Le point x_0 appartiendrait au segment (β,γ), et comme il a en outre la propriété W_E, il aurait, en vertu de (2), la propriété $W_{(\beta,\gamma)\cdot E}$. L'ensemble $A_{(\beta,\gamma)\cdot E}$ serait alors non vide, ce qui est contraire à (11) en raison de l'égalité $(\beta,\gamma)\cdot E=H$. Les deux cas réunis montrent que $A_E\subset N_\infty$, ce qui donne, avec (10), que $A_E\subset N_\infty\cdot \mathrm{Fr}\, E=B$. Il résulte de cette dernière relation et de (9) que A_E est non-dense, c. q. f. d.

Puisque les points de l'ensemble E, pour lesquels la densité inférieure de E est nulle, constituent un sous-ensemble de A_E, on a le corollaire suivant:

COROLLAIRE. *Si la fonction $f(x)$ admet partout une dérivée, alors chaque sous-ensemble de l'ensemble $E=\{f'(x)>a\}$, composé des points pour lesquels la densité inférieure de E est égale à zéro, est non-dense.*

Voici à présent un exemple d'ensemble E de classe M_3 ne satisfaisant pas à la condition nécessaire, mentionnée dans le corollaire, pour l'ensemble $\{f'(x)>a\}$. Soit E_1 un ensemble de I-e catégorie, du type F_σ, ayant avec chaque intervalle une partie commune de mesure positive mais moindre que la longueur de cet intervalle. Les ensembles avec de telles propriétés peuvent être obtenus en sommant une quantité dénombrable d'ensembles parfaits non-denses, de mesure positive. Le complémentaire de l'ensemble E_1 a une partie commune de mesure positive avec chaque intervalle. Désignons par E_2 l'ensemble des points appartenant à CE_1 et qui sont des points de densité de CE_1. Aux points de E_2 la densité de l'ensemble E_1 est nulle. En vertu du théorème de Lebesgue sur les points de densité d'un ensemble, on a $|CE_1-E_2|=0$, donc E_2 ainsi que CE_1 ont dans chaque intervalle une partie de mesure positive. L'ensemble E_2 est donc dense. Choisissons dans E_2 une partie dénombrable, dense, et désignons-la par E_3. L'ensemble $E=E_1+E_3$ est du type F_σ et a avec chaque intervalle une partie commune de mesure positive, puisque E_1 l'avait déjà. Pour chaque intervalle (a,b) on a $|E\cdot(a,b)|>0$. L'ensemble E est donc de classe M_3. Il renferme un ensemble dense de points dont la densité est nulle, notamment tout point de E_3 a cette propriété. Il résulte du théorème démontré (du corollaire) qu'il n'existe aucune fonction $f(x)$ admettant partout une dérivée, pour laquelle $E=\{f'(x)>a\}$.

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On the extending of models (II) *

Common extensions

by

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In this paper we are concerned with the problem of the existence of common extensions of models.

Let $X=Cn_1(X)$ be a self-consistent system in E_1 and $\{\mathfrak{M}_t\}_{t\in T}$, some family of models of X. Under which conditions does there exist a model $\mathfrak{M}$ of X such that each model $\mathfrak{M}_t$ $(t\in T)$ is a submodel of $\mathfrak{M}$?

The model $\mathfrak{M}$ will be called a *common $\mathfrak{A}$-extension* of the models $\mathfrak{M}_t$ $(t\in T)$, where $\mathfrak{A}=\mathfrak{A}(X)$ is the class of all models of the system X [1]).

It is known that common $\mathfrak{A}$-extensions of models do not always exist. If, for example, $\mathfrak{A}$ is the class of all Boole'an algebras, $\mathfrak{M}_1\in\mathfrak{A}$ is an arbitrary algebra with only one element, and $\mathfrak{M}_2\in\mathfrak{A}$ is an arbitrary algebra with two or more elements, then the common $\mathfrak{A}$-extension of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ does not exist.

The solution of the problem mentioned above is given in the paragraph 2.

§ 1. The notion of the O-completeness of a system

An open disjunction $\alpha_1\vee\alpha_2\vee\ldots\vee\alpha_n$ is said to have *alternating variables*, if each variable, which occurs in some α_i $(i=1,2,\ldots,n)$, does not occur in every α_j for $j=1,2,\ldots,n$ and $j\neq i$.

A self-consistent system $X=Cn_1(X)$ is called *O-complete*, if it follows from the belonging to X of a disjunction $\beta\vee\gamma$ with alternating variables that either β or γ belongs to X also.

* This paper presents the second part of the paper [3]. We adopt here the notions and notations used in [3].

[1]) Making use of the distinction between sets and classes, one can avoid any antinomial construction and formulate all our considerations in the framework of the axiomatic set theory [1].

(1.1) *A self-consistent system* $X=Cn_1(X)\subset E_1$ *is O-complete if and only if there is a model* $\mathfrak{M}$ *such that*

$$O_1\cap X=O_1\cap E_1(\mathfrak{M})\ ^{2)}.$$

This follows easily from the theorem 5 in [2] which states that, if $X\subset O_1$ and X is closed under the so called O-consequence based on the propositional calculus and the rule of substitution for individual variables, then X is O-complete (pseudo-complete in the terminology of [2]) if and only if $X=O_1\cap E_1(\mathfrak{M})$ for some model $\mathfrak{M}$. It will be sufficient to remark that, if X is a system, then the set $O_1\cap X$ is closed under the O-consequence.

A model $\mathfrak{M}$ of a system X such that

$$O_1\cap X=O_1\cap E_1(\mathfrak{M})$$

may be called *functionally O-free in* $\mathfrak{A}$, where $\mathfrak{A}=\mathfrak{A}(X)$ is the class of all models of the system X.

(1.2) *A self-consistent system* $X=Cn_1(X)\subset E_1$ *is O-complete if and only if there exists a functionally O-free model in* $\mathfrak{A}=\mathfrak{A}(X)$.

(1.2) follows from (1.1) by the fact that the equality

$$(*)\qquad O_1\cap X=O_1\cap E_1(\mathfrak{M})$$

for some model $\mathfrak{M}$ implies the existence of a *functionally O*-free model $\mathfrak{M}^*$ in $\mathfrak{A}=\mathfrak{A}(X)$.

To show it we apply the theorem on the existence of extensions of models stated in [3] (Theorem 1, paragraph 5). From this theorem it follows that, if the model $\mathfrak{M}$ satisfies the conditions

$$(1)\qquad A\subset E_1(\mathfrak{M})$$

and

$$(2)\qquad O_1\cap Cn_1(A\cup C)\subset E_1(\mathfrak{M}),$$

there exists a model $\mathfrak{M}^*$ such that

(3) $\mathfrak{M}$ *is a submodel of* $\mathfrak{M}^*$

and

$$(4)\qquad A\cup C\subset E_1(\mathfrak{M}^*).$$

If $A=O_1\cap X$ and $C=X$, then $A\cup C=X=Cn_1(X)$ and by the equality $(*)$ the conditions (1) and (2) hold good. Therefore, by (4), there is

[2]) $E_1(\mathfrak{M})$ denotes the set of all formulas $\alpha\in E_1$ which are valid in $\mathfrak{M}$.

a model $\mathfrak{M}^*$ for the whole system X, *i. e.* $X\subset E(\mathfrak{M}^*)$. To show that the model $\mathfrak{M}^*$ is functionally O-free in $\mathfrak{A}=\mathfrak{A}(X)$, it will be enough to prove

$$(**)\qquad O_1\cap E_1(\mathfrak{M}^*)\subset X.$$

By (3) every open formula valid in $\mathfrak{M}^*$ is valid in $\mathfrak{M}$ also (see 3.1 in [3], p 43), *i. e.* $O_1\cap E_1(\mathfrak{M}^*)\subset O_1\cap E_1(\mathfrak{M})$. Taking account of $(*)$, we infer $(**)$.

§ 2. The theorem on the common extensions of models

Let $X=Cn_1(X)\subset E_1$ be a system and let $\{\mathfrak{M}_t\}_{t\in T}$ be a family of models of X contained in the class $\mathfrak{A}=\mathfrak{A}(X)$ of all models of X. We shall show the following

THEOREM. *The necessary and sufficient condition for the existence of a common* $\mathfrak{A}$*-extension of the models* $\mathfrak{M}_t$ $(t\in T)$ *is:*

there is in X *no such disjunction* $\beta_1\vee\beta_2\vee\ldots\vee\beta_n\in X$ *with alternating variables and there are no such models* $\mathfrak{M}_{t_1},\mathfrak{M}_{t_2},\ldots,\mathfrak{M}_{t_n}$ $(t_1,t_2,\ldots,t_n\in T)$ *that* $\beta_i\notin E_1(\mathfrak{M}_{t_i})$ *for* $i=1,2,\ldots,n$.

Proof. The necessity of condition is obvious. For proving its sufficiency let $D(\mathfrak{M}_t)$ denote, as in [3], the description of the model $\mathfrak{M}_t$. If the set $X\cup\bigcup_{t\in T}D(\mathfrak{M}_t)$ is self-consistent in E_3, then, in view of Gödel's theorem, there is a common $\mathfrak{A}$-extension of the family of models $\{\mathfrak{M}_t\}_{t\in T}$. The inconsistency in E_3 of the set $X\cup\bigcup_{t\in T}D(\mathfrak{M}_t)$ implies, however, the existence of some sentences $\gamma_1,\gamma_2,\ldots,\gamma_k$ belonging to $\bigcup_{t\in T}D(\mathfrak{M}_t)$ such that $(\gamma_1\wedge\gamma_2\wedge\ldots\wedge\gamma_k)'\in Cn_3(X)$.

Therefore, if there is no $\mathfrak{A}$-extension for the family $\{\mathfrak{M}_t\}_{t\in T}$, then there are some models

$$\mathfrak{M}_{t_1},\mathfrak{M}_{t_2},\ldots,\mathfrak{M}_{t_n}\qquad t_1,t_2,\ldots,t_n\in T$$

and some sentences

$$a_1,a_2,\ldots,a_n,$$

belonging to E_3, such that every a_j $(j=1,2,\ldots,n)$ is a conjunction of all γ_l $(l=1,2,\ldots,k)$, which belong to $D(\mathfrak{M}_{t_j})$, and

$$(a_1\wedge a_2\wedge\ldots\wedge a_n)'\in Cn_3(X)$$

or, equivalently,

$$a_1'\vee a_2'\vee\ldots\vee a_n'\in Cn_3(X).$$

Let β_j $(j=1,2,\ldots,n)$ be formula obtained from the formula a_j' by putting variables in place of all the constants by means of which the model $\mathfrak{M}_{t_j}$ is described. Clearly,

$$\beta_j\notin E_1(\mathfrak{M}_{t_j})\qquad j=1,2,\ldots,n.$$

If, however, different variables are put in place of different constants, then the formula

$$\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n$$

is an open disjunction with alternating variables and, in view of lemma (1.8) given in [3],

$$\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n \in Cn_1(X) = X.$$

§ 3. Corollaries

The theorem just proved implies following corollaries:

Corollary 1. *In order that every subclass $\{\mathfrak{M}_t\}_{t\in T}$ of the class $\mathfrak{A}=\mathfrak{A}(X)$ has a common $\mathfrak{A}$-extension, it is necessary and sufficient that $X=Cn_1(X)$ is a O-complete system.*

The sufficiency follows from the proof given above. It suffices to remark that if $\beta_j \notin E_1(\mathfrak{M}_{t_j})$ then $\beta_j \notin X$ for $j=1,2,\ldots,n$. But $\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n \in X$. Therefore X is not O-complete.

On the other hand it is easily seen that, if X is not O-complete, *i. e.* $\beta,\gamma \in O_1$, $\beta \vee \gamma \in X$, $\beta \notin X$ and $\gamma \notin X$, then there are models $\mathfrak{M}_1,\mathfrak{M}_2 \in \mathfrak{A}=\mathfrak{A}(X)$ such that $\beta \notin E_1(\mathfrak{M}_1)$ and $\gamma \notin E_1(\mathfrak{M}_2)$. Evidently, the models $\mathfrak{M}_1,\mathfrak{M}_2$ have no common $\mathfrak{A}$-extension.

Corollary 2. *The necessary and sufficient condition for the existence of common $\mathfrak{A}$-extensions for each subclass of models of a given elementarily definable class $\mathfrak{A}=\mathfrak{A}(X)$ is that there exists in $\mathfrak{A}$ a functionally O-free model.*

§ 4. The existence of $\mathfrak{A}$-free products

If no signs of relations r_i occur in E_1, *i. e.*, if among the primitive signs occur only the signs of functions f_i, then every model

$$\mathfrak{M} = \langle A, F_1, F_2, \ldots \rangle$$

for E_1 is called an algebra.

R. Sikorski has examined in the paper [4] the so called $\mathfrak{A}$-free products for the subclasses of a class $\mathfrak{A}$ of algebras. One of his results is as follows (see theorem VIII, p. 215 in [4]):

If $\mathfrak{A}$ is an equationally definable class of algebras and the subclass $\{\mathfrak{M}_t\}_{t\in T}$ of $\mathfrak{A}$ has a common $\mathfrak{A}$-extension, then the $\mathfrak{A}$-free product of all $\mathfrak{M}_t (t \in T)$ exists.

Now we can state that

(4.1) *If $\mathfrak{A}$ is a class of algebras definable by the set $X \subset E_1$ of equations, then for each subclass $\{\mathfrak{M}_t\}_{t\in T} \subset \mathfrak{A}$ the $\mathfrak{A}$-free product exists if and only if there exists a functionally O-free model in $\mathfrak{A}$ (i. e. if the system $Cn_1(X)$ is O-complete).*

If we remark that each $\mathfrak{A}$-free product is also a common $\mathfrak{A}$-extension, we infer (4.1) from our corollaries and from the Sikorski's result, mentioned above.

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