On the paradox of the sphere

by

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I have been interested in the problem whether it is possible to combine the results of three papers: W. Sierpiński [8], R. M. Robinson [4], and J. F. Adams [1]. It appears to be quite easy. Using a lemma from [8] we can considerably generalize the theorems of [4] and [1], and obtain by means of them a generalization of the main theorem of [8].

The aim of this paper is to formulate these generalizations and to prove three new theorems related to them (Theorems (T1), (T2) and (T3)).

All the theorems given in this paper, except the first — (S1) — are not effective (in their proofs the axiom of choice has been used).

(0) denotes congruence of sets. (1) denotes congruence of sets lying on the sphere, by a rotation around its centre.

All that will be said concerns the 3-dimensional Euclidean space. By a sphere and a solid sphere we understand the sets defined by $x^2+y^2+z^2=1$ and $x^2+y^2+z^2<1$ or $x^2+y^2+z^2<1$ respectively.

The lemma required from [8] (1) is the following:

(S1) There exist 2^n independent rotations of the sphere around its centre (i.e. rotations generating a free group).

The generalizations of the theorems of R. M. Robinson are the following:

(R1) Let $0<n<2^n$, $0<\mu<2^n$ and \( \overline{M} = m, \overline{N} = n; \) m independent rotations \( \{\varphi_m\}_{m \in M} \) of a sphere S around its centre, and \( m \) relations \( \{R_m\}_{m \in M} \), each having the set \( X \) as domain and range, are given.

Then the following propositions are equivalent:

1° We can decompose the sphere \( S \) into \( n \) disjoint sets \( \{A_n\}_{n \in N} \) in such a way that for this subdivision \( \varphi_n \) is compatible \(^7\) with \( R_m \) for each \( m \in M \).

2° Every product of any finite number of factors of the form \( R^m \) has a fixed point \(^4\).

(R2) Let \( m, n, M, N, \{\varphi_n\}_{n \in N} \) and \( S \) satisfy the hypothesis of (R1), and let \( \{P_m\}_{m \in M}; \{Q_n\}_{n \in N} \) be classes of non-empty subsets of \( N \) different from \( N^* \). Then the sphere \( S \) may be decomposed into \( n \) disjoint pieces \( \{A_n\}_{n \in N} \) satisfying the system of congruences

\[
\sum_{\varphi \in \varphi_\mu} A_\mu \cong \sum_{\varphi \in \varphi_\mu} A_\mu, \quad \mu \in M
\]

if and only if none of the congruences generated \(^4\) by these assert the congruence of two complementary portions of \( S \) (i.e. none are of the form \( \sum_{\varphi \in \varphi_\mu} A_\mu \cong \sum_{\varphi \in \varphi_\mu} A_\mu \)). Furthermore, if that decomposition is possible, it can be done in such a way that the \( \mu \)-th congruence can be effected by the rotation \( \varphi_\mu \) (i.e. \( \varphi_\mu \left( \sum_{\varphi \in \varphi_\mu} A_\mu \right) = \sum_{\varphi \in \varphi_\mu} A_\mu \)) for each \( \mu \in M \).

The generalization of the theorem of J. F. Adams (it concerns the congruence \( \cong_1 \), (R2) concerns \( \cong \)) is the following:

(A1) Let \( m, n, M, N, \{P_m\}_{m \in M}; \{Q_n\}_{n \in N} \) satisfy the hypothesis of (R2). Then the sphere \( S \) may be decomposed into \( n \) disjoint pieces \( \{A_n\}_{n \in N} \) satisfying the system of congruences

\[
\sum_{\varphi \in \varphi_\mu} A_\mu \cong \sum_{\varphi \in \varphi_\mu} A_\mu, \quad \mu \in M.
\]

The proofs of these generalizations are the same as the proofs of the original theorems in [4] and [1].

Now, applying (S1) and (R1), we easily obtain the following generalization of another result of [4] (§ 5):

1° The product \( RR' \) and the converse \( R^{-1} \) are defined by the equivalences:

\[
\begin{align*}
\{r, r' r'' \} & = \text{[there exists such a } r \in X \text{ that } r_1 r_2 r_3 \in N^*], \\
\{r, r' r'' \} & = \text{[there exists such a } r \in X \text{ that } r_1 r_2 r_3 \in N^*],
\end{align*}
\]

for each \( r_1, r_2, r_3 \in N \).

The relation \( R \) is said to have a fixed point if for a certain \( r \in N \) we have \( r R r \).

2° We do not suppose that \( \mu_1 \neq \mu_2 \) implies \( P_{\mu_1} \neq P_{\mu_2} \) or \( Q_{\mu_1} \neq Q_{\mu_2} \).

A congruence is said to be generated by a system of congruences \( \sigma \) if it belongs to \( \sigma \) or if it is obtainable by finite application to the congruences of \( \sigma \) of the rules of taking complements and of transitivity.

The rule of taking complements consists in making use of the implication

\[
\sum_{\varphi \in \varphi_\mu} A_\mu \cong \sum_{\varphi \in \varphi_\mu} A_\mu, \quad \mu \in M
\]

if \( \sum_{\varphi \in \varphi_\mu} A_\mu \cong \sum_{\varphi \in \varphi_\mu} A_\mu \), then \( \sum_{\varphi \in \varphi_\mu} A_\mu \cong \sum_{\varphi \in \varphi_\mu} A_\mu \).

The rule of transitivity consists in the application of the fact that if two congruences have one member in common, then the remaining members are congruent.

A corollary of this theorem will be given below — (A3).
(Rₙ) If 2 ≤ n ≤ 2ⁿ, then the sphere can be decomposed into n disjoint congruent sets.

It is interesting to compare this theorem with the following two theorems. One, which is much easier but also not effective, is an immediate generalization of the well-known construction of unmeasurable-L sets (Sierpiński [10], p. 64, or Ruziewicz [5]):

If 0 ≤ n ≤ 2ⁿ, then the straight line and the circumference of the circle can be decomposed into n disjoint congruent sets, and the semiclosed intervals (0, 1) can be decomposed into n disjoint sets of two each of which are \( \xi = \xi \).

The other proved by J. van Neumann [3] is much more difficult than the first and also not effective:

Each of the intervals (0, 1), (0, 1), (0, 1) can be decomposed into \( \kappa_n \) disjoint congruent sets.

The following generalization of (Rₙ) will be proved here:

\((T₁)\) There exists such a set \( E \) on a sphere that for each \( n \), for which \( 2 ≤ n ≤ 2^n \), that sphere can be decomposed into \( n \) disjoint sets congruent to \( E \) by a rotation around its center.

It is interesting to compare this theorem with the following theorem of W. Sierpiński ([6] or [7], p. 100, Proposition C₉):

The plane can be decomposed into \( 2^n \) disjoint sets congruent to a set \( E \) (which is a graphical representation of a function, i.e., it is defined by an equation of the form \( y = f(x) \)) and is the sum of \( n \) sets of the same kind as \( E \).

No proof of this theorem without the use of the continuum hypothesis \( 2^n = \kappa_n \) is known—our theorem \((T₁)\) is free of that assumption.

For proving \((T₁)\) we shall introduce certain symbols and a lemma.

For a given class of sets \( \{A_i\} \) of any set of congruences, of the form

\( \sum_{i=1}^{2^n} A_i \rightarrow \sum_{i=1}^{2^n} A_i \)

where \( P \), \( Q \), \( C \), \( N \), will be called a system of congruences.

The congruence \( \sum \rightarrow \sum \) will be denoted shortly by

\( P \rightarrow Q \)

and \( P \) and \( Q \) will be called its nominating sets.

If \( σ \) is a system of congruences we shall denote by \( G(σ) \) the system generated by \( σ \) by the rule of taking complements.

\( T(σ) \) the system generated by \( σ \) by the rule of transitivity.

\( H(σ) \) the system generated by \( σ \).

**Lemma.** For each system of congruences \( σ \) we have

\( H(σ) = T(σ) \).

Proof. Evidently \( T(C(σ)) \rightarrow H(σ) \); thus it is enough to verify, that the set \( T(C(σ)) \) is closed under the operations \( T \) and \( C \). It is closed under \( T \), because for any system of congruences \( σ \) we evidently have \( T(T(σ)) = T(σ) \). It is closed under the operation \( C \), because any complementary congruence to a congruence belonging to \( T(C(σ)) \) obviously belongs to \( T(C(σ)) \) also.

Proof of \((T₁)\). We shall use \((S₂)\) and \((Rₙ)\). For proving \((T₁)\) it is sufficient to verify that there exists a decomposition of the sphere into \( 2^n \) disjoint pieces \( \{A_i\}_{i=1}^{2^n} \) satisfying the system of congruences \( σ \):

\( (v) \rightarrow (v) \), \( v \in N \), \( (v) \rightarrow N \), \( 1 < i < 2^n \),

where \( v \) is a fixed element of \( N \) and for each ordinal number \( i \) for which \( 1 < i < 2^n \) we have

\( N \rightarrow C_i \) and \( N_i \rightarrow i \).

In fact it is sufficient to put \( E = A_0 \), and the assertion \( (Tₙ) \) is satisfied since we take when \( n = 2^n \) the decomposition \( \{A_i\}_{i=1}^{2^n} \) and when \( 2 ≤ n ≤ 2^n \) the decomposition into the sets \( \{A_i\}_{i=1}^{2^n} \) and the set \( \sum_{i=1}^{2^n} A_i \) where \( n \) is any ordinal number of the power \( n \).

Now let us observe that \( σ \) satisfies the hypothesis of \((Rₙ)\) because \( σ \) is \( 2^n \) and all the nominating sets occurring in \( σ \) are different from \( 0 \) and \( N \). Let \( \{σ_i\}_{i=1}^{2^n} \) be a set of independent rotations of the sphere (theorem \((S₂)\)).

Then to prove the existence of the decomposition \( \{A_i\}_{i=1}^{2^n} \) fulfilling \( σ \) it is sufficient to verify this that system of congruences satisfies the condition formulated in \((Rₙ)\), i.e. that none of the congruences belonging to \( H(σ) \) is of the form

\( N \rightarrow P \rightarrow P \)

where \( P \), \( C \), \( N \),

The system \( C(σ_i) \) consists of the congruences belonging to \( σ_i \) and of the congruences \( σ_i \):

\( (v) \rightarrow (v) \), \( v \in N \), \( (v) \rightarrow N \), \( 1 < i < 2^n \).

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Now it is obvious that the class of nominating sets occurring in $\sigma_1$ is disjoint with the class of nominating sets occurring in $\sigma_2$. This proves that

$$T(\sigma_1 \cup \sigma_2) = T(\sigma_1) \cup T(\sigma_2),$$

and then, by the lemma, that

$$H(\sigma_1) = T(\sigma_1) \cup T(\sigma_2).$$

(2)

It is obvious that no two nominating sets occurring in $\sigma_1$ are complementary in $N$. Therefore no two nominating sets occurring in $T(\sigma_1)$ are complementary in $N$. The same can be verified for $T(\sigma_2)$. Therefore, by (2), no congruence of $H(\sigma_1)$ is of the form (1), q.e.d.

Another result of the paper [4] can be generalized as follows:

(R4) If $1 < n < 2^n$, then the sphere $S$ may be decomposed into $n$ disjoint sets each of which is $S$.

Applying (R4) it is easy to prove that the solid sphere $S$ may be decomposed into $n$ disjoint sets one of which is $S$ and the others $S$. But following more closely the reasoning of [4] we can moreover find that

(S4) If $1 < n < 2^n$, the solid sphere $S$ may be decomposed into $n$ disjoint sets one of which is $S$ and all the others $S$.

This constitutes the above-mentioned generalization of the main result of [8] and [4].

Let us indicate the decomposition of $S$ which exists by (S4) and (R4) and which proves (R4):

$$(A_x)_{x \in N}, \quad \text{where} \quad N = n, \quad A_xA_{x'} = 0 \quad \text{for} \quad x \neq x', \quad A_x = A^+_x + A^-_x, \quad A^+_xA^-_x = 0$$

and

$$A^+_x = \sum_{x \in x} A^+_x, \quad A^-_x = \sum_{x \in x} A^-_x \quad \text{for each} \quad x \in N.$$

The argumentation for the existence of this decomposition is similar to the argumentation given in [4], § 5.

The proof of generalization (S4) is more sophisticated, but also by means of (S4) and (R4) almost the same as the proof of the original theorem given in [4], § 5, § 6; we omit it.

The second new theorem will be the following one:

(T4) Let $m, n, m, N_1, (\varphi_1)_{x \in N_1}$ and $S$ satisfy the hypothesis of (R4). Then for the system of congruences

$$\sum_{x \in x} A_x = \sum_{x \in x} A_x, \quad \mu \in M, \quad \text{where} \quad (P_x)_{x \in N_1} \text{ and } (Q_x)_{x \in N_1} \text{ are classes of non-empty subsets of } N,$$

there exists a set $\mathcal{B}$ such that is of the power $2^n$ and can be decomposed into $n$ disjoint sets $(A_x)_{x \in N_1}$ satisfying that system of congruences in such a way that the $\mu$th congruence can be effected by the rotation $\varphi_1$.

Proof. We shall use (S4), (R4), the notation of the proof of (T4) and the lemma.

Let $N'$ be a set of power $n$ disjoint with $N$, and let $\varphi$ be a one-one transformation of $N$ into $N'$. We put

$$P'_\mu = \varphi(P_\mu), \quad Q'_\mu = \varphi(Q_\mu) \quad \text{for} \quad \mu \in M.$$

Now to prove (T4) it is enough to verify that there exists a decomposition of the sphere $S$ into $\mu$ disjoint sets $(A_x)_{x \in N_1}$, $(A_x)_{x \in N'}$ satisfying the system of congruences $\varphi_1$:

$$P_\mu \sim Q_\mu, \quad P'_\mu \sim Q'_\mu, \quad \mu \in M.$$

In fact, one of the sets $\sum_{x \in x} A_x$, $\sum_{x \in x} A_x$ must be of the power $2^n$, since their sum is $S$. We can suppose that it is the first and we put

$$R = \sum_{x \in x} A_x.

We can also suppose by means of (R4) that

$$\varphi_1(\sum_{x \in x} A_x) = \sum_{x \in x} A_x, \quad \text{for each} \quad \mu \in M.$$

Then the set $R$ fulfills the assertion of (T4).

Therefore we must verify only that the system $\sigma_1$ satisfies the condition formulated in (R4), as evidently it fulfills the hypothesis of that theorem, i.e. we must verify that none of the congruences belonging to $H(\sigma_1)$ is of the form

$$(3) \quad N \sim N' \sim K \sim K,$$

where $\mathcal{K}_N + N'$.

The system $A$ consists of the congruences $\varphi_1$:

$$P_\mu \sim Q_\mu, \quad N \sim N' \sim P_\mu \sim Q_\mu, \quad \mu \in M,$$

and $\sigma_1$:

$$P'_\mu \sim Q'_\mu, \quad N' \sim N \sim P'_\mu \sim Q'_\mu, \quad \mu \in M.$$

It is obvious that the class of nominating sets occurring in $\sigma_1$ is disjoint with the class of nominating sets occurring in $\sigma_2$ because by hypothesis none of the sets $P_\mu, Q_\mu, P'_\mu, Q'_\mu$ is empty. Then using the lemma we find, as in the proof of (T4), that

$$(4) \quad H(\sigma_1) = T(\sigma_1) \cup T(\sigma_2).$$
No two nominating sets occurring in $\mathcal{Q}_2$ are complementary in $Y \times Y$ (because none of the sets $P$, $Q$, $P'$, $Q'$ is empty). Therefore no two nominating sets occurring in $T(\mathcal{Q}_2)$ are complementary in $Y \times Y$. The same can be verified for $T(\mathcal{Q}_3)$. Thus by (4) no congruence of $H(\mathcal{Q}_n)$ is of the form $(3)$, q. e. d.

Let us note at least the following corollaries of $(A_3)$ and $(T_2)$, which seem to us particularly strange.

$(A_3)$ The sphere can be decomposed into a sequence of disjoint sets $A_1, A_2, \ldots$ satisfying any congruence of the form

$$\sum_{x \in H} A_x = \sum_{x \in \mathcal{N}} A_x,$$

where $N_1$ and $N_2$ are arbitrary, not empty and not full sets of natural numbers.

$(T_2)$ There exists such a sequence $A_1, A_2, \ldots$ consisting of non-empty and disjoint subsets of the sphere that

$$\sum_{x \in H} A_x = \sum_{x \in \mathcal{N}} A_x,$$

for each non-empty sets $N_1$ and $N_2$ consisting of natural numbers$^*$. By means of $(T_2)$ we can obtain also for the sphere theorems similar to the following theorem of W. Sierpiński concerning the plane:

If $0 < n < 2^n$ there exists on the plane a set $P$ which can be decomposed into $n$ disjoint sets congruent with $P$.

Note added in proof. Recently appeared a paper by T. Dekker and J. de Groot, Decompositions of a sphere, Proc. Int. Math. Congress 2 (1964), p. 269 announcing a transfinte generalization of the theorems of B. M. Robinson and J. F. Adams similar to those given in this paper. Moreover T. Dekker and J. de Groot have elaborated a direct proof of the theorems $(R_2)$ and $(A_3)$ which will be published in the next volume of Fundamenta Mathematicae.

References


$^*$ It is interesting to compare this theorem with theorem 2 of my paper [3].
$^*$ See Sierpiński [9]. The proof of this theorem is an effective construction of $P$, such that $P = \mathcal{Q}_n$. 

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