Category theorems

by

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§ 1. Introduction. A *topology* for a set X is a set T of subsets of X such that: $X \in T$, the empty set $\emptyset$ is a member of T, the union of any set of members of T is a member of T, and the intersection of any finite set of members of T is a member of T.

Let T and T^* be two topologies for a set Y. T is *categorically related* to T^* if and only if for every topological space X and function f on X into Y, f T^*-continuous (*i. e.* continuous with respect to the topology T^*) at each point of X implies that f is T-continuous at each point of a residual subset of X.

In this paper a condition (α) is given for an ordered pair (T,T^*) of topologies for a set Y, and a proof is given for a rather general theorem which states that if (T,T^*) satisfies condition (α) then T is categorically related to T^*.

A number of examples of pairs (T,T^*) which satisfy condition (α) are given, and the basic theorem is interpreted for these examples. This procedure results in new proofs for several well known category theorems, and also a few new category theorems.

§ 2. Condition (α) and the basic theorem. Let T and T^* be topologies for a set Y. We say that the ordered pair (T,T^*) *satisfies condition* (α) if and only if there exist sequences $U_1,U_2,\ldots$ and $K_1,K_2,\ldots$ of subsets of Y such that:

(i) $U_n \subset K_n$ for each n;

(ii) if $p \in U \in T$, then there exists n such that $p \in U_n \subset K_n \subset U$;

(iii) if $q \in U_n$, then there exists $V \in T^*$ such that $q \in V$ and $V - K_n \in T^*$.

There are a number of natural examples of pairs (T,T^*) of topologies which satisfy condition (α). In all of the examples which we give in this paper, the sets U_n are members of T and therefore by (ii) form a base for the topology T. It is frequently true in the examples we consider that the set K_n is the T-closure of the set U_n. In some (but not all) examples,

the sets K_n are T^*-closed, and for these cases it is possible to verify that (iii) is satisfied by taking V to be Y.

We observe that if the pair (T,T^*) satisfies condition (α) and T' is a topology for Y such that $T' \supset T^*$, then the pair (T,T') also satisfies condition (α).

BASIC THEOREM. *If T and T^* are topologies for Y and (T,T^*) satisfies condition (α), then T is categorically related to T^*.*

Proof. Let f be a T^*-continuous function on a topological space into Y. Let $U_1,U_2,\ldots$ and $K_1,K_2,\ldots$ be sequences of subsets of Y which satisfy (i), (ii) and (iii) for the pair (T,T^*). For each positive integer n we define D_n to be the set of all points x in the domain of f such that in each neighborhood of x there exist points y and z for which $f(y) \in U_n$ and $f(z) \notin K_n$.

We first observe that if f is not T-continuous at a point p in the domain of f, then $p \in D_n$ for some value of n. This fact follows immediately from (ii) and the definition of continuity at p.

We next observe that D_n is closed for each n. This fact follows easily from the definition of D_n and the definition of closure.

We now prove by contradiction that each of the sets D_n is nowhere dense. Let us assume for a given n that D_n is not nowhere dense. Then, since D_n is closed, D_n must contain a non empty open set G_0. It follows from the definition of D_n that there exists $y \in G_0$ such that $f(y) \in U_n$. We use (iii) to obtain $V \in T^*$ such that $f(y) \in V$ and $V - K_n \in T^*$. Since f is T^*-continuous, there exists an open set G_1 such that $y \in G_1 \subset G_0$ and $f(G_1) \subset V$. It follows from the definition of D_n that there exists $z \in G_1$ such that $f(z) \notin K_n$. We see that $f(z) \in V - K_n$. Since $V - K_n \in T^*$ and f is T^*-continuous, there exists an open set G_2 such that $z \in G_2 \subset G_1$ and $f(G_2) \subset V - K_n$. The definition of D_n states that there exists $t \in G_2$ such that $f(t) \in U_n$, but this is impossible since $f(G_2) \subset V - K_n$ and $U_n \subset K_n$. We have thus obtained a contradiction, and it follows that each set D_n is nowhere dense.

We have proved that $\bigcup_{n=1}^{\infty} D_n$ is a first category set which contains the set of all points at which f is not T-continuous. Our theorem now follows from the fact that a subset of a first category set is a first category set.

§ 3. Real valued semi-continuous functions and the Baire theorem. Let R be the set of all real numbers, and let T be the usual topology for R. We define T^* to be the topology for R which is obtained by taking as base elements sets of the form $\{x \mid r < x\}$. It is obvious that a real valued function on a topological space is T^*-continuous if and only if is lower semi-continuous in the usual sense.

The open intervals with rational end points are denumerable and may be written in a sequence $U_1, U_2, \ldots$ We let $K_1, K_2, \ldots$ be the corresponding sequence of closed intervals. It is obvious that the sets U_n and K_n satisfy (i) and (ii) of condition (α). If $U_n=(a,b)$, we can verify (iii) by letting $V=\{x \mid a<x\}$. Thus the pair (T,T^*) satisfies condition (α). As a special case of our Basic Theorem we now obtain the following well known theorem:

Theorem 1. *If a real valued function on a topological space is lower semi-continuous, then it is continuous except at points of a first category set.*

The Baire category theorem for the limit of a sequence of continuous functions follows easily from the above theorem. Since this fact does not seem to be generally known, it seems worth while to include the proof here.

Theorem 2. *If $f_1, f_2, \ldots$ is a sequence of continuous functions on a topological space into a metric space which converges pointwise to a function g, then g is continuous except at points of a first category set.*

Proof. For each positive integer n and each x in the domain of the functions, we define $\varphi_n(x)$ to be the diameter of the set $\bigcup_{k=n}^{\infty} f_k(x)$. Each of the functions φ_n is easily seen to be a real valued lower semi-continuous function, and hence is continuous at points of a residual set C_n. We define $C=\bigcap_{n=1}^{\infty} C_n$. Suppose $p \in C$ and $\varepsilon>0$. Choose m so that $\varphi_m(p)<\varepsilon/3$. Because of the continuity of f_m and the continuity of φ_m at p, there exists a neighborhood V of p such that if $x \in V$ then $\varphi_m(x)<\varepsilon/3$ and the distance from $f_m(x)$ to $f_m(p)$ is less than $\varepsilon/3$. It follows easily that the distance from $g(x)$ to $g(p)$ is less than ε, and this proves that g is continuous at p. Since C is a residual set, this proves our theorem.

§ 4. The compact-open and point-open topologies. In this paragraph we let A be a locally compact, separable metric space, we let B be a separable metric space, and we let Y be the set of all continuous functions on A into B. If n is a positive integer, $C_1, \ldots, C_n$ are subsets of A and $G_1, \ldots, G_n$ are subsets of B, then we define $W(C_1, \ldots, C_n;\ G_1, \ldots, G_n) = \{f \mid f \in Y \text{ and } f(C_j) \subset G_j \text{ for } 1 \leqslant j \leqslant n\}$. The *compact-open topology* for Y is the topology obtained by taking as base elements all sets of the form $W(C_1, \ldots, C_n, G_1, \ldots, G_n)$ where each set C_j is compact and each set G_j is open. The *point-open topology* for Y is the topology obtained by taking as base elements sets of the form $W(C_1, \ldots, C_n; G_1, \ldots, G_n)$ where each set C_j contain only one point and each set G_j is open.

We let T be the compact-open topology for Y and we let T^* be the point-open topology for Y. There exists a countable base L for A such

that each member of L has a compact closure. We define M to be the collection of all sets which are the closures of members of L. We let N be a countable base for B.

There are only a countable number of sets of type $W(C_1, \ldots, C_n; G_1, \ldots, G_n)$ where n is a positive integer, each $C_j \in M$ and each $G_j \in N$. Therefore, these sets may be arranged in a sequence $U_1, U_2, \ldots$ If $U_n = W(C_1, \ldots, C_m; G_1, \ldots, G_m)$, then we define $K_n = W(C_1, \ldots, C_m; \overline{G}_1, \ldots, \overline{G}_m)$. The sets K_n are closed with respect to the point-open topology, and it is easy to see that (i), (ii) and (iii) are each satisfied. Thus the pair (T, T^*) satisfies condition (α).

If we apply the Basic Theorem to the above example, we obtain the following form of a well known theorem concerning functions of two variables.

Theorem 3. *Let P be a topological space, let A be a locally compact separable metric space, and let B be a separable metric space. If f is a function on $P \times A$ into B which is continuous in each variable separately, then there exists a residual subset Q of P such that f is continuous at each point of $Q \times A$.*

Proof. We define Y, T and T^* as above. We define a function F on P into Y by letting $F(u)(v)=f(u,v)$. Since $f(u,v)$ is continuous in v for each fixed u, it is clear that $F(u) \in Y$ for each $u \in P$. Since $f(u,v)$ is continuous in u for each fixed v, it follows that F is continuous with respect to the T^* topology for Y. Thus there exists a residual subset Q of P such that F is continuous with respect to the T topology for Y at each point of Q. It follows easily that f is continuous at each point of $Q \times A$.

We next prove a theorem about transformation groups.

Theorem 4. *Let S be a second category topological group and let A be a locally compact separable metric space. We assume that for each $s \in S$, F_s is a homeomorphism of A onto A; and we assume that $F_s F_t = F_{st}$ for all s and t in S. If $F_s(x)$ is continuous in s for each fixed x, then $F_s(x)$ is simultaneously continuous in s and x (i. e. F is continuous on $S \times A$ into A).*

Proof. Let H be the group of all homeomorphisms of A onto A. We define g on S into H by letting $g(s)=F_s$ for each $s \in S$. Then g is an algebraic homomorphism on S into H. Moreover, since $F_s(x)$ is continuous in s for each fixed x, g is continuous with respect to the point-open topology for H. It follows that there exists a residual set of points in S at which g is continuous with respect to the compact-open topology for H. Since S is second category this residual set is non empty.

The product operation in H is continuous with respect to the compact open topology, and hence continuity of the homomorphism g at one point of S implies continuity of g at every point of S. The continuity of g with respect to the compact-open topology, however, implies the continuity of F as a function on $S\times A$ into A.

§ 5. The overlap topology. Let C be the set of all continuous real valued functions of a real variable, let C^1 be the subset of C which consists of all functions which have continuous derivatives, and let D be the function on C^1 into C for which $D(f)$ is the derivative of f for each $f\in C^1$. The function D is not continuous relative to any of the standard non trivial topologies for C (*i. e.* uniform, compact-open, point-open). On the other hand, it would be useful to have a topology T^* for C with respect to which D would be continuous; and which would be such that (T,T^*) would satisfy condition (α), where T it the compact-open topology for C. We will now define such a topology.

Let A, B and Y be the same as in § 4. If U is an open set in A and V is an open set in B, we define

$$Z(U,V)=\{f\mid f\in Y \text{ and } f(U)\cap V\neq\emptyset\}.$$

The collection of all sets of the form $Z(U,V)$ forms a subbase for a topology T^* which we call the *overlap topology* for Y. The overlap topology is obviously at least as coarse as the point-open topology for Y(*i. e.* it is contained in the point-open topology).

Let us now prove that if T is the compact-open topology for Y, then (T,T^*) satisfies condition (α). We define sets U_n,V_n as in § 4. Since (i) and (ii) are obviously satisfied, it is sufficient to prove that each set K_n is closed in the T^* topology.

Let $K_n=W(C_1,\dots,C_m;\bar{G}_1,\dots,\bar{G}_m)$ and assume $f\notin K_n$. Then $f(C_j)$ non $\subset\bar{G}_j$ for some j. Since C_j is the closure of an open set, there exists a point p interior to C_j such that $f(p)\notin\bar{G}_j$. There is a neighborhood V of $f(p)$ such that $V\cap\bar{G}_j=\emptyset$, and there is a neighborhood U of p such that $U\subset C_j$. It is easy to see that $Z(U,V)$ is a T^* neighborhood of f, and that if $g\in Z(U,V)$ then $g\notin K_n$. This proves that K_n is closed in the T^* topology.

Lemma 1. *If both C and C^1 are topologized by the overlap topology, then D is continuous.*

Proof. Let $g\in C^1$ and let $Z(U,V)$ be a neighborhood of $D(g)$. (In order to prove D continuous at g, we may without loss of generality consider only subbasic neighborhoods of $D(g)$.) We let $g'=D(g)$. There exists $p\in U$ such that $g'(p)\in V$. We choose $\varepsilon>0$ so that the 2ε-neighborhood of $g'(p)$ is contained in V. Since g' is continuous, there exists an open interval $U^*\subset U$ such that $p\in U^*$ and $|g'(x)-g'(p)|<\varepsilon$ for all $x\in U^*$. We choose distinct points p_1 and p_2 in U^*. There exist open sets U_1,U_2,V_1,V_2 such that

$$p_1\in U_1\subset U^*,\quad p_2\in U_2\subset U^*,\quad U_1\cap U_2=\emptyset,\quad g(p_1)\in V_1,\quad g(p_2)\in V_2$$

and such that if

$$x_1\in U_1,\quad x_2\in U_2,\quad y_1\in V_1,\quad y_2\in V_2$$

then

$$\left|\frac{y_1-y_2}{x_1-x_2}-\frac{g(p_1)-g(p_2)}{p_1-p_2}\right|<\varepsilon.$$

Now let $f\in Z(U_1,V_1)\cap Z(U_2,V_2)$. There exist $x_1\in U_1$ and $x_2\in U_2$ such that $f(x_1)\in V_1$ and $f(x_2)\in V_2$. Therefore,

$$\left|\frac{f(x_1)-f(x_2)}{x_1-x_2}-\frac{g(p_1)-g(p_2)}{p_1-p_2}\right|<\varepsilon.$$

By the mean-value theorem, there exist q and r in U^* such that

$$f(x_1)-f(x_2)=(x_1-x_2)f'(q)\qquad\text{and}\qquad g(p_1)-g(p_2)=(p_1-p_2)g'(r).$$

Thus $|f'(q)-g'(r)|<\varepsilon$, and since $|g'(r)-g'(p)|<\varepsilon$ it follows that $|f'(q)-g'(p)|<2\varepsilon$. Therefore $f'(q)\in V$ and $f'(U)\cap V\neq\emptyset$. This proves that $D(f)=f'\in Z(U,V)$.

Theorem 5. *Let P be a topological space, let R be the real number system, and let f be a function on $P\times R$ into R which is continuous in each variable separately. If $g(u,v)=\partial^n f(u,v)/\delta v^n$ exists and is continuous in v for each $u\in P$, then there exists a residual subset Q of P such that g is continuous at each point of $Q\times R$.*

Proof. Let C and C^1 be as before. We define F on P into C^1 by letting $F(u)(v)=f(u,v)$. If D is the derivative function on C^1 into C, then D^nF is a function on P into C.

The continuity of $f(u,v)$ in u for fixed v implies that F is continuous with respect to the point-open topology for C. Hence F is continuous with respect to the overlap topology for C. Using Lemma 1 and induction, we see that D^nF is continuous on P into C with respect to the overlap topology. Therefore, there exists a residual set Q in P, at each point of which D^nF is continuous with respect to the compact-open topology. It follows that g is continuous at each point of $Q\times R$.

A slight modification of Lemma 1 and the above theorem yields the following lemma.

Lemma 2. *Let V_1 be the set $x_1^2+\dots+x_n^2<1$ in Euclidean n-space and let S be any topological space which is second category at each point. Let $F(g;x_1,\dots,x_n)$ be a continuous function defined on $S\times V_1$ and let $F_j(g;x_1,\dots,x_n)=\partial F(g;x_1,\dots,x_n)/\partial x_j$ exist and be continuous in x for any fixed g in S. There exists a set Q which is dense in S such that F_j is simultaneously continuous in all variables at each point of $Q\times V_1$.*

Proof. Let C be the set of all continuous real-valued functions on V_1, and let C_j be the subset of C consisting of those functions whose partial derivative with respect to the jth coordinate exists and is continuous. A subset U of V_1 will be called a *segment parallel to the* jth *axis* if and only if there exist $p \in V_1$ and $\varepsilon > 0$ such that

$$U = \{x \mid x_k = p_k \text{ for } k \neq j \text{ and } p_j - \varepsilon < x_j < p_j + \varepsilon\}.$$

We let T^* be the topology generated by subbase elements of the form $\{f \mid f(U) \cap V \neq \emptyset\}$ where $U \subset V_1$, U is a segment parallel to the jth axis, and V is an open subset of the real number system. We let T be the compact open topology for C. Now let D_j be the function on C_j into C for which $D_j(f)$ is the partial derivative of f with respect to the jth coordinate. If we topologize C and C_j by the topology T^*, then D_j is continuous on C_j into C (proof as in Lemma 1). We define Φ on S into C by letting $\Phi(g)(x) = F(g,x)$ for all $g \in S$ and $x \in V_1$. Φ is continuous with respect to the T^* topology for C_j, and thus $D_j\Phi$ is continuous with respect to the T^* topology. It is easily seen that (T, T^*) satisfies condition (α). Thus there exists a residual (and hence dense, since S is second category at each point) subset Q of S such that $D_j\Phi$ is continuous with respect to T at each point of Q. It follows that F_j is continuous at each point of $Q \times V_1$.

The above lemma is a generalization of a lemma due to Montgomery (see [8], p. 384, Lemma 1′). If we use our Lemma 2 instead of Montgomery's Lemma 1′, but otherwise follow the proof given in [8], we obtain the following generalization of Montgomery's Theorem 1.

THEOREM 6. *If G is a second category group which is a transformation group of a manifold of class C^1 and if each transformation of G is of class C^1, then the derivatives of the functions which define the transformations locally are continuous in all variables simultaneously.*

The above theorem is proved in [8] for locally compact groups G, and Montgomery observes that his proof also goes over for certain other groups G (including complete metric groups). It is not clear, however, that Montgomery's proof will hold for every second category group.

§ 6. The space of functions which are infinitely differentiable and have compact supports. In this section we let Y be the set of all functions f such that: f is a real valued function of a real variable, f is infinitely differentiable, and f has a compact support (*i. e.* f vanishes outside of some bounded interval). We let D be the derivative function on Y into Y.

We say that a sequence $f_1, f_2, \ldots$ in Y S-converges to f if and only if there exists a bounded interval J such that each f_n vanishes outside of J, and not only does the sequence $f_1, f_2, \ldots$ converge uniformly to f, but for each positive integer k the sequence $D^kf_1, D^kf_2, \ldots$ converges uniformly to D^kf.

THEOREM 7. *Let X be a topological space which satisfies the first axiom of countability, and let F be a function on the set X into the set Y. If F is continuous relative to pointwise convergence in Y, then there exists a residual subset Q of X at each point of which F is continuous relative to S-convergence in Y.*

Proof. Since X satisfies the first axiom of countability, sequential continuity and continuity are equivalent for F. Since D is continuous relative to the overlap topology for Y and F is continuous relative to the point-open (and hence the overlap) topology for Y, each of the functions $DF, D^2F, D^3F, \ldots$ is continuous relative to the overlap topology for Y. It follows that there exists a residual subset Q_1 of X such that at each point of Q_1 the functions $F, DF, D^2F, \ldots$ are all continuous relative to the compact-open topology for Y.

For each positive integer n, we define A_n to be the set of all points $x \in X$ such that in each neighborhood of x there exist points y and z such that $F(y)$ vanishes outside of the interval $[-n,n]$ and $F(z)$ does not vanish identically outside of the interval $[-n,n]$. It is obvious that each set A_n is closed. Since F is continuous with respect to the point-open topology for Y, it is easy to see that each set A_n does not contain a non empty open set. Thus $Q_2 = X - \bigcup_{n=1}^{\infty} A_n$ is a residual set. We define $Q = Q_1 \cap Q_2$.

Now let $p \in Q$ and let $x_1, x_2, \ldots$ be a sequence in X which converges to p. There exists a positive integer n such that $F(p)$ vanishes outside of $[-n,n]$. Since $p \notin A_n$, there exists a neighborhood N of P such that $F(x)$ vanishes outside of $[-n,n]$ for each $x \in N$. Therefore, there exists a bounded interval J such that each of the functions $F(x_1), F(x_2), \ldots$ vanishes outside of J. Since each of the functions $F, DF, D^2F, \ldots$ is continuous at p relative to the compact-open topology for Y, it follows that $F(x_1), F(x_2), \ldots$ converges uniformly to $F(p)$ and $D^nF(x_1), D^nF(x_2), \ldots$ converges uniformly to $D^nF(p)$ for each positive integer n. This proves that F is continuous at p relative to S-convergence in Y.

COROLLARY 1. *If X is a second category metric group and F is an algebraic homomorphism of the group X into the group Y (addition being the group operation in Y), then F is continuous relative to S-convergence in Y if and only if F is continuous relative to pointwise convergence in Y.*

§ 7. Some miscellaneous theorems. The theorems in this section are concerned with situations in which the closures of base elements for one topology for a space are also closed relative to a second coarser topology.

THEOREM 8. *Let f be a function on a topological space X into a separable metric space Y, and let B be a base for the topology for Y. If $f^{-1}(\overline{V})$ is closed for each $V \in B$, then f is continuous at each point of a residual subset of X.*

Proof. Let T be the given topology for Y. The set of all sets of the form $Y-\overline{V}$, $V \in B$, forms a subbase for a topology T^* for Y. It is obvious that (T, T^*) satisfy condition (α), and that f is T^*-continuous. It follows that f must be continuous (*i. e.* T-continuous) at points of a residual subset of X.

The following theorem is related to Theorem 2 of [9].

THEOREM 9. *If f is an algebraic homomorphism of a second category group X into a locally compact separable metric group Y, then f is continuous if and only if f^{-1} takes compact sets into closed sets.*

Proof. Suppose f^{-1} takes compact sets into closed sets. Choose a base B for Y such that each member of B has a compact closure. By Theorem 8, f is continuous at points of a residual subset of X. Since X is second category and f is an algebraic homomorphism, f is continuous.

THEOREM 10. *Let f be a function on a topological space X into a separable Banach space Y. If f is continuous with respect to the weak topology for Y, then there exists a residual subset of X at each of whose points f is continuous with respect to the norm topology.*

Proof. Since it is well known that (norm) closed spheres in Y are also closed in the weak topology, this theorem follows at once from Theorem 8.

The above theorem was proved by Alexiewicz and Orlicz (see [1], p. 108, Corollaire (1.2)).

For the special case of the L^p spaces, $p \geqslant 1$, we are able to obtain a slightly stronger theorem than Theorem 10 if we replace the weak topology by the topology of convergence in measure.

We restrict ourselves to real functions on the unit interval J, and we let m be the Lebesgue measure function. If $f \in L^p$ and $\varepsilon > 0$, we define $N_\varepsilon(f)$ to be the set of all functions $g \in L^p$ for which $m\{x \mid x \in J$ and $|f(x)-g(x)| \geqslant \varepsilon\} < \varepsilon$. The sets $N_\varepsilon(f)$ form a base for a topology T^* for L^p. We call T^* the *convergence in measure topology* for L^p.

If E is a measurable subset of J and $f \in L^p$, we define $L(f, E) = (\int_E |f|^p)^{1/p}$. Thus, $L(f, J)$ is the usual norm of f in L^p. We let T be the topology induced on L^p by this norm.

THEOREM 11. *If F is a function on a topological space X into L^p and F is continuous relative to the convergence in measure topology T^*, then there exists a residual subset of X at each point of which F is continuous relative to the norm topology T.*

Proof. We wish to show that (T, T^*) satisfies condition (α). Since the metric topology induced on L^p by the norm is separable, it is obvious that it is sufficient to prove that norm-closed spheres in L^p are also closed sets relative to the T^* topology.

Let S be a norm-closed sphere in L^p having center f and radius r. Suppose $g \notin S$. We define $\varepsilon = L(g-f, J) - r$. Because of absolute continuity, there exists $\delta_1 > 0$ such that if E is a measurable subset of J and $m(E) < \delta_1$, then $L(g-f, E) < \varepsilon/3$. There exists $\delta_2 > 0$ such that if K is a measurable subset of J and $|g(x)-h(x)| < \delta_2$ for all $x \in K$ then $L(g-h, K) < \varepsilon/3$. We let δ be the least of the numbers $\delta_1, \delta_2, \varepsilon/3$.

Now suppose that $h \in N_\delta(g)$. There exists a measurable subset E of J such that $m(E) < \delta$ and $|h(x)-g(x)| < \delta$ for all $x \in J-E$. Since $p \geqslant 1$, it is easy to see that $L(g-f, J) \leqslant L(g-f, E) + L(g-f, J-E)$. It follows that

$$L(h-f, J) \geqslant L(h-f, J-E) \geqslant L(g-f, J-E) - L(h-g, J-E)$$
$$\geqslant L(g-f, J) - L(g-f, E) - L(h-g, J-E) \geqslant r + \varepsilon - \varepsilon/3 - \varepsilon/3 > r.$$

Thus $h \notin S$, and hence $S \cap N_\delta(g) = \emptyset$. This proves S is closed in the T^* topology.

§ 8. Semi-continuous set-valued functions. Let S be a topological space, and let $C(S)$ be the set of all closed subsets of S. If D is a directed set and n is a net (see [6]) on D into $C(S)$, then we define:

$$\overline{\lim_{d \in D}}\, n(d) = \{x \mid \text{for each neighborhood } U \text{ of } x,\ n(d) \cap U \neq \emptyset \text{ for all } d \text{ in a cofinal subset of } D\};$$

$$\underline{\lim_{d \in D}}\, n(d) = \{x \mid \text{for each neighborhood } U \text{ of } X,\ n(d) \cap U \neq \emptyset \text{ for all } d \text{ in a residual subset of } D\}.$$

If X is a topological space and f is a function on X into $C(S)$, then,

(a) f is *upper semi-continuous* at $p \in X$ if and only if $\overline{\lim_{d \in D}} f(n(d)) \subset f(p)$ for each net n that converges to p.

(b) f is *lower semi-continuous* at $p \in X$ if and only if $\underline{\lim_{d \in D}} f(n(d)) \supset f(p)$ for each net n that converges to p.

The concepts of semi-continuity defined above have been investigated by Choquet in [2]. These types of semi-continuity differ in general from those types studied by the author in [3] and [4].

For the remainder of this section we assume that S is a separable, metrizable space, and that B is a countable base for the topology of S. If a is a subset of S, then we define

$$N_1(a) = \{b \mid b \in C(S) \text{ and } b \cap a = \emptyset\}$$

and

$$N_2(a)=\{b \mid b \in C(S) \text{ and } b \cap a \neq \emptyset\}.$$

The set of all sets $N_1(\overline{V}), V \in B$, forms a subbase for a topology T_1 for $C(S)$. The set of all sets $N_2(V), V \in B$, forms a subbase for a topology T_2 for $C(S)$. The set of all sets $N_1(V), V \in B$, forms a subbase for a topology T_3 for $C(S)$. The topology T_2 is independent of the particular base B used for its definition, but in general both T_1 and T_3 depend on B.

The proof of the following lemma is quite easy and is omitted.

Lemma 3. *If f is a function on a topological space X into $C(S)$ and $p \in X$, then*

(a) *f is lower semi-continuous at p if and only if f is T_2-continuous at p;*

(b) *if f is T_1-continuous at p, then f is upper semi-continuous at p;*

(c) *if $\overline{V}$ is compact for each $V \in B$, then upper semi-continuity of f at p implies T_1-continuity of f at p;*

(d) *if f is T_3-continuous at p, then f is upper semi-continuous at p.*

Lemma 4. *(T_2, T_1) satisfies condition (α).*

Proof. The sets of the form $\bigcap_{j=1}^{m} N_2(V_j)$, $V_j \in B$, form a countable base for T_2 and can be arranged in a sequence $U_1, U_2, \ldots$ If $U_n = \bigcap_{j=1}^{m} N_2(V_j)$, we define $K_n = \bigcap_{j=1}^{m} N_2(\overline{V}_j)$. The sets K_n are closed in the T_1 topology, and it is easy to verify that (i), (ii) and (iii) are satisfied.

Lemma 5. *If the boundary of V is compact for each $V \in B$, then (T_1, T_2) satisfies condition (α).*

Proof. The sets of the form $\bigcap_{j=1}^{m} N_1(\overline{V}_j)$, $V_j \in B$, form a countable base for T_1 and can be arranged in a sequence $U_1, U_2, \ldots$ If $U_n = \bigcap_{j=1}^{m} N_1(\overline{V}_j)$, then we define $K_n = \bigcap_{j=1}^{m} N_1(V_j)$. The sets K_n are closed in the T_2 topology, and it is easy to verify that (i), (ii) and (iii) are satisfied. The hypothesis that the members of B have compact boundaries is used in verifying (ii).

Lemma 6. *Both (T_2, T_3) and (T_3, T_2) satisfy condition (α).*

Proof. This lemma follows from the fact that each of the topologies T_2 and T_3 have countable bases whose members are closed with respect to the other topology. Thus in each case we can choose a base $U_1, U_2, \ldots$ and let $K_n = U_n$.

Not all of the results obtained in the preceding lemmas are needed for the proof of the next theorem. (In particular, Lemma 5 is not used at all.) These extra results have been included for the sake of completeness, and because they may be of some slight interest in themselves.

Theorem 12. *Let f be a function on a topological space X into $C(S)$. If f is lower semi-continuous on X, then there exists a residual set in X at each point of which f is upper semi-continuous. If S is locally compact and f is upper semi-continuous on X, then there exists a residual set in X at each point of which f is lower semi-continuous.*

Proof. Suppose that f is lower semi-continuous on X. Then by Lemma 3, f is T_2-continuous on X. Since (T_2, T_3) satisfies condition (α) by Lemma 6, it follows from our Basic Theorem that there exists a residual subset Q of X such that f is T_3-continuous at each point of Q. Lemma 3 then implies that f is upper semi-continuous at each point of Q.

Now let us assume that S is locally compact (as well as separable and metrizable) and that f is upper semi-continuous on X. Choose the base B so that $\overline{V}$ is compact for each $V \in B$. Then, by Lemma 3, f is T_1-continuous on X. Lemma 4 together with the Basic Theorem implies that there exists a residual subset Q of X such that f is T_2-continuous at each point of Q. By Lemma 3, f is lower semi-continuous at each point of Q.

§ 9. The space of measurable sets.

Let E be a Euclidean space, and let $\mu(A)$ be the Lebesgue measure of A for each measurable subset A of E. We let Y be the set of all subsets of E which are measurable and have finite measure. If we define $d(A,B)=\mu(A-B)+\mu(B-A)$ for all A and B in Y, then d is a metric for Y (provided we identify sets which differ only by a set of measure zero) and the resulting metric space is separable.

We now define two topologies for Y which are related to d in a fairly obvious way. If $A \in Y$ and $\varepsilon > 0$, we define:

$$N_1(A,\varepsilon)=\{B \mid B \in Y \text{ and } \mu(A-B)<\varepsilon\},$$

and

$$N_2(A,\varepsilon)=\{B \mid B \in Y \text{ and } \mu(B-A)<\varepsilon\}.$$

The sets $N_1(A,\varepsilon)$ form a base for a topology T_1 for Y, and the sets $N_2(A,\varepsilon)$ form a base for a topology T_2 for Y. Both T_1 and T_2 are perfectly separable. It is easy to prove that $\{B \mid B \in Y \text{ and } \mu(A-B) \leqslant \varepsilon\}$ is closed in the T_2 topology, and that $\{B \mid B \in Y \text{ and } \mu(B-A) \leqslant \varepsilon\}$ is closed in the T_1 topology. Using these facts, it is easy to show that both (T_1, T_2) and (T_2, T_1) satisfy condition (α). We thus obtain the following theorem:

Theorem 13. *Let f be a function on a topological space X into Y. If f is continuous with respect to either of the topologies T_1 or T_2, then there exists a residual subset Q of X such that f is continuous with respect to the metric d at each point of Q.*

Example. Let C be the set of all continuous real valued functions on the unit interval. We assume that C is metrized by the usual uniform metric. We define a function f on C by letting

$$f(u)=\{t\,|\,u(t)>0\} \quad \text{for each} \quad u\in C.$$

It is easy to verify that f is continuous with respect to the T_1 topology, but that f is not continuous with respect to the T_2 topology. It follows from Theorem 13 that f is continuous with respect to d on a residual subset of C. It is easy to see that f is d-continuous at u if and only if $\mu\{t\,|\,u(t)=0\}=0$.

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Über eine Dimensionstheorie in topologischen Verbänden

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Die mengentheoretische Topologie hat weitgehende Verallgemeinerungen zu einer Topologie der Vereine und Verbände erfahren (Nöbeling [3] und Sikorski [4]). Insbesondere ist auch die Menger-Urysohn'sche Dimensionstheorie von R. Sikorski auf gewisse topologische Verbände (C-Algebren (Sikorski [4] und [5])) übertragen worden. Allerdings verwendet Sikorski [5] einen *globalen* Dimensionsbegriff, mit dem sich nicht alle Sätze der Punktmengen-Dimensionstheorie formulieren lassen. Die vorliegende Arbeit hat nun zum Ziel, unter Zugrundelegung einer allgemeineren, *lokalen* Dimensionsdefinition eine Theorie zu entwickeln, in der noch fehlende Sätze bewiesen werden können. Es wird sich dabei zeigen, daß dieser lokale Dimensionsbegriff für C-Algebren, hier S-Verbände genannt, mit dem Sikorski'schen zusammenfällt.

Die vorliegende Arbeit stützt sich auf G. Nöbeling [3] und verwendet die dortigen Begriffe, Bezeichnungen und Sätze.

Ein für alle Mal sei ein klassisch-topologischer Boole-Verband $\mathfrak{B}$ vorgelegt.

Definition. $\mathfrak{B}$ heiße speziell ein Sikorski-Verband oder kurz ein *S-Verband*, wenn $\mathfrak{B}$ ein σ-Verband, regulär und T_1-topologisch ist und außerdem eine abzählbare Basis [1]) besitzt.

Es kann leicht gezeigt werden, daß jeder S-Verband eine abzählbare reguläre Basis besitzt. Ein S-Verband ist demnach dasselbe wie eine Sikorski'sche C-Algebra.

In Anlehnung an die Stoffeinteilung Mengers [2] behandelt die vorliegende Arbeit nach der Formulierung der Dimensionsdefinition (§ 1) die Dimension einzelner Somen (§ 2), Summen- und Zerspaltungssätze (§ 3), die lokale dimensionelle Struktur von S-Verbänden (§ 4), Überdeckungssätze (§ 5), die Beziehungen globaler Trennungs- und Zusammen-

[1]) „Basis" ist immer als offene Basis gemeint.