Category theorems

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§ 1. Introduction. A topology for a set $X$ is a set $T$ of subsets of $X$ such that: $X \in T$, the empty set $\emptyset$ is a member of $T$, the union of any set of members of $T$ is a member of $T$, and the intersection of any finite set of members of $T$ is a member of $T$.

Let $T$ and $T'$ be two topologies for a set $X$. $T$ is categorically related to $T'$ if and only if for every topological space $X$ and function $f$ on $X$ into $X$, $f$ is $T$-continuous (i.e. continuous with respect to the topology $T'$) at each point of $X$ implies that $f$ is $T$-continuous at each point of a residual subset of $X$.

In this paper a condition (a) is given for an ordered pair $(T, T')$ of topologies for a set $Y$, and a proof is given for a rather general theorem which states that if $(T, T')$ satisfies condition (a) then $T$ is categorically related to $T'$.

A number of examples of pairs $(T, T')$ which satisfy condition (a) are given, and the basic theorem is interpreted for these examples. This procedure results in new proofs for several well known category theorems, and also a few new category theorems.

§ 2. Condition (a) and the basic theorem. Let $T$ and $T'$ be topologies for a set $Y$. We say that the ordered pair $(T, T')$ satisfies condition (a) if and only if there exist sequences $U_n, U_{n+1}, \ldots$ and $K_1, K_2, \ldots$ of subsets of $Y$ such that:

(i) $U_n \subseteq K_n$ for each $n$;
(ii) if $p \in U \subseteq T$, then there exists $n$ such that $p \in U_n \subseteq K_n \subseteq U$;
(iii) if $q \in U_n$, then there exists $V \subseteq T'$ such that $q \in V$ and $V \subseteq K_n \subseteq T'$.

There are a number of natural examples of pairs $(T, T')$ of topologies which satisfy condition (a). In all of the examples which we give in this paper, the sets $U_n$ are members of $T$ and therefore by (ii) form a base for the topology $T$. It is frequently true in the examples we consider that the set $K_n$ is the $T$-closure of the set $U_n$. In some (but not all) examples,

... sets $K_n$ are $T$-closed, and for these cases it is possible to verify that (iii) is satisfied by taking $V$ to be $Y$.

We observe that if the pair $(T, T')$ satisfies condition (a) and $T'$ is a topology for $X$ such that $T \subseteq T'$, then the pair $(T, T')$ also satisfies condition (a).

Basic Theorem. If $T$ and $T'$ are topologies for $Y$ and $(T, T')$ satisfies condition (a), then $T$ is categorically related to $T'$.

Proof. Let $f$ be a $T'$-continuous function on a topological space into $Y$. Let $U_1, U_2, \ldots$ and $K_1, K_2, \ldots$ be sequences of subsets of $Y$ which satisfy (i), (ii) and (iii) for the pair $(T, T')$. For each positive integer $n$ we define $D_n$ to be the set of all points $x$ in the domain of $f$ such that in each neighborhood of $x$ there exist points $y$ and $z$ for which $f(y) \in U_n$ and $f(z) \in K_n$.

We first observe that if $f$ is not $T$-continuous at a point $p$ in the domain of $f$, then $p \not\in D_n$ for some value of $n$. This fact follows immediately from (ii) and the definition of continuity at $p$.

We next observe that $D_n$ is closed for each $n$. This fact follows easily from the definition of $D_n$ and the definition of closure.

We now prove by contradiction that each of the sets $D_n$ is nowhere dense. Let us assume for a given $n$ that $D_n$ is not nowhere dense. Then, since $D_n$ is closed, $D_n$ must contain a non-empty open set $G_n$. It follows from the definition of $D_n$ that there exists $y \in G_n$ such that $f(y) \in U_1$.

We use (ii) to obtain $V \subseteq T'$ such that $f(y) \in V$ and $V \subseteq K_n \subseteq T'$. Since $f$ is $T'$-continuous, there exists an open set $G_1$ such that $y \in G_1 \subseteq V$. We assume that $G_1 \subseteq G_n$.

It follows from the definition of $D_n$ that there exists $x \in G_1$ such that $f(x) \in K_n$. We see that if $f(x) \in V - K_n$. Since $V - K_n \subseteq T'$ and $f$ is $T'$-continuous, there exists an open set $G_2$ such that $x \in G_2 \subseteq V$ and $f(G_2) \subseteq V - K_n$.

The definition of $D_n$ states that there exists $t \in G_2$ such that $f(t) \in U_n$, but this is impossible since $f(G_2) \subseteq V - K_n$ and $U_n \subseteq V - K_n$. We have thus obtained a contradiction, and it follows that each set $D_n$ is nowhere dense.

We have proved that $D = \bigcup D_n$ is a first category set which contains the set of all points at which $f$ is not $T$-continuous. Our theorem now follows from the fact that a subset of a first category set is a first category set.

§ 3. Real valued semi-continuous functions and the Baire theorem. Let $E$ be the set of all real numbers, and let $T$ be the usual topology for $E$. We define $T'$ to be the topology for $E$ which is obtained by taking as base elements sets of the form $[x \leq \epsilon < x]$. It is obvious that a real valued function on a topological space is $T'$-continuous if and only if it is lower semi-continuous in the usual sense.
The open intervals with rational end points are denumerable and may be written in a sequence \( U_1, U_2, \ldots \). We let \( E_1, E_2, \ldots \) be the corresponding sequence of closed intervals. It is obvious that the sets \( U_n \) and \( E_n \) satisfy (i) and (ii) of condition (a). If \( U_n = (a, b) \), we can verify (iii) by letting \( V = (a, b) \). Thus the pair \((T, T^*)\) satisfies condition (a). As a special case of our Basic Theorem we now obtain the following well known theorem:

**Theorem 1.** If a real valued function on a topological space is lower semi-continuous, then it is continuous except at points of a first category set.

The Baire category theorem for the limit of a sequence of continuous functions follows easily from the above theorem. Since this fact does not seem to be generally known, it seems worth while to include the proof here.

**Theorem 2.** If \( f_1, f_2, \ldots \) is a sequence of continuous functions on a topological space into a metric space which converges pointwise to a function \( g \), then \( g \) is continuous except at points of a first category set.

Proof. For each positive integer \( n \) and each \( x \) in the domain of the functions, we define \( f_n(x) \) to be the diameter of the set \( \bigcup_{m=n}^{\infty} f_m(x) \). Each of the functions \( f_n \) is easily seen to be a real valued lower semi-continuous function, and hence is continuous at points of a residual set \( G_n \). We define

\[
G = \bigcap_{n=1}^{\infty} G_n.
\]

Suppose \( p \in G \) and \( \varepsilon > 0 \). Choose \( m \) so that \( f_m(p) < \varepsilon / 3 \). Because of the continuity of \( f_n \) and the continuity of \( f_m \) at \( p \), there exists a neighborhood \( V \) of \( p \) such that if \( x \in V \) then \( f_m(x) < \varepsilon / 3 \) and the distance from \( f_m(x) \) to \( f_m(p) \) is less than \( \varepsilon / 3 \). It follows easily that the distance from \( f_n(x) \) to \( f_n(p) \) is less than \( \varepsilon / 3 \), and this proves that \( g \) is continuous at \( p \). Since \( G \) is a residual set, this proves our theorem.

§ 4. The compact-open and point-open topologies. In this paragraph we let \( A \) be a locally compact, separable metric space, we let \( B \) be a separable metric space, and we let \( Y \) be the set of all continuous functions on \( A \) into \( B \). If \( s \) is a positive integer, \( C_1, \ldots, C_s \) are subsets of \( A \) and \( G_1, \ldots, G_s \) are subsets of \( B \), then we define \( W(C_1, \ldots, C_s; G_1, \ldots, G_s) = \{ f : f \in \mathcal{F}, f(C_i) \cap G_i \neq \emptyset \text{ for } 1 \leq i < s \} \). The compact-open topology for \( Y \) is the topology obtained by taking as base elements all sets of the form \( W(C_1, \ldots, C_s; G_1, \ldots, G_s) \) where each \( C_i \) is compact and each \( G_i \) is open. The point-open topology for \( Y \) is the topology obtained by taking as base elements sets of the form \( W(C_1, \ldots, C_s; G_1, \ldots, G_s) \) where each \( C_i \) contains only one point and each \( G_i \) is open.

We let \( T \) be the compact-open topology for \( Y \) and we let \( T^* \) be the point-open topology for \( Y \). There exists a countable base \( L \) for \( A \) such that each member of \( L \) has a compact closure. We define \( M \) to be the collection of all sets which are the closures of members of \( L \). We let \( N \) be a countable base for \( B \).

There are only a countable number of sets of type \( W(C_1, \ldots, C_s; G_1, \ldots, G_s) \) where \( s \) is a positive integer, each \( C_i \) is \( M \), and each \( G_i \) is \( N \). Therefore, these sets may be arranged in a sequence \( U_1, U_2, \ldots \). If \( U_i = W(C_1, \ldots, C_s; G_1, \ldots, G_s) \), then we define \( K_i = W(C_1, \ldots, C_s; G_1, \ldots, G_s) \). The sets \( K_i \) are closed with respect to the point-open topology, and it is easy to see that \( (i) \), \( (ii) \), and \( (iii) \) are satisfied. Thus the pair \((T, T^*)\) satisfies condition (a).

If we apply the Basic Theorem to the above example, we obtain the following form of a well known theorem concerning functions of two variables.

**Theorem 3.** Let \( P \) be a topological space, let \( A \) be a locally compact separable metric space, and let \( B \) be a separable metric space. If \( f \) is a function on \( P \times A \) into \( B \) which is continuous in each variable separately, then there exists a residual subset \( Q \) of \( P \) such that \( f \) is continuous at each point of \( Q \times A \).

Proof. We define \( Y, T \) and \( T^* \) as above. We define a function \( F \) on \( P \times Y \) by letting \( F(u)(v) = f(u, v) \). Since \( f(u, v) \) is continuous in \( v \) for each fixed \( u \), it is clear that \( F(u) = y \) for each \( u \in P \). Since \( f(u, v) \) is continuous in \( u \) for each fixed \( v \), it follows that \( F(u) \) is continuous with respect to the \( T \) topology for \( Y \). Thus there exists a residual subset \( Q \) of \( P \) such that \( F \) is continuous with respect to the \( T \) topology for \( Y \) at each point of \( Q \). It follows easily that \( f \) is continuous at each point of \( Q \times A \).

We next prove a theorem about transformation groups.

**Theorem 4.** Let \( S \) be a second category topological group and let \( A \) be a locally compact separable metric space. We assume that for each \( s \in S, F_s \) is a homeomorphism of \( A \) onto \( A_1 \); and we assume that \( F_s \) is continuous for all \( s \). If \( f(x) \) is continuous in \( x \) for each fixed \( s \), then \( F_s(x) \) is simultaneously continuous in \( s \) and \( x \) (i.e., \( F_s \) is continuous on \( S \times A \) into \( A_1 \)).

Proof. Let \( H \) be the group of all homeomorphisms of \( A \) onto \( A_1 \). We define \( g \) on \( S \) into \( H \) by letting \( g(s) = F_s \) for each \( s \in S \). Then \( g \) is an algebraic homomorphism on \( S \) into \( H \). Moreover, since \( F_s(x) \) is continuous in \( x \) for each fixed \( s \), \( g \) is continuous with respect to the point-open topology for \( H \). It follows that there exists a residual set of points in \( S \) at which \( g \) is continuous with respect to the compact-open topology for \( H \). Since \( S \) is second category this residual set is non-empty.

The product operation in \( H \) is continuous with respect to the compact open topology, and hence continuity of the homomorphism \( g \) at one point of \( S \) implies continuity of \( g \) at every point of \( S \). The continuity of \( g \) with
respect to the compact-open topology, however, implies the continuity of $f$ as a function on $S \times A$ into $A$.

§ 5. The overlap topology. Let $C$ be the set of all continuous real valued functions of a real variable, let $C'$ be the subset of $C$ which consists of all functions which have continuous derivatives, and let $D$ be the function on $C'$ into $C$ for which $D(f)$ is the derivative of $f$ for each $f \in C$. The function $D$ is not continuous relative to any of the standard non trivial topologies for $C$ (i.e. uniform, compact-open, point-open). On the other hand, it would be useful to have a topology $T^*$ for $C$ with respect to which $D$ would be continuous; and which would be such that $(T,T^*)$ would satisfy condition (a), where $T$ is the compact-open topology for $C$. We will now define such a topology.

Let $A$, $B$ and $Y$ be the same as in § 4. If $U$ is an open set in $A$ and $V$ is an open set in $B$, we define

$$Z(U,V) = \{ f \in Y \text{ and } f(U) \cap V \neq \emptyset \}.$$ 

The collection of all sets of the form $Z(U,V)$ forms a subbase for a topology $T^*$ which we call the overlap topology for $Y$. The overlap topology is obviously at least as coarse as the point-open topology for $Y$ (i.e. it is contained in the point-open topology).

Let us now prove that if $T$ is the compact-open topology for $Y$, then $(T,T^*)$ satisfies condition (a). We define sets $U_i,V_i$ as in § 4. Since (i) and (ii) are obviously satisfied, it is sufficient to prove that each $K_i$ is closed in the $T^*$ topology.

Let $K_i = W(C_1, \ldots, C_n; G_1, \ldots, G_n)$ and assume $f \in K_i$. Then $f(C_j)$ non $G_j$ for some $j$. Since $C_j$ is the closure of an open set, there exists a point $p$ interior to $C_j$ such that $f(p) \in G_j$. There is a neighborhood $V$ of $f(p)$ such that $V \cap G_j = \emptyset$, and there is a neighborhood $U$ of $p$ such that $U \cap C_j$. It is easy to see that $Z(U,V)$ is a $T^*$ neighborhood of $f$, and that if $g \in Z(U,V)$ then $g \in K_i$. This proves that $K_i$ is closed in the $T^*$ topology.

**Lemma 1.** If both $C$ and $C'$ are topologized by the overlap topology, then $D$ is continuous.

**Proof.** Let $g \in C'$ and let $Z(U,V)$ be a neighborhood of $D(g)$. (In order to prove $D$ continuous at $g$, we may without loss of generality consider only subbasic neighborhoods of $D(g)$.) Let $g' = D(g)$. There exists $p \in U$ such that $g'(p) \in V$. We choose $\epsilon > 0$ so that the $2\epsilon$-neighborhood of $g'(p)$ is contained in $V$. Since $g'$ is continuous, there exists an open interval $U^* \cap C(U)$ such that $g(x) \in \epsilon$ and $|g'(x) - g'(p)| < \epsilon$ for all $x \in U^*$. We choose distinct points $p_1$ and $p_2$ in $U^*$. There exist open sets $U_1, U_2, V_1, V_2$ such that $p_1 \in U_1 \cap U_2 = \emptyset$, $g(p_1) \in V_1$, $g(p_2) \in V_2$.

and such that if

$$x_1 \in U_1, \quad x_2 \in U_2, \quad y_1 \in V_1, \quad y_2 \in V_2$$

then

$$|x_1 - x_2 - (g(p_1) - g(p_2))| < \epsilon, \quad |y_1 - y_2 - (p_1 - p_2)| < \epsilon.$$ 

Now let $f \in Z(U_1,V_1) \cap Z(U_2,V_2)$. There exist $x_i \in U_i$ and $y_i \in V_i$ such that $f(x_i) \in V_i$ and $f(x_2) \in V_2$. Therefore,

$$|f(x_1) - f(x_2) - (g(p_1) - g(p_2))| < \epsilon, \quad |y_1 - y_2 - (p_1 - p_2)| < \epsilon.$$ 

By the mean-value theorem, there exist $q$ and $r$ in $U^*$ such that

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(q) \quad \text{ and } \quad \frac{g(p_1) - g(p_2)}{p_1 - p_2} = g'(r).$$

Thus $f'(q) - g'(r) < \epsilon$, and since $|g'(r) - g'(p)| < \epsilon$ it follows that $|f'(q) - f'(p)| < 2\epsilon$. Therefore $f'(q) < \epsilon$ and $f(U) \cap V \neq \emptyset$. This proves that $D(f) \in Z(U,V)$.

**Theorem 5.** Let $P$ be a topological space, let $R$ be the real number system, and let $f$ be a function on $P \times R$ into $R$ which is continuous in each variable separately. If $g(u,v) = f(u,v)$ for exists and is continuous in $v$ for each $u \in P$, then there exists a residual subset $Q$ of $P$ such that $g$ is continuous at each point of $Q \times R$.

**Proof.** Let $C$ and $C'$ be as before. We define $F$ on $P \times C'$ by letting $F((u,v)) = f(u,v)$. If $D$ is the derivative function on $C'$ into $C$, then $DF$ is a function on $P \rightarrow C$.

The continuity of $F(u,v)$ in $u$ for fixed $v$ implies that $F$ is continuous with respect to the point-open topology for $C$. Hence $F$ is continuous with respect to the overlap topology for $C$. Using Lemma 1 and induction, we see that $DF$ is continuous on $P \times C$ with respect to the overlap topology. Therefore, there exists a residual set $Q$ in $P$, at each point of which $DF$ is continuous with respect to the compact-open topology. It follows that $g$ is continuous at each point of $Q \times R$.

A slight modification of Lemma 1 and the above theorem yields the following lemma.

**Lemma 2.** Let $V_i$ be the set $x_1 + \ldots + x_i < 1$ in Euclidean $n$-space and let $S$ be any topological space which is second category at each point. Let $F(g(x_1, \ldots, x_n))$ be a continuous function defined on $S \times V_i$ and let $F(g(x_1, \ldots, x_n):=g(x_1, \ldots, x_n)|x_1 \leq \frac{1}{2}$ exist and be continuous in $x$ for any fixed $g$ in $S$. There exists a set $Q$ which is dense in $S$ such that $F_q$ is simultaneously continuous in all variables at each point of $Q \times V_i$. 


Proof. Let $O$ be the set of all continuous real-valued functions on $V_1$, and let $O_f$ be the subset of $O$ consisting of those functions whose partial derivative with respect to the $j$th coordinate exists and is continuous. A subset $U$ of $V_1$ will be called a segment parallel to the $j$th axis if and only if there exist $p \in V_1$ and $\epsilon > 0$ such that

$$U = \{x; x_i = p_i \text{ for } k \neq j \text{ and } p_j - \epsilon < x_j < p_j + \epsilon \}.$$ 

We let $T^*$ be the topology generated by subbase elements of the form $(f(U) \cap V_1)\lfloor V_1$, $U$ is a segment parallel to the $j$th axis, and $V_1$ is an open subset of the real number system. We let $T$ be the compact open topology for $C$. Now let $D_j$ be the function on $C_j$ into $C$ for which $D_j(\beta)$ is the partial derivative of $\beta$ with respect to the $j$th coordinate. If we topologize $O$ by the topology $T^*$, then $D_j$ is continuous on $C_j$ into $C$ (proof as in Lemma 1). We define $\Phi$ on $S$ into $C$ by letting $\Phi(g)(x) = \Phi(g)(x_0)$ for all $g \in S$ and $x \in V_1$. $\Phi$ is continuous with respect to the $T^*$ topology for $C_j$, and thus $D_j \Phi$ is continuous with respect to $T^*$ topology. It is easily seen that $(T, T^*)$ satisfies condition (a). Thus there exists a residual (and hence dense, since $S$ is second category at each point) subset $Q$ of $S$ such that $D_j \Phi$ is continuous with respect to $T^*$ at each point of $Q$. It follows that $F_1$ is continuous at each point of $Q \times V_1$.

The above lemma is a generalization of a lemma due to Montgomery (see [8], p. 384, Lemma 1). If we use our Lemma 2 instead of Montgomery's Lemma 1, but otherwise follow the proof given in [8], we obtain the following generalization of Montgomery's Theorem 1.

**Theorem 6.** If $G$ is a second category group which is a transformation group of a manifold of class $C_0$ and if each transformation of $G$ is of class $C$, then the derivatives of the functions which define the transformations locally are continuous in all variables simultaneously.

The above theorem is proved in [8] for locally compact groups $G$, and Montgomery observes that his proof also goes over for certain other groups $G$ (including complete metric groups). It is not clear, however, that Montgomery's proof will hold for every second category group.

§ 6. The space of functions which are infinitely differentiable and have compact supports. In this section we let $Y$ be the set of all functions $f$ such that: $f$ is a real valued function of a real variable, $f$ is infinitely differentiable, and $f$ has a compact support (i.e. $f$ vanishes outside of some bounded interval). We let $D$ be the derivative function on $Y$ into $Y$.

We say that a sequence $f_1, f_2, \ldots$ in $Y$ $S$-converges to $j$ if and only if there exists a bounded interval $J$ such that each $f_j$ vanishes outside of $J$, and not only does the sequence $f_1, f_2, \ldots$ converge uniformly to $f$, but for each positive integer $k$ the sequence $D^k f_1, D^k f_2, \ldots$ converges uniformly to $D^k f$.

**Theorem 7.** Let $X$ be a topological space which satisfies the first axiom of countability, and let $F$ be a function on the set $X$ into the set $Y$. If $F$ is continuous relative to pointwise convergence in $X$, then there exists a residual subset $Q$ of $X$ at each point of which $F$ is continuous relative to $S$-convergence in $Y$.

Proof. Since $X$ satisfies the first axiom of countability, sequential continuity and continuity are equivalent for $F$. Since $F$ is continuous relative to the overlap topology for $Y$ and $F$ is continuous relative to the point-open (and hence the overlap) topology for $Y$, each of the functions $D^k F, DF, D^2 F, \ldots$ is continuous relative to the overlap topology for $Y$. It follows that there exists a residual subset $Q$ of $X$ such that at each point of $Q$, the functions $F, DF, D^2 F, \ldots$ are all continuous relative to the compact-open topology for $Y$.

For each positive integer $n$, we define $A_n$ to be the set of all points $x \in X$ such that at each neighborhood of $x$ there exist points $y$ and $z$ such that $F(y)$ vanishes outside of the interval $[-n, n]$ and $F(z)$ does not vanish identically outside of the interval $[-n, n]$. It is obvious that each set $A_n$ is closed. Since $F$ is continuous with respect to the point-open topology for $Y$, it is easy to see that each set $A_n$ does not contain a non empty open set. Thus $Q = \bigcap_{n=1}^{\infty} A_n$ is a residual set. We define $Q = \bigcap_{n=1}^{\infty} A_n$. Now let $p \in Q$ and let $x_1, x_2, \ldots$ be a sequence in $X$ which converges to $p$. There exists a positive integer $n$ such that $F(x_j)$ vanishes outside of $[-n, n]$. Since $p \in A_n$, there exists a neighborhood $N$ of $p$ such that $F(x)$ vanishes outside of $[-n, n]$ for each $x \in N$. Therefore, there exists a bounded interval $J$ such that each of the functions $F(x_1), F(x_2), \ldots$ vanishes outside of $J$. Since each of the functions $F, DF, D^2 F, \ldots$ is continuous at $p$ relative to the compact-open topology for $Y$, it follows that $F(x_1), F(x_2), \ldots$ converges uniformly to $F(p)$ and $D^k F(x_1), D^k F(x_2), \ldots$ converges uniformly to $D^k F(p)$ for each positive integer $k$. This proves that $F$ is continuous at $p$ relative to $S$-convergence in $Y$.

**Corollary.** If $X$ is a second category metric group and $F$ is an algebraic homomorphism of the group $X$ into the group $Y$ (addition being the group operation in $Y$), then $F$ is continuous relative to $S$-convergence in $Y$ if and only if $F$ is continuous relative to pointwise convergence in $Y$.

§ 7. Some miscellaneous theorems. The theorems in this section are concerned with situations in which the closures of base elements for one topology for a space are also closed relative to a second coarser topology.
Theorem 8. Let \( f \) be a function on a topological space \( X \) into a separable metric space \( Y \), and let \( B \) be a base for the topology for \( Y \). If \( f^{-1}(B) \) is closed for each \( V \in B \), then \( f \) is continuous at each point of a residual subset of \( X \).

Proof. Let \( T \) be the given topology for \( Y \). The set of all sets of the form \( Y - V \), \( V \in B \), forms a subbase for a topology \( T^* \) for \( Y \). It is obvious that \((T, T^*)\) satisfy condition (a), and that \( f \) is \( T^*\)-continuous. It follows that \( f \) must be continuous (i.e. \( T\)-continuous) at points of a residual subset of \( X \).

The following theorem is related to Theorem 2 of [9].

Theorem 9. If \( f \) is an algebraic homomorphism of a second category group \( X \) into a locally compact separable metric group \( Y \), then \( f \) is continuous if and only if \( f^{-1} \) takes compact sets into closed sets.

Proof. Suppose \( f^{-1} \) takes compact sets into closed sets. Choose a base \( B \) for \( Y \) such that each member of \( B \) has a compact closure. By Theorem 8, \( f \) is continuous at points of a residual subset of \( X \). Since \( X \) is second category and \( f \) is an algebraic homomorphism, \( f \) is continuous.

Theorem 10. Let \( f \) be a function on a topological space \( X \) into a separable Banach space \( Y \). If \( f \) is continuous with respect to the weak topology for \( Y \), then there exists a residual subset of \( X \) at each of whose points \( f \) is continuous with respect to the norm topology for \( Y \).

Proof. Since it is well known that (norm) closed spheres in \( Y \) are also closed in the weak topology, this theorem follows at once from Theorem 8.

The above theorem was proved by Alexiewicz and Orlicz (see [1], p. 108, Corollaire I.2.2)).

For the special case of the \( L^p \) spaces, \( p \geq 1 \), we are able to obtain a slightly stronger theorem than Theorem 10 if we replace the weak topology by the topology of convergence in measure.

We restrict ourselves to real functions on the unit interval \( J \), and let \( m \) be the Lebesgue measure function. If \( f \in L^p \) and \( \epsilon > 0 \), we define \( N(f, \epsilon) \) to be the set of all functions \( g \in L^p \) for which \( m(\{x : |g(x) - f(x)| > \epsilon \}) < \epsilon \). The sets \( N(f, \epsilon) \) form a base for a topology \( T^* \) for \( L^p \). We call \( T^* \) the convergence in measure topology for \( L^p \).

If \( f \) is a measurable subset of \( J \) and \( f \in L^p \), we define \( L(f, J) \) as the usual norm of \( f \) in \( L^p \). We let \( T \) be the topology induced on \( L^p \) by this norm.

Theorem 11. If \( F \) is a function on a topological space \( X \) into \( L^p \) and \( F \) is continuous relative to the convergence in measure topology \( T^* \), then there exists a residual subset of \( X \) at each point of which \( F \) is continuous relative to the norm topology \( T \).

Proof. We wish to show that \((T, T^*)\) satisfy condition (a). Since the metric topology induced on \( L^p \) by the norm is separable, it is obvious that it is sufficient to prove that norm-closed spheres in \( L^p \) are also closed sets relative to the \( T^* \) topology.

Let \( S \) be a norm-closed sphere in \( L^p \) having center \( f \) and radius \( r \). Suppose \( g \in S \) we define \( \epsilon = L(g - f, J) - r \). Because of absolute continuity, there exists \( \delta_1 > 0 \) such that if \( E \) is a measurable subset of \( J \) and \( m(E) < \delta_1 \), then \( L(g - f, E < \epsilon/3 \). There exists \( \delta_2 > 0 \) such that if \( K \) is a measurable subset of \( J \) and \( |g(x) - h(x)| < \delta_2 \) for all \( x \in K \) then \( L(g - h, K) < \epsilon/3 \). We let \( \delta \) be the least of the numbers \( \delta_1, \delta_2, \epsilon/3 \).

Now suppose that \( h \in X(f, \epsilon) \). There exists a measurable subset \( E \) of \( J \) such that \( m(E) < \delta \) and \( |h(x) - g(x)| < \delta \) for all \( x \in E \). Since \( p \geq 1 \), it is easy to see that \( L(g - f, J) < L(g - f, E) + L(g - f, J - E) \). It follows that

\[
L(h - f, J) = L(h - f, J - E) + L(h - f, J - E) > L(g - f, E) + L(g - f, J - E) \geq r + \epsilon/3 - \epsilon/3 \geq r.
\]

Thus \( h \in S \), and hence \( S \cap X(f, \epsilon) = \emptyset \). This proves \( S \) is closed in the \( T^* \) topology.

§ 8. Semi-continuous set-valued functions. Let \( S \) be a topological space, and let \( O(S) \) be the set of all closed subsets of \( S \). If \( D \) is a directed set and \( s \) is a net (see [5]) on \( O(S) \), then we define:

\[
\lim_{\alpha \in D} s(\alpha) = \{x \in X : \forall U \in O(S), s(\alpha) \cap U \neq \emptyset \}
\]

for all \( \alpha \in D \) in a cofinal subset of \( D \):

\[
\lim_{\alpha \in D} s(\alpha) = \{x \in X : \forall U \in O(S), s(\alpha) \cap U \neq \emptyset \}
\]

for all \( \alpha \in D \) in a residual subset of \( D \).

If \( X \) is a topological space and \( f \) is a function on \( X \) into \( O(S) \), then,

(a) \( f \) is upper semi-continuous at \( p \in X \) if and only if \( \lim_{n \to \infty} f(n(\alpha)) \subseteq f(p) \) for each net \( n \) that converges to \( p \).

(b) \( f \) is lower semi-continuous at \( p \in X \) if and only if \( \lim_{n \to \infty} f(n(\alpha)) \supseteq f(p) \) for each net \( n \) that converges to \( p \).

The concepts of semi-continuity defined above have been investigated by Choquet in [3]. These types of semi-continuity differ in general from those types studied by the author in [3] and [4].

For the remainder of this section we assume that \( S \) is a separable, metrizable space, and that \( B \) is a countable base for the topology of \( S \). If \( a \) is a subset of \( S \), then we define

\[
N_s(a) = \{b : b \in O(S) \text{ and } b \cap a = \emptyset \}
\]
and 

\[ N(a) = \{ b \in C(S) \mid b \wedge a \neq \emptyset \} \]

The set of all sets \( N(V) \), \( V \in B \), forms a subbase for a topology \( T_1 \) for \( C(S) \). The set of all sets \( N(V) \), \( V \in B \), forms a subbase for a topology \( T_1 \) for \( C(S) \). The set of all sets \( N(V) \), \( V \in B \), forms a subbase for a topology \( T_2 \) for \( C(S) \). The topology \( T_2 \) is independent of the particular base \( B \) used for its definition, but in general both \( T_1 \) and \( T_2 \) depend on \( B \).

The proof of the following lemma is quite easy and is omitted.

**Lemma 3.** If \( f \) is a function on a topological space \( X \) into \( C(S) \) and \( p \in X \), then

(a) \( f \) is lower semi-continuous at \( p \) if and only if \( f \) is \( T_1 \)-continuous at \( p \);

(b) \( f \) is \( T_2 \)-continuous at \( p \), then \( f \) is upper semi-continuous at \( p \);

(c) \( f \) is compact for each \( V \in B \), then \( f \) is upper semi-continuity of \( f \) at \( p \)

implies \( T_2 \)-continuity of \( f \) at \( p \);

(d) \( f \) is \( T_2 \)-continuous at \( p \), then \( f \) is upper semi-continuous at \( p \).

**Proof.** The sets of the form \( \bigcap_{j=1}^{n} N_j(V_j) \), \( V_j \in B \), form a countable base

for \( T_1 \) and can be arranged in a sequence \( U_1, U_2, \ldots \). If \( U_n = \bigcap_{j=1}^{m} N_j(V_j) \),

we define \( K_n = \bigcap_{j=1}^{m} N_j(V_j) \). The sets \( K_n \) are closed in the \( T_1 \) topology, and it is easy to verify that (i), (ii) and (iii) are satisfied.

**Lemma 4.** If the boundary of \( V \) is compact for each \( V \in B \), then \( (T_1, T_2) \)

satisfies condition (a).

Proof. The sets of the form \( \bigcap_{j=1}^{n} N_j(V_j) \), \( V_j \in B \), form a countable base

for \( T_2 \) and can be arranged in a sequence \( U_1, U_2, \ldots \). If \( U_n = \bigcap_{j=1}^{m} N_j(V_j) \),

then we define \( K_n = \bigcap_{j=1}^{m} N_j(V_j) \). The sets \( K_n \) are closed in the \( T_2 \) topology, and it is easy to verify that (i), (ii) and (iii) are satisfied. The hypothesis that the members of \( B \) have compact boundaries is used in verifying (ii).

**Lemma 5.** Both \( (T_1, T_2) \) and \( (T_1, T_3) \) satisfy condition (a).

Proof. This lemma follows from the fact that each of the topologies \( T_1 \) and \( T_2 \) have countable bases whose members are closed with respect to the other topology. Thus in each case we can choose a base \( U_1, U_2, \ldots \) and let \( K_n = U_n \).

Not all of the results obtained in the preceding lemmas are needed for the proof of the next theorem. (In particular, Lemma 5 is not used

at all.) These extra results have been included for the sake of completeness, and because they may be of some slight interest in themselves.

**Theorem 12.** Let \( f \) be a function on a topological space \( X \) into \( C(S) \).

If \( f \) is lower semi-continuous on \( X \), then there exists a residual set in \( X \) at each point of which \( f \) is upper semi-continuous. If \( S \) is locally compact and \( f \) is upper semi-continuous on \( X \), then there exists a residual set in \( X \) at each point of which \( f \) is lower semi-continuous.

Proof. Suppose that \( f \) is lower semi-continuous on \( X \). Then by Lemma 3, \( f \) is \( T_1 \)-continuous on \( X \). Since \( (T_1, T_2) \) satisfies condition (a) by Lemma 6, it follows from our Basic Theorem that there exists a residual subset \( Q \) of \( X \) such that \( f \) is \( T_2 \)-continuous at each point of \( Q \).

Lemma 3 then implies that \( f \) is upper semi-continuous at each point of \( Q \).

Now let us assume that \( S \) is locally compact (as well as separable and metrizable) and that \( f \) is upper semi-continuous on \( X \). Choose the base \( B \) so that \( B \) is compact for each \( V \in B \). Then, by Lemma 3, \( f \) is \( T_2 \)-continuous on \( X \). Lemma 4 together with the Basic Theorem implies that there exists a residual subset \( Q \) of \( X \) such that \( f \) is \( T_2 \)-continuous at each point of \( Q \). By Lemma 3, \( f \) is lower semi-continuous at each point of \( Q \).

§ 9. The space of measurable sets. Let \( B \) be a Euclidean space, and let \( \mu(A) \) be the Lebesgue measure of \( A \) for each measurable subset \( A \) of \( E \). We let \( Y \) be the set of all subsets of \( B \) which are measurable and have finite measure. If we define \( d(A, B) = \mu(A-B) + \mu(B-A) \) for all \( A \) and \( B \) in \( Y \), and \( d \) is a metric for \( Y \) (provided we identify sets which differ only by a set of measure zero) and the resulting metric space is separable.

We now define two topologies for \( Y \) which are related to \( d \) in a fairly obvious way. If \( A \subset Y \) and \( \varepsilon > 0 \), we define:

\[ N_1(A, \varepsilon) = \{ B \mid B \subset Y \text{ and } \mu(A-B) < \varepsilon \}, \]

and

\[ N_2(A, \varepsilon) = \{ B \mid B \subset Y \text{ and } \mu(B-A) < \varepsilon \}. \]

The sets \( N_1(A, \varepsilon) \) form a base for a topology \( T_1 \) for \( Y \), and the sets \( N_2(A, \varepsilon) \) form a base for a topology \( T_2 \) for \( Y \). Both \( T_1 \) and \( T_2 \) are perfectly separable. It is easy to prove that \( B \mid B \subset Y \text{ and } \mu(A-B) < \varepsilon \) is closed in the \( T_2 \) topology, and that \( B \mid B \subset Y \text{ and } \mu(B-A) < \varepsilon \) is closed in the \( T_2 \) topology. Using these facts, it is easy to show that both \( (T_1, T_2) \) and \( (T_1, T_3) \) satisfy condition (a). We thus obtain the following theorem:

**Theorem 13.** Let \( f \) be a function on a topological space \( X \) into \( Y \).

If \( f \) is continuous with respect to either of the topologies \( T_1 \) or \( T_2 \), then there exists a residual subset \( Q \) of \( X \) such that \( f \) is continuous with respect to the metric \( d \) at each point of \( Q \).

*Fundamenta Mathematicae*. T. XXII, 19
Example. Let $C$ be the set of all continuous real valued functions on the unit interval. We assume that $C$ is metrized by the usual uniform metric. We define a function $f$ on $C$ by letting

$$f(w) = \{t \mid w(t) > 0\} \quad \text{for each} \quad w \in C.$$  

It is easy to verify that $f$ is continuous with respect to the $T_1$ topology, but that $f$ is not continuous with respect to the $T_2$ topology. It follows from Theorem 13 that $f$ is continuous with respect to $d$ on a residual subset of $C$. It is easy to see that $f$ is $d$-continuous at $w$ if and only if

$$\mu(f(w(t) = 0)) = 0.$$  

References


Über eine Dimensionstheorie in topologischen Verbänden

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Ein für alle Mal sei ein klassisch-topologischer Boole-Verband $\mathfrak{B}$ vorgestellt.

Definition. $\mathfrak{B}$ heißt speziell ein Sikorski-Verband oder kurz ein $S$-Verband, wenn $\mathfrak{B}$ ein $\sigma$-Verband, regulär und $T_1$-topologisch ist und außerdem eine abzählbare Basis $1)$ besitzt.


In Anlehnung an die Stoffteilung Mengers [2] behandelt die vorliegende Arbeit nach der Formulierung der Dimensionsdefinition (§ 1) die Dimension einzelner Somen (§ 2), Summen- und Zerlegungssätze (§ 3), die lokale dimensionelle Struktur von $S$-Verbänden (§ 4), Überdeckungssätze (§ 5), die Beziehungen globaler Trennungs- und Zusammen-

1) „Basis“ ist immer als offene Basis gemeint.