From theorems 4, 8, and 10 we obtain the following partial answers to that problem: if \( X_0 \) contains only the number 0, \( X_1 \) is the set of all non-negative integers, and \( X_2 \) is the set of all non-negative rationals, then \( P(X_0) = Q^0 \), \( P(X_1) = Q^0 \), \( P(X_2) = P^0 \).

References


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Contributions to the theory of definable sets and functions

by

A. Mostowski (Warszawa)

In this paper we collect some scattered results concerning sets and functions definable in elementary arithmetic. We shall use consistently the terminology and notations of the paper [2], with which we, assume, the reader is acquainted. In particular we denote by \( R_k \) the set of \( k \)-ples \((x_1, x_2, \ldots, x_k) = m\), where the \( x_i \)'s are non-negative integers, and by \( P^0_k \) (or \( Q^0_k \)) the set of functions from \( R_k \) to \( R_l \) whose graphs are in \( P^0_{k+l} \) (or in \( Q^0_{k+l} \)).

1. We begin by establishing some simple properties of the classes \( P^0_k \) and \( Q^0_k \).

**Theorem 1.** \( P^0_k \subset Q^0_k \).

**Proof.** The theorem is evident in case \( k = 0 \). Let us, therefore, assume that \( k > 0 \) and let \( f \in P^0_k \). It follows from the definitions that there exists a set \( B \in Q^0_{k+1} \) such that

\[
\{(m, n) | n = \sum \}_{x \in B}(m, x, 0) + B \}.
\]

Hence

\[
\{(m, n) | n \neq m \} = \sum \}_{x \in B}(m, x, 0) + B \} \cdot (p \neq m)
\]

which proves that the graph of \( f \) is in \( Q^0_{k+1} \), q. e. d.

**Theorem 2.** If \( n > 1 \), then \( P^0_n \cap Q^0_n \neq 0 \).

**Proof.** It is well known that there are sets \( M \) which belong to \( P^0_2 \) and \( Q^0_2 \) without belonging to \( Q^0_2 \). Let \( f \) be the characteristic function of such a set \( M \). The graph of \( f \) is in \( P^0_3 \), since

\[
\{(y = f(m)) = \{(y = 0), (m \in M) \cap \{y = 1\} \cdot (m \in M)\} \}
\]

As \( (m \in M) = \{(m, n) \in Q^0 \} \), the graph of \( f \) is not in \( Q^0_2 \). Hence \( f \in P^0_3 - Q^0_2 \).

Slightly more intricate is the proof that \( Q^0_n - P^0_n \neq 0 \). Let \( C \in Q^0_n - P^0_n \) and let \( B \) be a set in \( Q^0_{n+1} \) such that

\[
m \notin C = \sum \}_{x \in B}(m, x, 0) + B \}.
\]
We select an arbitrary point \( m_0 \) outside \( C \) and put
\[
\begin{align*}
\delta(m) &= \varphi(m_0), \\
\eta(m) &= \varphi(m_0),
\end{align*}
\]
where \( \varphi \) and \( \eta \) are primitive recursive functions with the property that the formula \( \varphi = \varphi(m), \eta(m) \) establishes a one-one correspondence between elements of \( \varphi(m) \) and elements of \( \varphi(m) \).

The set \( R = \varphi(m) \) coincides with the set of values of the function \( \varphi \).

The graph of \( \varphi \) in \( P_{k+1}^\varphi \cdot Q_{k+1}^\varphi \), since
\[
\varphi(m) = \{ \delta_2(m_0), \delta_3(m_0) \} \not\in B,
\]
where \( \delta_2 \) and \( \delta_3 \) are primitive recursive functions with the property that the formula \( \varphi = \varphi(m), \eta(m) \) establishes a one-one correspondence between elements of \( \varphi(m) \) and elements of \( \varphi(m) \).

Let us put
\[
F = \varphi(m) \cdot (x = 0) + (x > 0) \cdot [m = \varphi(x - 1)] \cdot \prod_{1 < x < m}[m \not\in \varphi(x - 1)],
\]

\[
\text{We have then } F \in P_{k+1}^\varphi \cdot Q_{k+1}^\varphi
\]

\[
\text{and } g \text{ a recursive function such that } f(m) < g(m). \text{ Without loss of generality we may assume } B \text{ to be primitive recursive. We denote by } h \text{ the characteristic function of } B \text{ and put}
\]
\[
h(m, m, 0) = \varphi(m, m, 0),
\text{ then } h(m, m, 0) = \varphi(m, m, 0)
\]

\[
h(m, m, m, x + 1) = \varphi(m, m, m, 0) + \varphi(m, m, m, x + 1)
\]

Thus \( k \) is a primitive recursive function which vanishes everywhere except in points \( (m, m, x) \), where \( x \) is the least integer such that \( (m, m, x) \in B \).

If \( m \not\in f(m) \), then there exists an \( x \) such that \( h(m, m, x) = 1 \); no such \( x \) exists if \( m = f(m) \).

\[
r(m) = (\varphi(\sum_{x=0}^{m} k(m, m, x) = g(m))
\]

\[
\text{is (general) recursive.}
\]

If \( m \not\in f(m) \) and \( m \not\in \langle m \rangle \), then \( \sum_{x=0}^{m} k(m, m, x) = 1 \); if \( m = f(m) \), then

\[
\sum_{x=0}^{m} k(m, m, x) = 0.
\]

\[
f(m) = \langle \text{sum}\rangle = \sum_{x=0}^{m} k(m, m, x) = 0
\]
in which \((\mu m)_m[\ldots]\) denotes the least \(m\), satisfying the inequality \(m < a\) and the condition \([\ldots]\) (or 0, if no such \(m\) exists). This formula proves that \(f\) is recursive.

3. Let a set \(X \subseteq \mathbb{N}\) be such that \(\prod \sum (m, m) \in X\). A function \(f\) is called a selector of \(X\) if \(\prod \sum (m, f(m)) \in X\).

**Theorem 5.** Recursively enumerable sets possess recursive selectors.

**Proof.** Let \(X \subseteq \mathbb{N}\) and let \(B\) be a recursive set such that

\[ (m, m) \in X \iff \sum (m, m, x) \in B. \]

The function

\[ f(m) = \mu z [\sum (m, z) \in B] \]

is the required recursive selector of \(X\).

**Theorem 6.** \(g \in Q^{(0)}\) and \(g\) is not recursive, the set \(\prod \sum (m, g(m))\) is in \(Q^{(1)}\) and has no recursive selector.

This theorem follows immediately from the theorems 3 and 4.

4. Kleene [1] has constructed two disjoint recursively enumerable sets \(X, Y\), such that there is no recursive set \(Z\) satisfying the conditions

\[ XCZ, YZ = 0. \]

If in this proof we change the words “recursively enumerable” into “element of \(P^{(0)}\)” and “recursive” into “element of \(P^{(0)} \cdot Q^{(0)}\),” we obtain the proof of

**Theorem 7.** For each \(n > 0\) there are disjoint sets \(X, Y \subseteq \mathbb{N}\) such that the formulas \(X \subseteq \mathbb{N}, Y \subseteq \mathbb{N}\) are not satisfied by any set \(Z \subseteq \mathbb{N} \cdot P^{(0)} \cdot Q^{(0)}\). Theorem 7 is true for sets of class \(Q^{(0)}\). On the contrary, we shall prove

**Theorem 8.** If \(X, Y \subseteq \mathbb{N} \cdot Q^{(0)}\) and \(XY = 0\), then there is a set \(Z \subseteq \mathbb{N} \cdot P^{(0)} \cdot Q^{(0)}\) such that \(XZ \subseteq \mathbb{N} \cdot Q^{(0)}\) and \(YZ = 0\).

Since the case \(n = 0\) is evident, we assume that \(n > 0\) and denote by \(M\) and \(N\) two sets in \(P^{(0)} \cdot Q^{(0)}\) such that

\[ (m \in X) = \prod \sum (m, x) \epsilon M, \quad (m \in Y) = \prod \sum (m, x) \epsilon N. \]

It follows from \(XY = 0\) that \(R_a = (R_a - X) + (R_a - Y)\) and hence

\[ \prod \sum (m, x) \epsilon M \cdot (m, x) \epsilon N. \]

The graph of the function

\[ f(m) = \mu z [\sum (m, z) \epsilon M] \cdot (m, z) \epsilon N. \]

It may be represented in the form

\[ \prod \sum (x = f(m)) = A - B \quad \text{where} \quad A, B \subseteq \mathbb{N} \cdot P^{(0)} \cdot Q^{(0)}. \]

This follows from the equivalence

\[ x = f(m) = \prod \sum (m, x) \epsilon M \cdot N \cdot \sum (m, x) \epsilon M \cdot (m, x) \epsilon N, \]

and the observation that the sets,

\[ \prod \sum (m, x) \epsilon M \cdot N \quad \text{and} \quad \prod \sum (m, x) \epsilon M \cdot (m, x) \epsilon N, \]

belong to the classes \(P^{(0)} \cdot Q^{(0)}\) and \(Q^{(0)} \cdot P^{(0)}\) (cf. [3], theorem 3.3).

Let us put

\[ U = \prod \sum (m, f(m)) \epsilon M, \quad V = \prod \sum (m, f(m)) \epsilon N, \]

and hence \(X \subseteq \mathbb{N}, Y \subseteq \mathbb{N}\).

This means that \(X \subseteq \mathbb{N}, Y \subseteq \mathbb{N}\).

We shall prove

\[ U \subseteq \mathbb{N} \cdot P^{(0)} \cdot Q^{(0)} \]

and

\[ V \subseteq \mathbb{N} \cdot P^{(0)} \cdot Q^{(0)} \]

These equivalences prove that \(U \subseteq \mathbb{N} \cdot P^{(0)} \cdot Q^{(0)} \cdot P^{(0)} \cdot Q^{(0)}\). It follows from (5) that \(Z \subseteq \mathbb{N} \cdot P^{(0)} \cdot Q^{(0)} \). Theorem 8 is thus proved.

**References**


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