A remark on order-types

by

K. Padmavally (Madras)

1. 1. Sierpiński [5] has shown that

(a) *every order-type C of power $\aleph_l$ is imbeddable in the "complete power" $2((\omega_l))$* (Hausdorff [1], p. 460), *(i. e. the set of all transfinite sequences of zeros and ones, of length ω_l), ω_l denoting* [1]) *the initial ordinal corresponding to the cardinal number $\aleph_l$.*

J. Novák [2] has refined this result as follows:

(b) *If $\aleph_m$ denotes the minimum cardinal number of a dense subset of the continuous order-type C, then C is imbeddable in the complete power $2((\omega_m))$.*

In the paper [4] (which was submitted for publication before I came across J. Nováks result) I had proved (Theorem 3.6) that

(c) *If ω_n is an initial ordinal* [2]) *such that neither ω_n nor its inverse ω_n^* is imbeddable in the order-type C, then C is imbeddable in the complete power $2((\omega_n))$.*

The purpose of this note is to point out some relations between these results.

2. It may be noted that (b) follows immediately from (a) by observing that if a dense subset D of a continuous order-type C is imbeddable in an order-type A which has no gap sections and which possesses both extremities, (*i. e.* which is $=\bar{A}$, its completion — cf. Hausdorff [1], p. 448), then C is imbeddable in A. This can be proved by considering a subset B of A similar to D, and the set L of limit points of B in A. Since $A=\bar{A}$, corresponding to every gap or missing extremity g of B, there exists a point g' of L. Hence $B\cup L$ and therefore A contains a subset similar to C. Hence, in view of $2((\omega_m))$ being its own completion ([6], Theorem 1), (b) follows from (a).

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[1]) This notation is adhered to in what follows, *i. e.* ω_k denotes the initial ordinal corresponding to $\aleph_k$.

[2]) The result is proved for any indecomposable ordinal λ such that neither λ nor λ^* is imbeddable in C. Therefore, since an initial ordinal is indecomposable, (c) follows.

2. In what follows the initial ordinal ω_m corresponding to $\aleph_m$, the minimum cardinal number of a dense subset of C, is denoted by $\mu(C)$, the smallest initial ordinal ω_k such that C is imbeddable in $2((\omega_k))$ by $\nu(C)$, and the smallest initial ordinal ω_n such that neither ω_n nor its inverse ω_n^* is imbeddable in C by $\lambda(C)$. Then if $\nu(C)=\omega_k$ $(=\omega_k(C))$, we have the following

2.1. THEOREM. *A necessary and sufficient condition for $\mu(C)$ to be $\leqslant\nu(C)$ for the continuous order-type C is that $\sum_{\varkappa<k}2^{\aleph_\varkappa}=\aleph_k$.*

The proof depends on the fact that the minimum cardinal of a dense subset of $2((\omega_k))$ is $\sum_{\varkappa<k}2^{\aleph_\varkappa}$ which is $\geqslant\aleph_k$. This follows by arguments similar to those used by Hausdorff ([1], p. 490-493) in proving that the set $S(\sigma)$ has power $(\aleph_\alpha+\aleph_\beta)_\sigma=\sum_{d<\aleph_\sigma}(\aleph_\alpha+\aleph_\beta)^d$. This former result, namely that the minimum cardinal of a dense subset of $2((\omega_k))$ is $(\sum_{\varkappa<k}2^{\aleph_\varkappa}\geqslant\aleph_k)$, evidently implies that the condition is necessary, since $\nu[2((\omega_k))]\leqslant\omega_k$. The converse depends on the fact that if C is any continuous order-type imbeddable in $2((\omega_i))$ and D any dense subset of $2((\omega_i))$, then C has a dense subset D' which is imbeddable in D. This can be proved as follows. Corresponding to every element $d\in(D-C)$ there is a maximal segment I_d of $2((\omega_i))$ containing d and free from points of the subset C of $2((\omega_i))$. Since the order-type C is continuous, one of the extremities of I_d, say l_d, belongs to C. The union of $D\cap C$ and the set of elements $\{l_d\}_{d\in D}$ is dense in C, for, since C is continuous, between any two elements p,q of C, there is a non-empty interval of $2((\omega_i))$ and hence an element d of the dense subset D. Hence either $d\in C$, or there exists a segment of the form I_d and therefore an element l_d between p,q. Also the union of $D\cap C$ and of the set $\{l_d\}_{d\in D}$ is similar to a subset of D, since each l_d can be made to correspond to a d belonging to the corresponding $I_d\cap D$. Hence $(C\cap D)\cup\sum_{d\in D}l_d$ is the required dense subset of C imbeddable in D. Hence if $\sum_{\varkappa<i}2^{\aleph_\varkappa}=\aleph_i$ so that $2((\omega_i))$ has a dense subset of power $\aleph_i$, a necessary condition for C to be imbeddable in $2((\omega_i))$, *i. e.* for ω_i to be $\geqslant\nu(C)$, is that C have a dense subset of power $\leqslant\aleph_i$, or equivalently, that $\mu(C)\leqslant\omega_i$. Hence if $\sum_{\varkappa<k}2^{\aleph_\varkappa}=\aleph_k$, $\mu(C)\leqslant\nu(C)=\omega_k$. This proves that the condition is sufficient.

2.2. THEOREM. *A necessary and sufficient condition for $\mu(C)$ to be $\leqslant\lambda(C)$ for all continuous order-types C is that $2^{\aleph_\varkappa}\leqslant\aleph_n$ for every $\varkappa<n$, for all n.*

It may be noted that this condition is equivalent to the one stated above, extended to all ω_k (*i. e.* to all continuous order-types C).

Since, on account of result (c), $\lambda(C)\geqslant\nu(C)$, it follows immediately from the foregoing result that the condition is sufficient. To prove that the condition is necessary, suppose that for some n there exists $\varkappa<n$ such that $2^{\aleph_\varkappa}>\aleph_n$. Consider any ordinal ξ such that $\omega_\varkappa<\xi<\omega_n$. Then for the order-type $2((\xi))=C$, $\lambda(C)\leqslant\omega_n$; for, by Theorem of Hausdorff ([1], p. 472), stating that every series of $2((\xi))$ is cofinal with an "argumental" or a "basic" series, it follows that every well-ordered (inversely well-ordered) subset of $2((\xi))$ is imbeddable in $\xi(\xi^*)$. Hence neither ω_n nor ω_n^* is imbeddable in $2((\xi))$. Thus for the continuous order-type C_1 formed from C by coalescing consecutive points ([3], p. 253), we have $\lambda(C_1)\leqslant\omega_n$. But $\mu(C)$, and hence $\mu(C_1)$ is $>\omega_n$, for $2((\omega_\varkappa))$ is similar to a set of disjoint proper segments of $2((\xi))$ (namely the "complete segments" of order $\omega_\varkappa$, as defined in [4]) and hence to a set of disjoint proper segments of C_1 also. Hence every dense subset of C_1 contains a set similar to $2((\omega_\varkappa))$ (as can be seen if we take one element from each of the set of disjoint proper segments of order-type $2((\omega_\varkappa)))$, *i. e.* every dense subset of C_1 has power $2^{\aleph_\varkappa}>\aleph_n$, and thus $\mu(C_1)>\omega_n$. Hence for the continuous order-type C_1, we have $\mu(C_1)>\lambda(C_1)$, which proves that the condition is necessary.

2.3. Remark. For $n=0$ the condition referred to above holds, *i. e.* $\sum_{i<\omega}2^i=\aleph_0$. This is in accordance with J. Novák's observation that $\mu(C)=\nu(C)$ for the case where C is the real number space which is formed from $2((\omega))$ by coalescing consecutive elements. Here $\mu(C)$ is also the smallest ordinal ξ such that C is imbeddable in $2((\xi))$.

3. An example of an order-type P for which $\lambda(P)=\nu(P)$ is the smallest ordinal ξ such that P is imbeddable in $2((\xi))$ can be constructed as follows. The construction depends on the following lemma:

3.1. *If C is an order-type for which $\lambda(C)$ exists and $>\omega$, and in which every well-ordered or inversely well-ordered series has a limit, (i. e. has no gap sections and possesses both extremities), then C contains no isolated subset similar to itself.*

Proof. Let E be an isolated subset of C similar to C. Let f be a one-one order-preserving map of C onto E. Consider any element p of C for which $f(p)>p$. Then a well-ordered series $\{p_\beta\}$ of C can be defined by induction as follows. Let $p_0=p$. Suppose that the well-ordered set $\{p_\beta\}$ has been defined for all $\beta<\gamma$, so that $f(p_\beta)>p_\beta$ and $p_{\beta+1}'=f(p_\beta)$. Then if γ is a non-limiting ordinal $=\delta+1$, $p_\gamma=p_{\delta+1}=f(p_\delta)>p_\delta$. Since f is one-one order-preserving, it follows that $f(p_\gamma)>f(p_\delta)=p_\gamma$. Let $p_{\gamma+1}=f(p_\gamma)$. If γ is a limiting ordinal, let p_γ be the limit in C (which exists by hypothesis) of the well-ordered series $\{p_\beta\}_{\beta<\gamma}$. Then, since E is an isolated subset of C, $p_\gamma\notin E$. Therefore, since f is order-preserving, $f(p_\gamma)>p_\gamma$, and $p_{\gamma+1}$ can be defined as $f(p_\gamma)$. This definition of p_γ can evidently be applied for all ordinals γ, *i. e.* C contains any desired well-ordered series, contrary to the hypothesis that $\lambda(C)$ exists. A similar argument holds for the case where we start with a $p\in C$ for which $f(p)<p$. Also either of these cases must hold for at least one $p\in C$ since $f(p)=p$ for all $p\in C$ would imply that C is finite, contrary to the hypothesis that $\lambda(C)>\omega$. This proves the lemma.

3.2. Let ω_{n+1} be any non-limiting regular initial ordinal. Consider the "general product" P with argument $2((\omega_n))$ whose base consists of the elements $0,1$, as defined by Hausdorff ([1], p. 461). P is the set of all complexes $X=\{x_y\}$ where, for each X, the index y varies in the argument $2((\omega_n))$. The coordinate $x_y=0$ for all values of y except a well-ordered subset $F=F(X)$ of values of y, and $x_y=1$ for $y\in F(X)$. The set $F(X)$ varies unrestrictedly in $2((\omega_n))$ as X varies in P. The set P is ordered according to first differences, *i. e.* $X_1=\{x_y^1\}<X_2=\{x_y^2\}$ if there exists $z\in 2((\omega_n))$ such that $x_y^1=x_y^2$ for $y<z$, $x_z^1<x_z^2$ (*i. e.* $x_z^1=0$, $x_z^2=1$). The existence of z is ensured by the hypothesis that $F(X)$ is well-ordered for every $X\in P$. Hence P is a completely ordered set. It can be shown, as below that P is not imbeddable in any complete power $2((\xi))$, $\xi<\omega_{n+1}$, while $\lambda(P)=\omega_{n+1}$.

To prove the former result, *i. e.* that P is not imbeddable in any $2((\xi))$, $\xi<\omega_{n+1}$, by the foregoing lemma, it is sufficient to prove that every $2((\xi))$, $\xi<\omega_{n+1}$ is imbeddable in P. For, assuming this latter result, suppose that P is imbeddable in $2((\eta))$, $\eta<\omega_{n+1}$. Then $2((\eta+3))$, being imbeddable in P, would be imbeddable in $2((\eta))$. Therefore, since $2((\eta))$ is similar to an isolated subset of $2((\eta+3))$ (or equivalently to the set of all elements of $2((\eta+3))$ given by $x_{\eta+2}=0$), $2((\eta))$ is similar to an isolated subset of itself. But this contradicts the lemma, since $2((\eta))$ has no gaps and possesses both extremities ([3], Theorem 1).

To show that $2((\xi)),\xi<\omega_{n+1}$ is imbeddable in P, consider the subset S of the argument $2((\omega_n))$ (regarded as the set of all $X=\{x_\beta\}$, $\beta<\omega_n$, where each x_β is 0 or 1) consisting of all $X=\{x_\beta\}$ such that $x_\beta=0$ for even β, *i. e.* β of the form 2γ, while $x_\beta=1$ for all odd β (which are not even). This set is evidently an e_n-set (Hausdorff [1], p. 487) since every element has character $\omega_n\omega_n^*$ and every gap is either cofinal with ω_n or coinitial with ω_n^*. Hence by Theorem 18 of Hausdorff [1] it follows that every set of power $\leqslant\aleph_n$ and in particular every ordinal $\xi<\omega_{n+1}$, is imbeddable in S, and hence in $2((\omega_n))$. Consider the set of all $X\in P$ such that $F(X)$ is a given subset of $2((\omega_n))$ similar to ξ. This set has order-type $2((\xi))$. It follows that every $2((\xi))$, $\xi<\omega_{n+1}$, is imbeddable in P. Hence, by the foregoing, P is not imbeddable in any $2((\xi))$, $\xi<\omega_{n+1}$.

The result that $\lambda(P)=\omega_{n+1}$ follows from the fact that for the argument $2((\omega_n))$ we obtain $\lambda[2((\omega_n))]=\omega_{n+1}$, by using an extension of Theorem 14 of Hausdorff [1]. This theorem states that every well-ordered (inversely well-oredred) series of a product whose argument is well--ordered is cofinal (coinitial) with an argumental or a basic series (or the inverse of an argumental series). Applying this result to the product $2((\omega_n))$, we immediately find that every well-ordered (inversely well--ordered) series of $2((\omega_n))$ is cofinal (coinitial) with a regular initial ordinal (or its inverse) $\leqslant \omega_n$, *i. e.* that neither ω_{n+1} nor ω_{n+1}^* is imbeddable in $2((\omega_n))$, so that $\lambda[2((\omega_n))] = \omega_{n+1}$. Theorem 14 of Hausdorff can be extended as follows to the case where the argument is not necessarily well-ordered:

3.3. Consider the general product P whose argument $E = \bar{E}$ (which is not necessarily well-ordered) is such that $\lambda(E) \leqslant \omega_{n+1}$, and the base $C(y)$ corresponding to each $y \in E$ is also such that $\lambda[C(y)] \leqslant \omega_{n+1}$. P consists of the set of all $X = \{x_y\}$ where, for each $X \in P$, y varies in E and x_y varies in an ordered set $C(y)$ depending only on the element $y \in E$, and where $x_y = t_y$ is a fixed element of $C(y)$ (independent of X) for all but a well-ordered set $F = F(X)$ of values of y, x_y being unrestricted in $C(y)$ for $y \in F(X)$. The set $F(X)$ of E, as in the previous definition, is an unrestrictedly varying well-ordered subset of E as X varies in P. The set P is ordered according to first differing coordinates. Then if $\{X_\beta\}_{\beta<\alpha} (X_\beta = \{x_y^\beta\})$ is a well-ordered (inversely well-ordered) subset of P, $\alpha < \omega_{n+1}$.

Following Hausdorff, let us denote by (β, γ) for each pair β, γ of ordinals $< \alpha$ the place of first differing coordinates of X_β, X_γ. Then, as has been pointed out by Hausdorff, for a given β, for $\gamma > \beta$ (β, γ) is a non--increasing function of γ, and since $E = \bar{E}$, has a minimum $\pi(\beta)$ in E. As has also been shown by Hausdorff, $\pi(\beta)$ is a non-decreasing function of β. Therefore

Case 1. There exists $\beta < \alpha$ for which the minimum $\pi(\beta)$ of $\{(\beta, \gamma)\}_{\beta<\gamma<\alpha}$ is not attained. $\{(\beta, \gamma)\}_{\beta<\gamma<\alpha}$ is an inversely well-ordered subset S of E having no lowest element. Hence there exists a series $\{\gamma_\xi\}_{\xi<\eta}$ cofinal with α such that each (β, γ_ξ) is $< (\beta, \gamma_{\xi+1})$, so that η is similar to S^* and cofinal with α.

Case 2. Case 1 does not hold, *i. e.* $\pi(\beta)$ is attained for all $\beta < \alpha$. The same argument as that given by Hausdorff ([1], p. 472-473) is used to show that α is cofinal with some η such that either η or η^* is imbeddable in either E or some $C(y), y \in E$. Hence $\alpha < \omega_{n+1}$.

The product P considered in 3.2 above, whose argument is $2((\omega_n))$ and each base $C(y)$ is the ordinal two, evidently satisfies these conditions, since $\lambda[2((\omega_n))] = \omega_{n+1}$ and $\lambda(2) = 3$. Hence $\lambda(P)$ is ω_{n+1}, while P is not imbeddable in any $2((\xi))$, $\xi < \omega_{n+1}$. It follows

3.4. Theorem. *The ordinal λ given by Theorem 3.6 of* [4] *cannot be lowered.*

The above generalization of Theorem 14 of Hausdorff also leads to a generalization of a result of Sierpiński's [6], namely

3.5. Theorem. *For every non-limiting regular initial ordinal ω_{n+1}, there exists a transfinite ascending sequence of power equal to the cardinal number next higher than $2^{\aleph_n}$, (i. e. of type α where $|\alpha| =$ the cardinal next higher than $2^{\aleph_n}$) of order-types $\{C_\beta\}_{\beta<\alpha}$, each of which has power $\leqslant 2^{\aleph_{n+1}}$, and where each $\lambda(C_\beta) \leqslant \omega_{n+1}$.*

(The order-types are ordered according to the definition given by Sierpiński [7], *i. e.* $C_\beta < C_\gamma$ if C_β is similar to a subset of C_γ, but not conversely.)

Let C_0 be defined as θ, the order-type of the real number segment $0 \leqslant x \leqslant 1$. Suppose that the members $\{C_\beta\}$ of the required sequence have been defined for all $\beta < \gamma$, where the power of γ is $\leqslant 2^{\aleph_n}$, and each $C_\beta = \bar{C}_\beta$, its completion. Then if γ is a non-limiting ordinal $= \delta + 1$, let C_γ be defined as $3 \cdot C_\delta$. C_δ is evidently a subset of C_γ, while C_γ is not imbeddable in C_δ: otherwise this would give an isolated subset of C_δ similar to itself (C_δ being similar to an isolated subset of C_γ and hence of C_δ), contradicting our lemma since $C_\delta = \bar{C}_\delta$. Hence $C_\delta < C_\gamma$. If γ is a limiting ordinal, then, the power of γ being $\leqslant 2^{\aleph_n}$, and the power of $2((\omega_n))$ being $2^{\aleph_n}$, a subset S of $2((\omega_n))$ of distinct points $\{y_\beta\}_{\beta<\gamma}$ can be found. Consider the product P, whose argument is $2((\omega_n))$ and each base $C(y)$ is the corresponding C_β for $y = y_\beta \in S$, while for $y \notin S$, $C(y)$ is the ordinal two. Then, since each $\lambda(C_\beta)$ as well as $\lambda[2((\omega_n))] \leqslant \omega_{n+1}$, by the foregoing extension of Theorem 14 of Hausdorff, therefore $\lambda(P) = \lambda(\bar{P}) \leqslant \omega_{n+1}$. Hence, by Theorem 3.6 of [4], $\bar{P}$ is imbeddable in $2((\omega_{n+1}))$, and thus $\bar{P}$ has power $\leqslant 2^{\aleph_{n+1}}$. Also P (and hence $\bar{P}$) can easily be seen to be greater than each C_β, $\beta < \gamma$, since each subset T_β of P consisting of all $X = \{x_y\}$ where x_y is fixed, (*i. e.* independent of the particular $X \in T_\beta$), for all $y \neq y_\beta$, while x_y for $y = y_\beta$ varies unrestrictedly in C_β, is evidently similar to the corresponding C_β. Hence each C_β, $\beta < \gamma$, is imbeddable in $\bar{P}$. It follows that $\bar{P}$ is not imbeddable in any C_β, $\beta < \gamma$, since otherwise, *i. e.* if $\bar{P}$ were imbeddable in C_β, every C_ξ, $\xi < \gamma$, would be imbeddable in C_β, contrary to the hypothesis that $\{C_\xi\}_{\xi<\gamma}$ is an increasing sequence. Hence $\bar{P} > C_\beta$ for all $\beta < \gamma$. Let $\bar{P}$ be defined as C_γ. This defines C_γ for all γ such that $|\gamma| \leqslant 2^{\aleph_n}$, *i. e.* for $|\gamma| <$ the smallest cardinal higher than $2^{\aleph_n}$.

3.6. Using the same method as above, we can show that at least one of the following two results holds for each ω_{n+1}.

If every member C_β of the sequence defined above has power $\leqslant 2^{\aleph_n}$, it is clear that

(d) *There exists an ascending transfinite sequence of power equal to the smallest cardinal* $\aleph$ *higher than* $2^{\aleph_n}$, *of order-types of power* $\leqslant 2^{\aleph_n}$ *each.*

If (d) is not true, so that some C_β has power $\geqslant \aleph$, *i. e.* $> 2^{\aleph_n}$, then if we use this C_β instead of $2((\omega_n))$ in the foregoing, it is clear that members $\{C_\xi\}$ of the ascending sequence can be defined for all γ having power $\leqslant \aleph$, *i. e.*

(e) *There exists an ascending transfinite sequence of power equal to the smallest cardinal number* $\aleph'$ *higher than* $\aleph$, *(the smallest cardinal higher than* $2^{\aleph_n}$*) of order-types each having power* $\leqslant \aleph$.

These can be looked upon as the analogues of the result proved by Sierpiński [6], namely that there exists a descending sequence of power $2^{\aleph_0}$ of order-types having power $2^{\aleph_0}$ each.

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RAMANUJAN INSTITUTE OF MATHEMATICS, MADRAS

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Sur certains ensembles indénombrables singuliers de nombres irrationnels

par

J. Popruženko (Łódź)

Dans ce Mémoire, je m'occupe de certains phénomènes mathématiques qui ont été découverts et étudiés par d'autres auteurs à l'aide de l'hypothèse du continu. On trouve ces questions dans les sections 2 et 3 du présent travail.

Les raisonnements qui vont suivre et dont le but principal est d'éliminer les prémisses hypothétiques des démonstrations de l'existence de ces phénomènes reposent sur la notion d'ordre de croissance des suites infinies d'entiers positifs, donc — au fond — sur les propriétés des relations d'ordre partiel. Nous commençons par établir un théorème général à ce sujet.

1. Ensembles fondamentaux. Soit $\mathcal{M}$ un espace abstrait indénombrable quelconque. Soit ϱ une relation dont le champ d'existence est un certain ensemble de couples formés d'éléments de $\mathcal{M}$. Il est supposé que ϱ établit dans $\mathcal{M}$ un ordre partiel. Une telle relation est donc par hypothèse

1° *non-réflexive*

et

2° *transitive.*

Ces deux conditions supposées remplies, nous pouvons préciser certaines notions, dont nous nous servirons constamment dans la suite.

Un ensemble N sera dit *borné selon la relation* ϱ lorsqu'il existe un élément q de $\mathcal{M}$ tel que $p\varrho q$ quel que soit $p \in N$.

Une suite transfinie $\{p_\xi\}_{\xi<\delta}$ formée de certains éléments de $\mathcal{M}$ sera dite *bien ordonnée selon la relation* ϱ *en type d'ordre* δ lorsque l'inégalité $\xi < \xi' < \delta$ entraîne toujours $p_\xi \varrho p_{\xi'}$.

Elle sera dite *saturée selon la relation* ϱ si, en outre, l'ensemble de ses termes n'est pas borné selon la même relation.

Nous supposons que la relation ϱ est assujettie à une condition supplémentaire, essentielle pour nos raisonnements.