)$ is closed under the operations T and C. It is closed under T, because for any system of congruences τ we evidently have $T\big(T(\tau)\big)=T(\tau)$. It is closed under the operation C, because any complementary congruence to a congruence belonging to $T\big(C(\sigma)\big)$ obviously belongs to $T\big(C(\sigma)\big)$ also.

Proof of (T$_1$). We shall use (S$_1$) and (R$_2$). For proving (T$_1$) it is sufficient to verify that there exists a decomposition of the sphere into $2^{\aleph_0}$ disjoint pieces $\{A_\nu\}_{\nu\in N}$ ($\overline{\overline{N}}=2^{\aleph_0}$) satisfying the system of congruences σ_1:

$$(\nu_0)\sim(\nu),\qquad \nu\in N,$$
$$(\nu_0)\sim N-N_\xi,\qquad 1<\bar{\xi}<2^{\aleph_0}.$$

where ν_0 is a fixed element of N and for each ordinal number ξ for which $1<\bar{\xi}<2^{\aleph_0}$ we have

$$N_\xi\subset N \qquad\text{and}\qquad \overline{\overline{N}}_\xi=\bar{\xi}.$$

In fact it is sufficient to put $E=A_{\nu_0}$, and the assertion (T$_1$) is satisfied since we take when $\mathfrak{n}=2^{\aleph_0}$ the decomposition $\{A_\nu\}_{\nu\in N}$, and when $2<\mathfrak{n}<2^{\aleph_0}$ the decomposition into the sets $\{A_\nu\}_{\nu\in N_\alpha}$ and the set $\sum_{\nu\in N-N_\alpha} A_\nu$ where α is any ordinal number of the power $\mathfrak{n}-1$.

Now let us observe that σ_1 satisfies the hypothesis of (R$_2$) because $\bar{\bar{\sigma}}_1=2^{\aleph_0}$ and all the nominating sets occuring in σ_1 are different from 0 and N. Let $\{\varphi_\mu\}_{\mu\in M}$ ($\overline{\overline{M}}=2^{\aleph_0}$) be a set of independent rotations of the sphere (theorem (S$_1$)).

Then to prove the existence of the decomposition $\{A_\nu\}_{\nu\in N}$ fulfilling σ_1 it is sufficient to verify that this system of congruences satisfies the condition formulated in (R$_2$), *i. e.* that none of the congruences belonging to $H(\sigma_1)$ is of the form

$$(1)\qquad N-P\sim P$$

where $P\subset N$.

The system $C(\sigma_1)$ consists of the congruences belonging to σ_1 and of the congruences σ_2:

$$N-(\nu_0)\sim N-(\nu),\qquad \nu\in N;$$
$$N-(\nu_0)\sim N_\xi,\qquad 1<\bar{\xi}<2^{\aleph_0}.$$

[7]) The symbol $\underset{n}{=}$ denotes congruence by decomposition into n parts (for exact explanation see Sierpiński [8], p. 235, or [10]).

[8]) I have generalized recently this result for arbitrary cardinals $\mathfrak{n}$ satisfying the inequality $\aleph_0\leqslant\mathfrak{n}\leqslant 2^{\aleph_0}$ (to appear in Colloquium Mathematicum).

Now it is obvious that the class of nominating sets occuring in σ_1 is disjoint with the class of nominating sets occuring in σ_2. This proves that

$$T(\sigma_1 \cup \sigma_2) = T(\sigma_1) \cup T(\sigma_2),$$

and then, by the lemma, that

$$H(\sigma_1) = T(\sigma_1) \cup T(\sigma_2). \tag{2}$$

It is obvious that no two nominating sets occuring in σ_1 are complementary in N. Therefore no two nominating sets occuring in $T(\sigma_1)$ are complementary in N. The same can be verified for $T(\sigma_2)$. Therefore, by (2), no congruence of $H(\sigma_1)$ is of the form (1), q. e. d.

Another result of the paper [4] can be generalized as follows:

(R_4) *If $1 < \mathfrak{n} \leqslant 2^{\aleph_0}$, then the sphere S may be decomposed into $\mathfrak{n}$ disjoint sets each of which is $\underset{2}{=} S$.*

Applying (R_4) it is easy to prove that the solid sphere $\boldsymbol{S}$ may be decomposed into $\mathfrak{n}$ disjoint sets one of which $\underset{2}{=}\boldsymbol{S}$ and all the others $\underset{4}{=}\boldsymbol{S}$, but following more closely the reasoning of [4] we can moreover find that

(S_2) *If $1 < \mathfrak{n} \leqslant 2^{\aleph_0}$, the solid sphere $\boldsymbol{S}$ may be decomposed into $\mathfrak{n}$ disjoint sets one of which $\underset{2}{=}\boldsymbol{S}$ and all the others $\underset{3}{=}\boldsymbol{S}$.*

This constitutes the above-mentioned generalization of the main result of [8] and [4].

Let us indicate the decomposition of S which exists by (S_1) and (R_2) and which proves (R_4):

$$\{A_\nu\}_{\nu \in N}, \quad \text{where} \quad \overline{\overline{N}} = \mathfrak{n}, \qquad A_\nu A_{\nu'} = 0 \quad \text{for} \quad \nu \neq \nu',$$

$$A_\nu = A_\nu^1 + A_\nu^2, \qquad A_\nu^1 A_\nu^2 = 0$$

and

$$A_\nu^1 \underset{\text{rot}}{\simeq} \sum_{\nu \in N} A_\nu^2, \qquad A_\nu^2 \underset{\text{rot}}{\simeq} \sum_{\nu \in N} A_\nu^1 \quad \text{for each} \quad \nu \in N.$$

The argumentation for the existence of this decomposition is similar to the argumentation given in [4], § 5.

The proof of generalization (S_2) is more sophisticated, but also by means of (S_1) and (R_2) almost the same as the proof of the original theorem given in [4], § 5, § 6; we omit it.

The second new theorem will be the following one:

(T_2) *Let $\mathfrak{m}$, $\mathfrak{n}$, M, N, $\{\varphi_\nu\}_{\nu \in N}$ and S satisfy the hypothesis of (R_1). Then for the system of congruences*

$$\sum_{\nu \in P_\mu} A_\nu \underset{\text{rot}}{\simeq} \sum_{\nu \in Q_\mu} A_\nu, \qquad \mu \in M.$$

where $\{P_\mu\}_{\mu \in M}$ and $\{Q_\mu\}_{\mu \in M}$ are classes of non-empty subsets of N, there exists a set $R \subset S$ which is of the power $2^{\aleph_0}$ and can be decomposed into $\mathfrak{n}$ disjoint sets $\{A_\nu\}_{\nu \in N}$ satisfying that system of congruences in such a way that the μth congruence can be effectuated by the rotation φ_μ.

Proof. We shall use (S_1), (R_2), the notation of the proof of (T_1) and the lemma.

Let N' be a set of power $\mathfrak{n}$ disjoint with N, and let φ be a one-one transformation of N into N'. We put

$$P'_\mu = \varphi(P_\mu), \qquad Q'_\mu = \varphi(Q_\mu) \qquad \text{for} \quad \mu \in M.$$

Now to prove (T_2) it is enough to verify that there exists a decomposition of the sphere S into $2\mathfrak{n}$ disjoint sets $\{A_\nu\}_{\nu \in N}$, $\{A_\nu\}_{\nu \in N'}$ satisfying the system of congruences σ_1:

$$P_\mu \sim Q_\mu, \qquad P'_\mu \sim Q'_\mu, \qquad \mu \in M.$$

In fact, one of the sets $\sum_{\nu \in N} A_\nu$, $\sum_{\nu \in N'} A_\nu$ must be of the power $2^{\aleph_0}$, since their sum is S; we can suppose that it is the first and we put

$$R = \sum_{\nu \in N} A_\nu.$$

We can also suppose by means of (R_2) that

$$\varphi_\mu\Big(\sum_{\nu \in P_\mu} A_\nu\Big) = \sum_{\nu \in Q_\mu} A_\nu \qquad \text{for each} \quad \mu \in M.$$

Then the set R fulfils the assertion of (T_2).

Therefore we must verify only that the system σ_1 satisfies the condition formulated in (R_2), as evidently it fulfils the hypothesis of that theorem, *i. e.* we must verify that none of the congruences belonging to $H(\sigma_1)$ is of the form

$$N + N' - K \sim K \tag{3}$$

where $K \subset N + N'$.

The system $C(\sigma_1)$ consists of the congruences σ_2:

$$P_\mu \sim Q_\mu, \qquad N + N' - P'_\mu \sim N + N' - Q'_\mu, \qquad \mu \in M,$$

and σ_3:

$$P'_\mu \sim Q'_\mu, \qquad N' + N - P_\mu \sim N' + N - Q_\mu, \qquad \mu \in M.$$

It is obvious that the class of nominating sets occuring in σ_2 is disjoint with the class of nominating sets occuring in σ_3 because by hypothesis none of the sets P_μ, Q_μ, P'_μ, Q'_μ is empty. Then using the lemma we find, as in the proof of (T_1), that

$$H(\sigma_1) = T(\sigma_2) \cup T(\sigma_3) \tag{4}$$

No two nominating sets occuring in σ_2 are complementary in $N+N'$ (because none of the sets P_μ, Q_μ, P'_μ, Q'_μ is empty). Therefore no two nominating sets occuring in $T(\sigma_2)$ are complementary in $N+N'$. The same can be verified for $T(\sigma_3)$. Thus by (4) no congruence of $H(\sigma_1)$ is of the form (3), q. e. d.

Let us note at least the following corollaries of (A_1) and (T_2), which seem to us particularly strange.

(A_2) *The sphere can be decomposed into a sequence of disjoint sets $A_1, A_2, \ldots$ satisfying any congruence of the form*

$$\sum_{n\in N_1} A_n \simeq \sum_{n\in N_2} A_n .$$

where N_1 and N_2 are arbitrary, not empty and not full sets of natural numbers.

(T_3) *There exists such a sequence $A_1, A_2, \ldots$ consisting of non-empty and disjoint subsets of the sphere that*

$$\sum_{n\in N_1} A_n \underset{\mathrm{rot}}{\simeq} \sum_{n\in N_2} A_n$$

for each non-empty sets N_1 and N_2 consisting of natural numbers [9]).

By means of (T_2) we can obtain also for the sphere theorems similar to the following theorem of W. Sierpiński concerning the plane:

If $0<\mathfrak{n}\leqslant 2^{\aleph_0}$ *there exists on the plane a set P which can be decomposed into* $\mathfrak{n}$ *disjoint sets congruent with P* [10]).

Note added in proof. Recently appeared a paper of T. Dekker and J. de Groot, (*Decompositions of a sphere*, Proc. Int. Math. Congress 2 (1954), p. 209) announcing a transfinite generalization of the theorems of R. M. Robinson and J. F. Adams similar to those given in this paper. Moreover T. Dekker and J. de Groot have elaborated a direct proof of the theorems (R_1) and (A_1) which will be published in the next volume of Fundamenta Mathematicae.

References

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[2] Jan Mycielski, *About sets with strange isometrical properties (I)*, this volume, p. 1-10.

[3] J. von Neumann, *Die Zerlegung eines Intervalles in abzählbar viele kongruente Teilmengen*, Fund. Math. 11 (1928), p. 230-238.

[9]) It is interesting to compare this theorem with theorem 2 of my paper [2].

[10]) See Sierpiński [9]. The proof of this theorem is an effective construction of P, such that $\overline{\overline{P}}=\aleph_0\mathfrak{n}$.

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