From theorems 4, 8, and 10 we obtain the following partial answers to that problem: if X_0 contains only the number 0, X_1 is the set of all non-negative integers, and X_2 is the set of all non-negative rationals, then $F(X_0)=Q_2^{(1)}$, $F(X_1)=Q_2^{(1)}$, $F(X_2)=P_3^{(1)}$.

References

[1] W. Markwald, *Zur Eigenschaft primitiv-rekursiver Funktionen, unendlich viele Werte anzunehmen*, this volume, p. 166-167.

[2] A. Mostowski, *On definable sets of positive integers*, Fund. Math. 34 (1947), p. 81-112.

Reçu par la Rédaction le 20.9.1954

Contributions to the theory of definable sets and functions

by

A. Mostowski (Warszawa)

In this paper we collect some scattered results concerning sets and functions definable in elementary arithmetic. We shall use consistently the terminology and notations of the paper [2], with which, we assume, the reader is acquainted. In particular we denote by R_k the set of k-ples $(x_1,x_2,\dots,x_k)=\mathfrak{m}$, where the x_j's are non-negative integers, and by $P_n^{(kl)}$ (or $Q_n^{(kl)}$) the set of functions from R_k to R_l whose graphs are in $P_n^{(k+l)}$ (or in $Q_n^{(k+l)}$).

1. We begin by establishing some simple properties of the classes $P_n^{(k1)}$ and $Q_n^{(k1)}$.

THEOREM 1. $P_n^{(k1)}\subset Q_n^{(k1)}$.

Proof. The theorem is evident in case $n=0$. Let us, therefore, assume that $n>0$ and $f\in P_n^{(k1)}$. It follows from the definitions that there exists a set $B\in Q_{n-1}^{(k+2)}$ such that

$$\{f(\mathfrak{m})=m\}\equiv\sum_x\{(\mathfrak{m},m,x)\in B\}.$$

Hence

$$\{f(\mathfrak{m})\neq m\}\equiv\sum_{p,x}\{[(\mathfrak{m},p,x)\in B]\cdot(p\neq m)\}$$

which proves that the graph of f is in $Q_n^{(k+1)}$, q. e. d.

THEOREM 2. *If* $n\geqslant 1$, *then* $P_{n+1}^{(k1)}-Q_n^{(k1)}\neq 0\neq Q_n^{(k1)}-P_n^{(k1)}$.

Proof. It is well known that there are sets M which belong to $P_{n+1}^{(k)}\cdot Q_{n+1}^{(k)}$ without belonging to $Q_n^{(k)}$. Let f be the characteristic function of such a set M. The graph of f is in $P_{n+1}^{(k+1)}$, since

$$\{y=f(\mathfrak{m})\}\equiv\{(y=0)\cdot(\mathfrak{m}\notin M)+(y=1)\cdot(\mathfrak{m}\in M)\}.$$

As $\{\mathfrak{m}\in M\}\equiv\{f(\mathfrak{m})=1\}$, the graph of f is not in $Q_n^{(k+1)}$. Hence $f\in P_{n+1}^{(k1)}-Q_n^{(k1)}$.

Slightly more intricate is the proof that $Q_n^{(k1)}-P_n^{(k1)}\neq 0$. Let $C\in Q_n^{(k)}-P_n^{(k)}$ and let B be a set in $Q_{n-1}^{(k+1)}$ such that

$$\mathfrak{m}\notin C\equiv\sum_x[(\mathfrak{m},x)\in B].$$

We select an arbitrary point $\mathfrak{m}_0$ outside C and put

$$h(\mathfrak{m})=\mathfrak{m}_0 \quad \text{if} \quad \big(s_1^{(k)}(\mathfrak{m}),s_2(\mathfrak{m})\big)\notin B,$$
$$h(\mathfrak{m})=s_1^{(k)}(\mathfrak{m}) \quad \text{if} \quad \big(s_1^{(k)}(\mathfrak{m}),s_2(\mathfrak{m})\big)\in B,$$

where $s_1^{(k)}$ and s_2 are primitive recursive functions with the property that the formula $\mathfrak{m}\rightleftarrows\big(s_1^{(k)}(\mathfrak{m}),s_2(\mathfrak{m})\big)$ establishes a one-one correspondence between elements of R_1 and elements of R_{k+1}.

The set R_k-C coincides with the set of values of the function h. The graph of h is in $\boldsymbol{P}_n^{(k+1)}\cdot\boldsymbol{Q}_n^{(k+1)}$, since

$$\{\mathfrak{m}=h(\mathfrak{m})\}\equiv(\mathfrak{m}=\mathfrak{m}_0)\cdot\Big[\big(s_1^{(k)}(\mathfrak{m}),s_2(\mathfrak{m})\big)\notin B\Big]+$$
$$+[\mathfrak{m}=s_1^{(k)}(\mathfrak{m})]\cdot\Big[\big(s_1^{(k)}(\mathfrak{m}),s_2(\mathfrak{m})\big)\in B\Big].$$

Let us put

$$F=\underset{(\mathfrak{m},x)}{E}\{(\mathfrak{m}\in C)\cdot(x=0)+(x>0)\cdot[\mathfrak{m}=h(x-1)]\cdot\prod_{1\leqslant z<x}[\mathfrak{m}\neq h(z-1)]\}.$$

We have then $F\in\boldsymbol{Q}_n^{(k+1)}$, because the set

$$\underset{(\mathfrak{m},x)}{E}\{(x>0)\cdot[\mathfrak{m}=h(x-1)]\cdot\prod_{1\leqslant z<x}[\mathfrak{m}\neq h(z-1)]\}$$

belongs to $\boldsymbol{P}_n^{(k+1)}\cdot\boldsymbol{Q}_n^{(k+1)}$ (see [3], theorem 3.3). From $\mathfrak{m}\in C\equiv(\mathfrak{m},0)\in F$ we infer that $F\notin\boldsymbol{P}_n^{(k+1)}$.

We shall show that F is the graph of a function. If $\mathfrak{m}\in C$, then $(\mathfrak{m},0)\in F$, and hence $\sum_x(\mathfrak{m},x)\in F$. If $\mathfrak{m}\notin C$, there is an integer y such that $\mathfrak{m}=h(y)$. Assuming that y is the smallest integer with this property and putting $x=y+1$, we obtain again $(\mathfrak{m},x)\in F$. Hence the formula $\sum_x(\mathfrak{m},x)\in F$ is true for every $\mathfrak{m}$.

It remains to prove that

$$[(\mathfrak{m},x_1)\in F]\cdot[(\mathfrak{m},x_2)\in F]\to x_1=x_2\,.$$

If $\mathfrak{m}\in C$, then $x_1=0$ and $x_2=0$. If $\mathfrak{m}\notin C$, then $x_1>0$, $x_2>0$, $\mathfrak{m}=h(x_1-1)=h(x_2-1)$, and $\mathfrak{m}\neq h(z)$, for every $z<x_1-1$ and every $z<x_2-1$. It can be easily seen that either of the assumptions, $x_1<x_2$, $x_2<x_1$, leads to contradictions.

The set F is thus shown to be the graph of a function. Since $F\in\boldsymbol{Q}_n^{(k+1)}-\boldsymbol{P}_n^{(k+1)}$, this function is in $\boldsymbol{Q}_n^{(k1)}$ but not in $\boldsymbol{P}_n^{(k1)}$, q. e. d.

2. In this section we shall establish some properties of the functions of the class $\boldsymbol{Q}_1^{(k1)}$.

THEOREM 3. *If $f\in\boldsymbol{Q}_1^{(k1)}$, then the set $\underset{(\mathfrak{m},m)}{E}[m\leqslant f(\mathfrak{m})]$ is recursively enumerable (i. e., belongs to the class $\boldsymbol{P}_1^{(k+1)}$).*

Proof. Let B be a recursive set such that

$$\{f(\mathfrak{m})=m\}\equiv\prod_x\{(\mathfrak{m},m,x)\in B\}.$$

We have then the equivalence

$$\{m\leqslant f(\mathfrak{m})\}\equiv\prod_{j<m}\sum_x[(\mathfrak{m},j,x)\notin B],$$

which proves the theorem.

Remark. If f is the characteristic function of a set $M\in\boldsymbol{P}_2^{(k)}\cdot\boldsymbol{Q}_2^{(k)}-\boldsymbol{P}_1^{(k)}$, then the set $A=\underset{(\mathfrak{m},m)}{E}[m\leqslant f(\mathfrak{m})]$ is not recursively enumerable since $\mathfrak{m}\in M\equiv(\mathfrak{m},1)\in A$. This shows that theorem 1 is, in general, false for functions $f\in\boldsymbol{P}_2^{(k1)}$.

THEOREM 4. *If a function $f\in\boldsymbol{Q}_1^{(k1)}$ is majorized by a recursive function, then f is recursive (i. e. belongs to $\boldsymbol{P}_0^{(k1)}$).*

Proof. Let B be a recursive set such that

$$\{m=f(\mathfrak{m})\}\equiv\prod_x[(\mathfrak{m},m,x)\in B],\qquad B\in\boldsymbol{P}_0^{(k+2)}$$

and g a recursive function such that $f(\mathfrak{m})\leqslant g(\mathfrak{m})$. Without loss of generality we may assume B to be primitive recursive. We denote by h the characteristic function of B and put

$$h'(\mathfrak{m},m,0)=1\dot{-}h(\mathfrak{m},m,0),$$
$$h'(\mathfrak{m},m,x+1)=[1\dot{-}h(\mathfrak{m},m,x+1)]\dot{-}\sum_{y=0}^{x}h'(\mathfrak{m},m,y).$$

Thus h' is a primitive recursive function which vanishes everywhere except in points $(\mathfrak{m},m,x)$, where x is the least integer such that $(\mathfrak{m},m,x)\notin B$.

If $m\neq f(\mathfrak{m})$, then there is an x such that $h'(\mathfrak{m},m,x)=1$; no such x exists if $m=f(\mathfrak{m})$. Hence

$$\sum_{m=0}^{g(\mathfrak{m})}\sum_{x=0}^{\infty}h'(\mathfrak{m},m,x)=g(\mathfrak{m})$$

and the function

$$\gamma(\mathfrak{m})=(\mu y)[\sum_{m=0}^{g(\mathfrak{m})}\sum_{x=0}^{y}h'(\mathfrak{m},m,x)=g(\mathfrak{m})]$$

is (general) recursive.

If $m\neq f(\mathfrak{m})$ and $m\leqslant g(\mathfrak{m})$, then $\sum_{x=0}^{\gamma(\mathfrak{m})}h'(\mathfrak{m},m,x)=1$; if $m=f(\mathfrak{m})$, then $\sum_{x=0}^{\gamma(\mathfrak{m})}h'(\mathfrak{m},m,x)=0$. Hence we obtain the formula

$$f(\mathfrak{m})=(\mu m)_{g(\mathfrak{m})}[\sum_{x=0}^{\gamma(\mathfrak{m})}h'(\mathfrak{m},m,x)=0]$$

in which $(\mu m)_a[\ldots]$ denotes the least m, satisfying the inequality $m \leqslant a$ and the condition $[\ldots]$ (or 0, if no such m exists). This formula proves that f is recursive.

3. Let a set $X \subset R_{k+1}$ be such that $\prod_{\mathfrak{m}} \sum_{m} (\mathfrak{m}, m) \in X$. A function f is called a *selector* of X if $\prod_{\mathfrak{m}} (\mathfrak{m}, f(\mathfrak{m})) \in X$.

THEOREM 5. *Recursively enumerable sets possess recursive selectors.*

Proof. Let $X \in \boldsymbol{P}_1^{(k+1)}$ and let B be a recursive set such that

$$(\mathfrak{m}, m) \in X \equiv \sum_{x} (\mathfrak{m}, m, x) \in B.$$

The function

$$f(\mathfrak{m}) = s_1^{(1)}\Big\{(\mu z)\Big[\big(\mathfrak{m}, s_1^{(1)}(z), s_2(z)\big) \in B\Big]\Big\}$$

is the required recursive selector of X.

THEOREM 6. *$g \in \boldsymbol{Q}_1^{(k1)}$ and g is not recursive, the set $\underset{(\mathfrak{m},m)}{E}[m > g(\mathfrak{m})]$ is in $\boldsymbol{Q}_1^{(k+1)}$ and has no recursive selector.*

This theorem follows immediately from the theorems 3 and 4.

4. Kleene [1] has constructed two disjoint recursively enumerable sets X, Y, such that there is no recursive set Z, satisfying the conditions $X \subset Z$, $YZ = 0$. If in this proof we change the words "recursively enumerable" into "element of $\boldsymbol{P}_n^{(k)}$" and "recursive" into "element of $\boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$", we obtain the proof of

THEOREM 7. *For each $n > 0$ there are disjoint sets $X, Y \in \boldsymbol{P}_n^{(k)}$ such that the formulas $X \subset Z$, $YZ = 0$ are not satisfied by any set $Z \in \boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$.*

Theorem 7 is not true for sets of class $\boldsymbol{Q}_n^{(k)}$. On the contrary, we shall prove

THEOREM 8. *If $X, Y \in \boldsymbol{Q}_n^{(k)}$ and $XY = 0$, then there is a set $Z \in \boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$ such that $X \subset Z$ and $YZ = 0$.*

Since the case $n = 0$ is evident, we assume that $n > 0$ and denote by M and N two sets in $\boldsymbol{P}_{n-1}^{(k+1)}$ such that

$$(1) \qquad \{\mathfrak{m} \in X\} \equiv \prod_{x} \{(\mathfrak{m}, x) \in M\}, \qquad \{\mathfrak{m} \in Y\} \equiv \prod_{x} \{(\mathfrak{m}, x) \in N\}.$$

It follows from $XY = 0$ that $R_k = (R_k - X) + (R_k - Y)$ and hence

$$(2) \qquad \prod_{\mathfrak{m}} \sum_{x} \{[(\mathfrak{m}, x) \notin M] + [(\mathfrak{m}, x) \notin N]\}.$$

The graph of the function

$$f(\mathfrak{m}) = (\mu x)\{[(\mathfrak{m}, x) \notin M] + [(\mathfrak{m}, x) \notin N]\}$$

may be represented in the form

$$(3) \qquad \underset{(\mathfrak{m},x)}{E}[x = f(\mathfrak{m})] = A - B \quad \text{where} \quad A, B \in \boldsymbol{P}_{n-1}^{(k+1)}.$$

This follows from the equivalence

$$x = f(\mathfrak{m}) \equiv \prod_{z<x} [(\mathfrak{m}, z) \in M \cdot N] \cdot \{[(\mathfrak{m}, x) \notin M] + [(\mathfrak{m}, x) \notin N]\}$$

and the observation that the sets,

$$\underset{(\mathfrak{m},x)}{E} \prod_{z<x} [(\mathfrak{m}, z) \in M \cdot N] \quad \text{and} \quad \underset{(\mathfrak{m},x)}{E} \{[(\mathfrak{m}, x) \notin M] + [(\mathfrak{m}, x) \notin N]\},$$

belong to the classes $\boldsymbol{P}_{n-1}^{(k+1)}$ and $\boldsymbol{Q}_{n-1}^{(k+1)}$ (cf. [3], theorem 3.3).

Let us put

$$(4) \qquad U = \underset{\mathfrak{m}}{E}\Big[\big(\mathfrak{m}, f(\mathfrak{m})\big) \notin M\Big], \qquad V = \underset{\mathfrak{m}}{E}\Big[\big(\mathfrak{m}, f(\mathfrak{m})\big) \notin N\Big],$$

$$(5) \qquad Z = V - U.$$

Formula (2) proves that $U + V = R_k$. Using (1), we obtain

$$\mathfrak{m} \in X \rightarrow \big(\mathfrak{m}, f(\mathfrak{m})\big) \in M \rightarrow \mathfrak{m} \notin U$$

and hence $X \subset R_k - U = (U + V) - U = V - U = Z$. In a similar way we show that $Y \subset U - V$ and hence $YZ = 0$.

It remains to evaluate the class of the set Z. From (3) and (4) we obtain

$$\begin{aligned}
\mathfrak{m} \in U &\equiv \sum_{x} \{[x = f(\mathfrak{m})] \cdot [(\mathfrak{m}, x) \notin M]\} \\
&\equiv \sum_{x} [(\mathfrak{m}, x) \in (A - B) - M] \\
&\equiv \prod_{x} \{[x = f(\mathfrak{m})] \rightarrow [(\mathfrak{m}, x) \notin M]\} \\
&\equiv \prod_{x} [(\mathfrak{m}, x) \in B + (R_{k+1} - A) + (R_{k+1} - M)].
\end{aligned}$$

These equivalences prove that $U \in \boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$. In a similar way we prove that $V \in \boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$. It follows from (5) that $Z \in \boldsymbol{P}_n^{(k)} \cdot \boldsymbol{Q}_n^{(k)}$. Theorem 8 is thus proved.

References

[1] S. C. Kleene, *A symmetric form of Gödel's theorem*, Indag. Math. 12 (1950), p. 244-246.

[2] A. Mostowski, *On definable sets of positive integers*, Fund. Math. 34 (1947), p. 81-112.

[3] — *On a set of integers not definable by means of one-quantifier predicates*, An. de la Soc. Pol. de Math. 21 (1948), p. 114-119.

Reçu par la Rédaction le 27.9.1954