Remark. None of the constructions on the sphere which we have described above can be done in this way on the plane, because there exists no free group of isometries of the plane with more then one generator. It follows that for each pair a,b of similarities of the plane the following relation holds:

$$a^2b^2a^{-2}b^{-4}a^2b^2a^{-2}b^2a^2b^{-4}a^{-2}b^2=1 \tag{x}$$ [14].

We shall prove a certain generalization of the relation (x).

Let us take the notation $(a,b)=aba^{-1}b^{-1}$ — it is the so called commutator of the elements a and b. We have the following assertion:

(T_2) *For each four similarities of the plane φ,ψ,χ,η the following relation holds:*

$$\big((\varphi^2,\psi^2),(\chi^2,\eta^2)\big)=1.$$

(For example the relation (x) follows by the substitution $\varphi=a$, $\psi=b$, $\chi=b^{-1}$, $\eta=a$).

Proof. Let ζ be a similarity of the plane with a complex coordinate z. Then ζ^2 is a similarity without reflexion (preserving orientation), *i. e.*

$$\zeta^2(z)=a_{\zeta^2}z+b_{\zeta^2}, \tag{34}$$

where a_{ζ^2} and b_{ζ^2} are complex numbers uniquely defined by ζ (so $a_{\zeta^2}\neq 0$). Thus we have also

$$a_{\zeta^{-2}}=\frac{1}{a_{\zeta^2}}. \tag{35}$$

From (34) and (35) it follows that for each two similarities σ and τ the similarity (σ^2,τ^2) is of the form $z+b$ (where b is a complex number), *i. e.* it is a translation. The product of translations is commutative; this proves (T_2), because the commutator of commutative elements vanishes.

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[14]) This follows from a similar relation given by Sierpiński in his paper [5], p. 1.

Continuous functions in the logarithmic-power classification according to Hölder's conditions

by

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Introduction

We shall denote by $\omega(h)$ functions defined and never assuming zero for $h>0$, monotonic, non-decreasing and tending to zero for $h\rightarrow 0$. In addition we shall suppose that

$$\lim_{h\to+0}\Lambda(h)<\infty \quad \text{where} \quad \Lambda(h)=\sup_{0<t\leqslant h}\frac{t}{\omega(t)}. \tag{1}$$

As regards functions denoted in the sequel by $f(x)$ we shall always suppose that they are continuous, defined and bounded in the interval $(-\infty,+\infty)$.

Let H_ω denote the class of functions which for every x and every h [1]) *satisfy the generalized condition of Hölder*

$$|f(x+h)-f(x)|\leqslant M\omega(|h|), \tag{2}$$

where M denotes a constant dependent only on $f(x)$. We shall suppose that $\omega(h)$ satisfies the condition (1) [2]).

[1]) If condition (2) is satisfied for every h where $|h|<a$ for a certain positive constant a, then $f(x)$ will belong to class H_ω.

[2]) In the case of $\lim\Lambda(h)=\infty$ only constant functions would belong to class H_ω.

In the case of the inequality $\omega_1(h) \leqslant \omega_2(h)$, satisfied for $0<h<a$ with a certain constant a, we have $H_{\omega_1} \subset H_{\omega_2}$. This leads to the classification of functions $f(x)$ with regard to $\omega(h)$. For example, taking in (2) $\omega(h) = h^\delta |\log h|^\gamma$ we obtain a logarithmic-power scale of the classification of functions $f(x)$, which for $\gamma=0$ becomes a power scale. The meaning of this type of classification is clear, if only from the classical Jackson and Bernstein Theorems on the approximation continuous functions by polynomials.

Independently of the classification of functions with regard to (2), *i. e.* to their degree of continuity, they can be classified with regard to the singularities which they display.

We shall denote by H_ω^∞ the class of functions $f(x)$ satisfying the condition

$$(3)\qquad \overline{\lim_{h\to 0}} \frac{|f(x+h)-f(x)|}{\omega(|h|)} = \infty$$

for every x. We shall suppose that $\omega(h)$ satisfies condition (1)[3].

We note that $H_\omega^\infty \subset C - H_\omega$, where C denotes the set of all continuous functions $f(x)$, and that in the case of the inequality $\omega_1(h) \leqslant \omega_2(h)$ being satisfied for $0<h<a$ with a certain constant a we shall have $H_{\omega_1}^\infty \supset H_{\omega_2}^\infty$.

It could be asked what is the necessary and sufficient condition which $\omega_1(h)$ and $\omega_2(h)$ must satisfy in order that there exist a function $f(x)$ belonging to both H_{ω_1} and $H_{\omega_2}^\infty$. W. Orlicz gives this condition in the following form:

$$(4)\qquad \underline{\lim_{h\to +0}} \frac{\omega_2(h)}{h} \Lambda_1(h) = 0.$$

In paragraphs 2, 3 and 4 we use the results obtained by W. Orlicz [5], giving them a more simplified form.

In this paper we examine *functions $f(x)$ of type O*, by which we understand functions of the following form:

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi(b_n x),$$

where

$$a_n > 0, \qquad 0 < b_n < b_{n+1}, \qquad b_n \to \infty, \qquad \sum_{n=1}^{\infty} a_n < \infty$$

and where *$\varphi(x)$ is defined for every x, non-constant, periodic with period l and satisfies Lipschitz's condition.*

[3]) This supposition, in the case of continuous functions $f(x)$ satisfying (3) for every x, is explained in Remark 3, § 4.

We also take into account *functions of type W*, *i. e.* functions of type O for which $a_n = a^n$, $b_n = b^n$ where $0<a<1$, $ab \geqslant 1$.

Classes H_ω and H_ω^∞ are subjected to closer analysis and the sufficient conditions under which a function $f(x)$ of type O belongs to those classes are established.

For classes H_ω and H_ω^∞ in the logarithmic-power classification, *i. e.* in the case of

$$\omega(h) = h^\delta |\log h|^\gamma,$$

symbols $H(\delta,\gamma)$ and $H^\infty(\delta,\gamma)$ respectively are used. Having fixed the range of validity of the scale (the range of the values of the parameters δ and γ) for the classes $H(\delta,\gamma)$, who are proper parts of one another for different values of pairs of the parameters, we establish for the coefficients a_n, b_n the conditions under which a function $f(x)$ of type O belongs simultaneously to classes $H(\delta_1,\gamma_1)$ and $H^\infty(\delta_2,\gamma_2)$. These conditions permit the construction of various examples, and also of the universal example of a function $f(x;\delta,\gamma)$ of type O. For the values of the parameters δ_1,γ_1 arbitrarily chosen from the whole range of the logarithmic-power scale this function belongs simultaneously to classes $H(\delta_1,\gamma_1)$ and $H^\infty(\delta_2,\gamma_2)$ for any δ_2 satisfying the inequality $\delta_2 > \delta_1$, and in the case of $\delta_2 = \delta_1$ for any γ_2 satisfying the inequality $\gamma_2 < \gamma_1$. Thus, belonging to class $H(\delta_1,\gamma_1)$, this function does not simultaneously belong to any class of that scale which is a proper part of class $H(\delta_1,\gamma_1)$.

Dealing with functions of type O we shall, in certain cases, impose upon the function $\varphi(x)$ an additional condition, which we shall call condition T.

Function $g(x)$ satisfies condition T if there exists, for every x, such a number h_x of constant (independent of x) absolute value h^* that the inequality

$$|g(x+h_x) - g(x)| \geqslant d > 0$$

is satisfied for every x with a certain constant d.

Lemmata 2, 3 give the sufficient conditions under which the continuous and periodic function $g(x)$ satisfies condition T.

Condition T permits the effective use of a general method for the construction of examples of functions showing for a given degree of continuity (belonging to H_ω) a certain degree of singularity (belonging to $H_{\omega_1}^\infty$). The efficiency of the method is obvious when we apply it to the case of continuous functions nowhere possessing a derivative[4]). Results can be obtained in this way by a general method without a detailed analysis of function $\varphi(x)$.

[4]) Many results in this field are quoted by Knopp [4].

Besides the notation introduced above:

$$f(x),\quad \varphi(x),\quad \omega(h),\quad \Lambda(h),\quad \mathrm{H}_\omega,\quad \mathrm{H}_\omega^\infty,\quad \mathrm{H}(\delta,\gamma),\quad \mathrm{H}^\infty(\delta,\gamma),$$

we repeatedly use in this paper the following symbols:

a_n, b_n coefficients used in the definition of a function of type O,

a, b coefficients used in the definition of a function of type W,

K Lipschitz's constant of function $\varphi(x)$,

D oscillation of function $\varphi(x)$ in the interval $0 \leqslant x < l$, where l is the period of function $\varphi(x)$,

d, h^* constants appearing with condition T for the function which satisfies it,

r, s constants whose meaning has been defined in Lemma 1 (§ 1) for function $g(u)$. In the next paragraphs they retain that meaning for functions $\varphi(x)$.

§ 1. Lemmata

LEMMA 1. *Let $g(u)$ be a continuous function, periodic with period l_0 but non-constant, defined for every value u. For the function $g(u)$ we can determine two numbers r, s, satisfying the inequality*

$$0 < r \leqslant s \leqslant l_0,$$

with the following property: for every u there exists a number h_u, satisfying

$$r \leqslant |h_u| \leqslant s,$$

such that

$$|g(u+h_u) - g(u)| \geqslant \frac{D_0}{2},$$

where D_0 is the oscillation of function $g(u)$ in the interval $\langle 0, l_0\rangle$.

Proof. We shall denote by $u^{\min}, u^{\max}, u^{\mathrm{m}}$ all those points in $\langle 0, l_0\rangle$ for which $g(u)$ takes, respectively, the following values:

$$g(u^{\min}) = \min_{0 \leqslant u < l_0} g(u); \qquad g(u^{\max}) = \max_{0 \leqslant u < l_0} g(u);$$

$$g(u^{\mathrm{m}}) = \frac{1}{2}\Big(\max_{0 \leqslant u < l_0} g(u) + \min_{0 \leqslant u < l_0} g(u)\Big).$$

We shall denote by $E^{\min}$, $E^{\max}$ and E^{m} the sets of all points $u^{\min}$, $u^{\max}$ and u^{m} respectively; we shall denote by δ' and δ'' the distance from the set E^{m} of the sets $E^{\max}$ and $E^{\min}$ respectively.

We assume that

$$r = \min(\delta', \delta'').$$

We shall denote by E_1 the set of those values of u in $\langle 0, l_0\rangle$ for which $g(u) \leqslant g(u^{\mathrm{m}})$, and by E_2 the set of those values of u in $\langle 0, l_0\rangle$ for which $g(u) \geqslant g(u^{\mathrm{m}})$. Denoting by δ_1 the upper bound of the distances of points of the set E_1 from the set $E^{\max}$, and by δ_2 the upper bound of the distances of points of the set E_2 from the set $E^{\min}$, we take

$$s = \max(\delta_1, \delta_2).$$

If

$$g(u) \leqslant g(u^{\mathrm{m}}) < g(u^{\max}),$$

then, denoting by $u_0^{\max}$ the point of $E^{\max}$ nearest to the point u, we write

$$h_u = u_0^{\max} - u,$$

and if

$$g(u) > g(u^{\mathrm{m}}) > g(u^{\min}),$$

then, denoting by $u_0^{\min}$ the point of $E^{\min}$ nearest to the point u, we write

$$h_u = u_0^{\min} - u.$$

It is easy to verify that all the conditions of the Lemma are satisfied in this manner; thus Lemma 1 is proved.

LEMMA 2. *Let $g(u)$ be a continuous function, periodic with period l_0 but non-constant, defined for every u, and let it have a derivative which is not defined or becomes zero at most at a finite number of points. We shall suppose that the derivative $g'(u)$ satisfies the following conditions:*

1\. *In the interval $\langle 0, l_0\rangle$ there exist $(n+1)$ points u_k $(k = 1, 2, \ldots, n+1)$ in whose one-side neighbourhoods the function $|g'(u)|$ is monotonic: namely it is non-decreasing in the right-hand neighbourhood of the point u_k, Δu_k^+ (whose length will be denoted by δu_k^+), and non-increasing in the left-hand neighbourhood, Δu_k^-, (whose length will be denoted by δu_k^-). Let $u_1 = 0$; $u_{n+1} = l_0$.*

2\. *Let the closed interval Δ_k^+ (of the length δ_k^+) adjoining the right-hand neighbourhood Δu_k^+ on the right and the closed interval Δ_k^- (of the length δ_k^-) adjoining the left-hand neighbourhood Δu_k^- on the left be such that the values which the function $|g'(u)|$ assumes in them are not smaller than the values of that function in the neighbourhoods adjoining those intervals.*

Let us suppose that $\delta u_k^+ \leqslant \delta_k^+$ and $\delta u_k^- \leqslant \delta_k^-$.

Suppose further that the above mentioned intervals and neighbourhoods have no common points other than, at most, the boundary points of intervals $\Delta_k^+, \Delta_{k+1}^-$, and that their sum together with the set of all points u_k completely covers the interval $\langle 0, l_0\rangle$.

3\. *We shall suppose that $g'(u)$ becomes zero at most at points u_k, and that it is undefined at most at points u_k or at most at the boundary points which are simultaneously common to Δ_k^+ and Δ_{k+1}^-.*

Under these conditions the function $g(u)$ satisfies condition T since for every u we can choose such an h_u with a constant absolute value h^* for each u that we shall have

$$|g(u+h_u)-g(u)| \geqslant qh^* = d.$$

Numbers h^*, q can be chosen as follows:

Let h^* be an arbitrary positive number satisfying the inequality

$$h^* \leqslant \frac{1}{2} \min_{k=1,2,\ldots,n} (\delta u_k^+ + \delta_k^+;\ \delta u_k^- + \delta_k^-). \tag{5}$$

Having chosen the values of h^* we determine q. We decrease the neighbourhoods of points u_k (mentioned in 1 and in 2) so that the length of each does *not exceed* h^*, thus increasing simultaneously the adjoining intervals. For these new neighbourhoods and intervals we retain the former notation. Inequality (5) remains valid. We take

$$q = \min_{k=1,2,\ldots,n} \left(\frac{|g(u_k+\delta u_k^+)-g(u_k)|}{\delta u_k^+}, \frac{|g(u_k-\delta u_k^-)-g(u_k)|}{\delta u_k^-} \right)^{5)}. \tag{6}$$

For such h^* and q and the newly defined neighbourhoods and intervals we shall prove Lemma 2.

Proof. For the proof we shall consider Δu_k^+, the right-hand neighbourhood of u_k, of the length δu_k^+, and the adjoining closed interval Δ_k^+, of the length δ_k^+, in which $g'(u)$ satisfies the conditions of the assumption and, as follows from those conditions, has the same sign. We shall have

$$\delta u_k^+ \leqslant h^*; \qquad \delta_k^+ \geqslant h^*; \qquad \delta u_k^+ + \delta_k^+ \geqslant 2h^*.$$

We shall show that, for u_k and for every point u contained in $\Delta u_k^+ + \Delta_k^+$, we can choose such an h_u that $|h_u| = h^*$ and that

$$\left|\frac{g(u+h_u)-g(u)}{h_u}\right| \geqslant \frac{|g(u_k+\delta u_k^+)-g(u_k)|}{\delta u_k^+}. \tag{7}$$

Let us consider the following cases:

1° $u=u_k$ and $\delta u_k^+ = h^*$. Here we take $h_u = h^* = \delta u_k^+$. In this case inequality (7) is obviously satisfied.

2° $u_k < u < u_k+\delta u_k^+$. Here we take $h_u = h^*$.

a. We note that

$$\frac{g(u_k+\delta u_k^+)-g(u_k)}{\delta u_k^+}$$

[5] For $k=1$ the left-hand neighbourhood of point u_1 and its adjoining interval should be replaced by the respective neighbourhood of point $u_{n+1}=l_0$ and the adjoining interval.

has a value intermediate between

$$\frac{g(u)-g(u_k)}{u-u_k} = g'(\eta_1) \quad \text{and} \quad \frac{g(u_k+\delta u_k^+)-g(u)}{(u_k+\delta u_k^+)-u} = g'(\eta_2),$$

where $\eta_1 < \eta_2$. In view of the function $|g'(u)|$ being monotonic it follows hence that

$$\left|\frac{g(u_k+\delta u_k^+)-g(u_k)}{\delta u_k^+}\right| \leqslant \left|\frac{g(u_k+\delta u_k^+)-g(u)}{(u_k+\delta u_k^+)-u}\right|. \tag{8}$$

b. Let us now consider three points satisfying the inequality

$$u < u_k+\delta u_k^+ < u+h_u.$$

We note that

$$\frac{g(u+h_u)-g(u)}{h_u}$$

has a value intermediate between

$$\frac{g(u_k+\delta u_k^+)-g(u)}{(u_k+\delta u_k^+)-u} = g'(\eta_3) \quad \text{and} \quad \frac{g(u+h_u)-g(u_k+\delta u_k^+)}{(u+h_u)-(u_k+\delta u_k^+)} = g'(\eta_4),$$

where $\eta_3 < u_k+\delta u_k^+ < \eta_4$. Hence $|g'(\eta_3)| \leqslant |g'(\eta_4)|$ and thus

$$\left|\frac{g(u_k+\delta u_k^+)-g(u)}{(u_k+\delta u_k^+)-u}\right| \leqslant \left|\frac{g(u+h_u)-g(u)}{h_u}\right|.$$

If we join the above inequality with inequality (8), inequality (7) is proved for case 2°.

3° $u=u_k$ and $\delta u_k^+ < h^*$. We then take $h_u = h^*$. This case can obviously be reduced to part b of case 2°, which we have already considered.

4° $u_k+\delta u_k^+ \leqslant u \leqslant u_k+\delta u_k^+ + \delta_k - h^*$. We then take $h_u = h^*$. In this case we obtain

$$\left|\frac{g(u+h_u)-g(u)}{h_u}\right| = |g'(\eta_6)| \geqslant |g'(\eta_5)| = \left|\frac{g(u_k+\delta u_k^+)-g(u_k)}{\delta u_k^+}\right|,$$

considering that $\eta_5 < u_k+\delta u_k^+ < \eta_6 < u_k+\delta u_k^+ + \delta_k^+$.

5° $u_k+\delta u_k^+ + \delta_k^+ - h^* < u \leqslant u_k+\delta u_k^+ + \delta_k^+$. We take $h_u = -h^*$.

In view of

$$u+h_u > u_k+\delta u_k + \delta_k^+ - 2h^* \geqslant u_k,$$

we can reduce this case to one of those considered above, changing only the roles of points u and $u+h$.

In this manner we have exhausted all the possible cases and inequality (7) is thus fully proved.

If we applied similar reasoning to the left-hand neighbourhood of u_k and the adjoining interval Δ_k^-, we should obtain

$$\left|\frac{g(u+h_u)-g(u)}{h_u}\right| \geqslant \left|\frac{g(u_k-\delta u_k^-)-g(u_k)}{\delta u_k^-}\right|.$$

From the above inequality and from inequality (7) follows the truth of Lemma 2.

We note that the conditions of the assumption of Lemma 2 are satisfied by every periodic function $g(u)$ whose graph is a continuous polygonal line, composed of a finite number of segments none of which is parallel to axis u. In the case of such a function we can take as u_k the abscissa of every second vertex of the polygonal line. If we denote the lengths of the projections of these segments on axis u by δ_i and the absolute values of their angular coefficients by q_i, it is sufficient to take

$$|h_u| = h^* = \frac{1}{2}\min_i \delta_i \quad \text{and} \quad q = \min_i q_i.$$

In any case it is immediately obvious that every such function satisfies condition T.

Lemma 3. *If a periodic function $g(u)$ (with a period l_0) has continuous first and second derivatives, and these derivatives have a finite number of zero places in the interval $\langle 0, l_0\rangle$ then the function $g(u)$ satisfies condition* T.

Proof. We denote the zero places of the function $g'(u)$ by u_k $(k=1,2,\ldots,n+1)$ ordered so that $u_1<u_2<\ldots<u_{n+1}$ and $u_1=0$, $u_{n+1}=l_0$.

From the conditions of the assumption it follows that there exist a certain right-hand neighbourhood of points u_k in which $|g'(u)|$ increases and a left-hand neighbourhood in which $|g'(u)|$ decreases. Let us take any $h^*>0$ provided it satisfies the inequality

$$h^* \leqslant \frac{1}{3}\min_{k=1,2,\ldots,n}(u_{k+1}-u_k).$$

Having deleted from intervals (u_k, u_{k+1}) the above mentioned one-side neighbourhoods of points u_k, u_{k+1}, we obtain closed intervals Δ_k, in which the function $|g'(x)|$ assumes a certain minimum value M_k not equal to zero. We shall now decrease the left-hand neighbourhoods of u_{k+1} and the right-hand neighbourhoods of u_k so that

a) the value of function $|g'(u)|$ in each of these neighbourhoods will not exceed the number M_k,

b) the length of each of these neighbourhoods will not exceed h^*, and thus the length of the interval Δ_k will not be smaller than h^*.

Denoting the boundary points of the newly obtained right-hand neighbourhood of the points u_k by $u_k+\delta u_k^+$ and of the left-hand neighbourhood by $u_k-\delta u_k^-$, let us define q as in (6).

For numbers h^* and q thus defined all the conditions of Lemma 2 are satisfied, *i. e.* Lemma 3 is proved,

Let it be noted that Lemmata 1, 2 and 3 will be applied in the sequel to the function $\varphi(u)$ defined in the Introduction. In that case the symbols $g(u)$, D_0, l_0 used in the Lemmata should be replaced by the symbols $\varphi(u)$, D, l.

§ 2. Sufficient conditions under which a function $f(x)$ of type O belongs to class H_ω

Theorem 1. *If the coefficients a_n, b_n of a function $f(x)$ of type O satisfy the condition*

$$\sum_{n=1}^{\infty} a_n b_n \Lambda\left(\frac{l}{b_n}\right) < \infty, \tag{9}$$

the function $f(x)$ belongs to class H_ω.

Proof. Let us write

$$\frac{b_n h}{l} - \left[\frac{b_n h}{l}\right] = \theta,$$

and suppose that $h>0$. We find that

$$\left|\frac{\varphi\big(b_n(x+h)\big)-\varphi(b_n x)}{\omega(h)}\right| = \left|\frac{\varphi(b_n x+l\theta)-\varphi(b_n x)}{l\theta}\cdot\frac{l\theta}{\omega(h)}\right| \leqslant K\frac{l\theta}{\omega(h)}.$$

In the case of $0<h<l/b_n$ we shall have $l\theta = b_n h$, and therefore

$$\frac{l\theta}{\omega(h)} \leqslant b_n \Lambda\left(\frac{l}{b_n}\right),$$

and in the case $h \geqslant l/b_n$ we shall have

$$\frac{l\theta}{\omega(h)} \leqslant \frac{l}{\omega\left(\frac{l}{b_n}\right)} \leqslant b_n \Lambda\left(\frac{l}{b_n}\right).$$

In both cases, *i. e.* for every $h>0$, we obtain

$$\frac{\big|\varphi\big(b_n(x+h)\big)-\varphi(b_n x)\big|}{\omega(|h|)} \leqslant K b_n \Lambda\left(\frac{l}{b_n}\right).$$

Since the above inequality remains valid also for $h<0$ (which is obvious if we substitute $x-h$ for x in it), therefore Theorem 1 is proved.

Remark. We note that in view of the inequality

$$b_n\Lambda\left(\frac{l}{b_n}\right)\geqslant\frac{l}{\omega\left(\frac{l}{b_n}\right)}>1,$$

true for $n\geqslant N$, from condition (9) follows the convergency of the series $\sum\limits_{n=1}^{\infty}a_n$.

THEOREM 2. *If the coefficients a_n,b_n of a function $f(x)$ of type O satisfy, for every $n\geqslant N$, the inequality*

$$K\Lambda\left(\frac{a}{b_n}\right)\sum_{i=1}^{n}a_ib_i+\frac{D}{\omega\left(\frac{a}{b_{n+1}}\right)}\sum_{i=n+1}^{\infty}a_i<C \tag{10}$$

with certain positive constants a,C independent of n, then $f(x)$ belongs to class H_ω.

Proof. In the case of $a/b_{n+1}<|h|\leqslant a/b_n$ $(n\geqslant N)$ the following inequalities are true

$$\sum_{i=1}^{n}a_ib_i\left|\frac{\varphi\big(b_i(x+h)\big)-\varphi(b_ix)}{b_ih}\cdot\frac{|h|}{\omega(|h|)}\right|\leqslant K\Lambda\left(\frac{a}{b_n}\right)\sum_{i=1}^{n}a_ib_i,$$

$$\sum_{i=n+1}^{\infty}a_i\frac{\left|\varphi\big(b_i(x+h)\big)-\varphi(b_ix)\right|}{\omega(|h|)}\leqslant\frac{D}{\omega\left(\frac{a}{b_{n+1}}\right)}\sum_{i=n+1}^{\infty}a_i.$$

And in the case of $h>a/b_N$, we have

$$\frac{|f(x+h)-f(x)|}{\omega(|h|)}<\frac{C_1}{\omega\left(\frac{a}{b_N}\right)}.$$

If we join the two cases, the theorem is proved.

§ 3. Sufficient conditions under which a function $f(x)$ of type O belongs to class H_ω^∞

THEOREM 3. *If a function $f(x)$ is of type O and its coefficients a_n,b_n satisfy the following conditions:*

$$\overline{\lim_{n\to\infty}}\frac{a_n}{\omega\left(\frac{s}{b_n}\right)}=\infty, \tag{11}$$

$$\overline{\lim_{n\to\infty}}\frac{\omega\left(\frac{s}{b_n}\right)}{\omega\left(\frac{r}{b_n}\right)}\cdot\frac{1}{a_nb_n}\sum_{i=1}^{n-1}a_ib_i<\theta\frac{D}{2sK}, \tag{12}$$

$$\overline{\lim_{n\to\infty}}\frac{\omega\left(\frac{s}{b_n}\right)}{\omega\left(\frac{r}{b_n}\right)}\cdot\frac{1}{a_n}\sum_{i=n+1}^{\infty}a_i<\frac{1-\theta}{2}, \tag{13}$$

where $0<\theta<1$, then $f(x)$ belongs to class H_ω^∞.

Proof. We shall carry out the proof applying Lemma 1 to the function $\varphi(u)$, symbols $g(u)$, l_0, D_0 being replaced by $\varphi(u)$, l, D. We substitute

$$u=b_nx,\qquad h_u=b_nh_x.$$

On the basis of Lemma 1 we can find two numbers r,s, satisfying the inequality $0<r\leqslant s\leqslant l$, such that for a certain fixed n there exists for every x a number h_x for which, with every x, the following two conditions are satisfied:

$$\frac{r}{b_n}\leqslant|h_x|\leqslant\frac{s}{b_n},$$

$$\left|\varphi\big(b_n(x+h_x)\big)-\varphi(b_nx)\right|\geqslant\frac{D}{2}. \tag{14}$$

Hence we obtain the inequality

$$a_n\frac{\left|\varphi\big(b_n(x+h_x)\big)-\varphi(b_nx)\right|}{\omega(|h_x|)}\geqslant\frac{D}{2}\cdot\frac{a_n}{\omega\left(\frac{s}{b_n}\right)}, \tag{15}$$

and moreover the inequalities

$$\sum_{i=1}^{n-1}a_i\frac{\left|\varphi\big(b_i(x+h_x)\big)-\varphi(b_ix)\right|}{b_i|h_x|}\cdot\frac{b_i|h_x|}{\omega(|h_x|)}\leqslant\frac{Ks}{b_n\omega\left(\frac{r}{b_n}\right)}\sum_{i=1}^{n-1}a_ib_i, \tag{16}$$

$$\sum_{i=n+1}^{\infty}a_i\frac{\left|\varphi\big(b_i(x+h_x)\big)-\varphi(b_ix)\right|}{\omega(|h_x|)}\leqslant\frac{D}{\omega\left(\frac{r}{b_n}\right)}\sum_{i=n+1}^{\infty}a_i. \tag{17}$$

Considering the last three inequalities we obtain

$$\frac{|f(x+h_x)-f(x)|}{\omega(|h_x|)}\geqslant\frac{a_n}{\omega\left(\frac{s}{b_n}\right)}\left(\frac{D}{2}-\frac{\omega\left(\frac{s}{b_n}\right)}{\omega\left(\frac{r}{b_n}\right)}\cdot\frac{Ks}{a_nb_n}\sum_{i=1}^{n-1}a_ib_i-\frac{\omega\left(\frac{s}{b_n}\right)}{\omega\left(\frac{r}{b_n}\right)}\cdot\frac{D}{a_n}\sum_{i=n+1}^{\infty}a_i\right). \tag{18}$$

Since the above inequality is true for every n and a suitably chosen h_x, therefore, conditions (11), (12) and (13) being satisfied, the function $f(x)$ will belong to class H_ω^∞.

THEOREM 4. *If the function $\varphi(x)$ satisfies condition T, and the function $f(x)$ is of type O and its coefficients a_n, b_n satisfy the following conditions:*

$$\overline{\lim_{n\to\infty}} \frac{a_n}{\omega\left(\frac{h^*}{b_n}\right)} = \infty, \tag{19}$$

$$\overline{\lim_{n\to\infty}} \frac{1}{a_n b_n} \sum_{i=1}^{n-1} a_i b_i < \theta \frac{d}{Kh^*}, \tag{20}$$

$$\overline{\lim_{n\to\infty}} \frac{1}{a_n} \sum_{i=n+1}^{\infty} a_i < (1-\theta)\frac{d}{D}, \tag{21}$$

where $0<\theta<1$, then the function $f(x)$ belongs to class H_ω^∞.

Proof. The proof immediately results from the proof of Theorem 3. It is sufficient, for this end, to take $r=s=h^*$ and to replace expression $D/2$ by d in (14) and (15) and consequently in (18).

THEOREM 5. *Suppose that $\varphi(x)$ satisfies T, and function $f(x)$ is of type W, where $ab>1$. If condition (19), in which $a_n=a^n$, $b_n=b^n$, and conditions*

$$\frac{1}{ab-1} < \theta \frac{d}{Kh^*}, \tag{22}$$

$$\frac{a}{1-a} < (1-\theta)\frac{d}{D}, \tag{23}$$

where $0<\theta<1$, are satisfied, then $f(x)$ belongs to H_ω^∞.

The above Theorem immediately results from Theorem 4.

A particular case of Theorem 5 is obtained when, applying Lemma 2 or 3, we choose h^* in such a manner that

$$h^* = \frac{m}{k} l,$$

where m, k are natural numbers and b is a multiple of number k. In this case, the left side of inequality (17) becomes zero. In connection with this, condition (23) is omitted and in (22) we can take $\theta=1$. Thus we can formulate the theorem as follows:

THEOREM 5*. *Let the function $\varphi(x)$ satisfy T and let*

$$h^* = \frac{m}{k} l,$$

where m, k are natural numbers. If the function $f(x)$ is periodic, of type W, where $ab>1$, b is a multiple of k and conditions (19) and

$$\frac{1}{ab-1} < \frac{d}{Kh^*} \tag{24}$$

are satisfied, then the function $f(x)$ belongs to class H_ω^∞.

§ 4. The necessary and sufficient condition of the existence of a function $f(x)$ belonging simultaneously to classes H_{ω_1} and $H_{\omega_1}^\infty$

THEOREM 6. *If a function $f(x)$ belongs to class H_{ω_1}, and the condition*

$$\underline{\lim_{h\to+0}} \frac{\omega_2(h)}{h} \Lambda_1(h) > 0, \tag{25}$$

is satisfied, then $f(x)$ belongs simultaneously to class H_{ω_2} [6].

Remark. If we take $\omega_2(h)=h$ in Theorem 6, the following conclusion results:

The necessary condition under which class H_{ω_1} is not contained in class $H(1,0)$ (that is, in the class of functions satisfying Lipschitz's condition) is that

$$\lim_{h\to+0} \frac{h}{\omega_1(h)} = 0.$$

THEOREM 7. *If condition (4) is satisfied, then, with the function $\varphi(x)$ given beforehand, we can so choose the coefficients a_n, b_n of a function $f(x)$ of type O that this function will belong simultaneously to H_{ω_1} and $H_{\omega_2}^\infty$.*

Proof. For the proof it is sufficient, (applying Theorems 1 and 3) to choose, with a given $\varphi(x)$, the coefficients a_n, b_n, in such a way that for $\omega(h)=\omega_1(h)$ condition (9) is satisfied, and for $\omega(h)=\omega_2(h)$ conditions (11), (12), (13) are satisfied. It will be seen that for b_n we can also choose an integer, so that the function $f(x)$ might be periodic.

Instead of the above-mentioned conditions we shall consider the following

$$\sum_{n=1}^{\infty} a_n b_n \Lambda_1\left(\frac{l}{b_n}\right) < \infty, \tag{26}$$

[6]) The proof of this theorem can be found on p. 22 of [5].

(27) $$\lim_{n\to\infty}\frac{a_n}{\omega_2\left(\dfrac{l}{b_n}\right)}=\infty,$$

(28) $$\lim_{n\to\infty}\frac{\omega_2\left(\dfrac{l}{b_n}\right)}{\omega_2\left(\dfrac{r}{b_n}\right)}\cdot\frac{1}{a_nb_n}\sum_{i=1}^{n-1}a_ib_i=0,$$

(29) $$\lim_{n\to\infty}\frac{\omega_2\left(\dfrac{l}{b_n}\right)}{\omega_2\left(\dfrac{r}{b_n}\right)}\cdot\frac{1}{a_n}\sum_{i=n+1}^{\infty}a_i=0.$$

Let us write

$$k_n=\frac{a_n}{\omega_2\left(\dfrac{l}{b_n}\right)}$$

and let, for example, $k_1=1$, $b_1=2$. Defining by induction, let us assume that we have already defined the coefficients k_i, b_i for $i<n$. Let us choose k_n so that

$$\frac{1}{k_n}\sum_{i=1}^{n-1}a_ib_i<\frac{1}{n}\qquad\text{and}\qquad k_n\geqslant n$$

and let $b_n>b_{n-1}$ and $b_n>n$.

In view of

$$\frac{\omega_2\left(\dfrac{l}{b_n}\right)}{\omega_2\left(\dfrac{r}{b_n}\right)}\cdot\frac{1}{a_nb_n}\sum_{i=1}^{n-1}a_ib_i<\frac{1}{r}\,\Lambda_2\left(\frac{l}{b_n}\right)\cdot\frac{1}{n}\to 0$$

not only condition (27), but also condition (28) is satisfied.

Suppose that b_n satisfies the additional inequality

(30) $$k_n\omega_2\left(\frac{l}{b_n}\right)<\frac{1}{2^{n-1}}k_{n-1}\omega_2\left(\frac{r}{b_{n-1}}\right).$$

It is obvious that condition (29) is satisfied and that the function $f(x)$ is of type O.

Thus, without using condition (4), we have constructed a function of type O (periodic, when b_n is an integer), which, in accordance with Theorem 3, belongs to $H^{\infty}_{\omega_2}$.

Now let us suppose that condition (4) is satisfied. According to (4), there exists a number $h_0>0$ so small that the inequality

(31) $$\frac{\omega_2(h_0)}{h_0}\Lambda_1(h_0)<\frac{1}{2^nk_n}\cdot\frac{1}{l}$$

is satisfied, and so small that if we take

$$b_n=\left[\frac{l}{h_0}\right]+1,$$

the relations $b_n>n$, $b_n>b_{n-1}$ and (30) are satisfied. Moreover, if we take $0<h_0<l/2$, then, in view of

$$b_n-1\leqslant\frac{l}{h_0}<b\qquad\text{and}\qquad\frac{b_n}{l}<\frac{2}{h_0},$$

we obtain

$$\left(\frac{l}{b_n}\right)^{-1}\omega_2\left(\frac{l}{b_n}\right)\Lambda_1\left(\frac{l}{b_n}\right)<2\frac{\omega_2(h_0)}{h_0}\Lambda_1(h_0).$$

Joining the above inequality with inequality (31) we obtain

$$a_nb_n\Lambda_1\left(\frac{l}{b_n}\right)<\frac{1}{2^{n-1}}.$$

Thus condition (26) is satisfied, *i. e.*, the Theorem is proved.

From joining Theorems 6 and 7 it follows that condition (4) is the necessary and sufficient condition of the existence of a function $f(x)$ (of type O), belonging simultaneously to classes $\mathrm{H}^{\infty}_{\omega_1}$ and $\mathrm{H}^{\infty}_{\omega_2}$.

Remark 1. Since in constructing the coefficients a_n, b_n in such a way that conditions (27), (28) and (29) be satisfied and the function $f(x)$ be of type O we have not used condition (4), therefore it follows hence, in view of the contents of Theorem 3, that none of the classes $\mathrm{H}^{\infty}_{\omega}$ is empty as it is possible to construct for each of them (the function $\varphi(x)$ being given beforehand) a function of type O belonging to that class [7]).

Remark 2. Suppose that class H_{ω_1} does not belong to class $\mathrm{H}(1,0)$: referring to the remark concerning Theorem 6, we obtain $\lim\limits_{h\to+0}\Lambda_1(h)=0$. Taking in turn $\omega_2(h)=h$ we see that condition (4) is satisfied. Hence it follows that in each class H_{ω} not contained in $\mathrm{H}(1,0)$ there exists a function $f(x)$ of type O which belongs to $\mathrm{H}^{\infty}(1,0)$ and thus has nowhere a finite derivative.

Remark 3. If we assumed that $\lim\limits_{h\to+0}\Lambda(h)=\infty$ then the function defined as $f(x)=|x|$ for $0\leqslant x<l_0$, $f(x)=f(x+l_0)$ $(l_0>0)$ for the remaining x, would already belong to class $\mathrm{H}^{\infty}_{\omega}$, *i. e.* it would have a singularity defined by formula (3) for every x. That is why we have made assumption (1) also in the classification of functions according to condition (3).

[7]) Examples of a function belonging to class $\mathrm{H}^{\infty}_{\omega}$ were given by Faber ([2] and [3]), also by Auerbach and Banach [1], by Ruziewicz [6]. In the example given by Ruziewicz the function is of type O, where $\varphi(x)=\cos x$.

§ 5. The logarithmic-power scale

Suppose that $\omega_2(h) \leqslant \omega_1(h)$ for $0 < h < a$ with a certain a; in that case $H_{\omega_2} \subset H_{\omega_1}$. Now, if $\omega_1(h)$ and $\omega_2(h)$ satisfied condition (4), then class H_{ω_2} would be a proper part of class H_{ω_1}. If, on the other hand, $\omega_1(h)$ and $\omega_2(h)$ did not satisfy (4), then, in view of Theorem 6, classes H_{ω_1} and H_{ω_2} would coincide. Hence is it obvious that (4) can be used to establish the scale of classification of functions $f(x)$.

Suppose that a function $\omega(h;\delta,\gamma)$, with fixed values of the parameters δ,γ, has the same properties as the function $\omega(h)$. Moreover, let classes $H_{\omega(h;\delta_1,\gamma_1)}$ and $H_{\omega(h;\delta_1,\gamma_2)}$ be proper parts of one another according to whether $\gamma_1 > \gamma_2$ or $\gamma_1 < \gamma_2$, and also let classes $H_{\omega(h;\delta_1,\gamma_1)}$ and $H_{\omega(h;\delta_2,\gamma_2)}$ be proper parts of one another independently of the values of γ_1 and γ_2 and according to whether $\delta_1 > \delta_2$ or $\delta_1 < \delta_2$. In that case we shall say that we have established the scale of classes H_ω according to the functions $\omega(h;\delta,\gamma)$.

Now let

$$\omega(h) = h^\delta |\log h|^\gamma \qquad (\delta > 0, \text{ and for } \delta = 0,\ \gamma < 0)$$

for $0 < h \leqslant a$. In this interval, with a suitable choice of number a, the function $\omega(h)$ is increasing; for $h > a$ let us define it in such a way that it be monotonic, non-decreasing [8]). The consideration of condition (4) in the case of function $\omega(h)$ defined in this way leads to the establishment of a logarithmic-power scale, and in the case of $\gamma = 0$ to the establishment of a power scale.

For classes H_ω and H_ω^∞ obtained in this manner we use the symbols $H(\delta,\gamma)$ and $H^\infty(\delta,\gamma)$. Class $H(\delta_2,\gamma_2)$ is a proper part of class $H(\delta_1,\gamma_1)$ if $\delta_1 < \delta_2$, and in the case of $\delta_1 = \delta_2$ if $\gamma_1 > \gamma_2$. Thus parameters δ,γ can assume the following values: $0 \leqslant \delta \leqslant 1$; $\gamma \geqslant 0$ if $\delta = 1$, $\gamma < 0$ if $\delta = 0$, arbitrary γ if $0 < \delta < 1$.

Applying (9) and (10) in the case of a logarithmic-power scale we obtain, on the basis of Theorems 1 and 2, the following theorem:

THEOREM 8. *A function $f(x)$ of type O belongs to class $H(\delta,\gamma)$ if the coefficients a_n, b_n of that function satisfy one of the following two inequalities:*

(32) $$\sum_{n=1}^{\infty} a_n b_n^\delta (\log b_n)^{-\gamma} < \infty \text{ [9]})$$

or for every $n \geqslant N$

(33) $$K b_n^{\delta-1} (\log b_n)^{-\gamma} \sum_{i=1}^{n} a_i b_i + D b_{n+1}^\delta (\log b_{n+1})^{-\gamma} \sum_{i=n+1}^{\infty} a_i < C,$$

where C is a constant independent of n.

[8]) For this purpose it is sufficient to take $a = e^{-\gamma/\delta}$ in the case of $\gamma > 0$ and to take *e. g.* $a = 1/2$ in the case of $\gamma < 0$; in both cases $\omega(h) = \omega(a)$ for $h > a$.

[9]) For simplicity of notation we shall continue to assume $b_n \geqslant 1$.

Before formulating Theorem 3 in application to the logarithmic-power scale, we note that in the case of that scale, *i. e.* if $\omega(h) = h^\delta |\log h|^\gamma$, function $h/\omega(h)$ is increasing for sufficiently small h $(h > 0)$. Hence, for sufficiently large n, we can replace $\omega(r/b_n)$, appearing on the right side of inequality (16), by $\omega(s/b_n)$, and consequently we can do the same in condition (12), simplifying it in this manner. After this remark, taking into account conditions (11) and (13), we formulate the Theorem as follows:

THEOREM 9. *In order that a function $f(x)$ of type O belongs to class $H^\infty(\delta,\gamma)$ it is sufficient that the coefficients a_n, b_n of that function satisfy the following relations:*

(34) $$\lim_{n\to\infty} a_n b_n^\delta (\log b_n)^{-\gamma} = \infty,$$

(35) $$\lim_{n\to\infty} \frac{1}{a_n b_n} \sum_{i=1}^{n-1} a_i b_i < \theta \frac{D}{2sK},$$

(36) $$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=n+1}^{\infty} a_i < \frac{1-\theta}{2} \left(\frac{r}{s}\right)^\delta,$$

where $0 < \theta < 1$.

We shall mention the following particular cases of Theorem 9:

a. Condition (35) in the theorem can be replaced in particular by condition

(37) $$\lim_{n\to\infty} \frac{a_{n-1} b_{n-1}}{a_n b_n} = 0.$$

Condition (36) can be replaced in particular by condition

(38) $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$$

b. If the function $\varphi(x)$ satisfies condition T, then relations (35) and (36) in Theorem 9 can be replaced by (20) and (21) respectively.

c. If the function $f(x)$ is of type W, where $ab > 1$, then relations (34), (35), (36) in Theorem 9 take the following form:

(39) $$(ab^\delta)^n (n \log b)^{-\gamma} \to \infty,$$

(40) $$\frac{1}{ab-1} < \theta \frac{D}{2sK},$$

(41) $$\frac{a}{1-a} < \frac{1-\theta}{2} \left(\frac{r}{s}\right)^\delta.$$

d. If the function $\varphi(x)$ satisfies condition T and the function $f(x)$ is of type W, where $ab > 1$, then (34), (35), (36) in Theorem 9 can be replaced by (39), (22), (23) respectively.

If $h^* = ml/k$ and b is a multiple of k, then, in view of Theorem 5*, conditions (22) and (23) are reduced to one condition (24).

In the relations of Theorem 9 let us replace in (34), (36) or in (39), (41) the symbols δ, γ by δ_1, γ_1. Applying simultaneously Theorems 8 and 9 and, if necessary, particular cases of the latter, we obtain the sufficient conditions under which a function $f(x)$ of type O belongs simultaneously to classes $H(\delta,\gamma)$ and $H^\infty(\delta_1,\gamma_1)$. We shall use them in the case where $f(x)$ is of type W.

Theorem 10. *If a function $f(x)$ is of type W, then it belongs to one of the classes $H(\delta,0)$, where $\delta>0$ and where*

a. *in the case of $ab>1$, δ is defined by the formula:*

$$ab^\delta = 1, \tag{42}$$

b. *in the case of $ab=1$, $f(x)$ belongs to class $H(1,1)$.*

Proof. Let us apply condition (33) of Theorem 8 and take in it $a_n = a^n$, $b_n = b^n$.

a. If $\gamma=0$, $ab^\delta=1$, $ab>1$, we obtain

$$Kb^{n(\delta-1)}\sum_{i=1}^{n}(ab)^i + Db^{(n+1)\delta}\sum_{i=n+1}^{\infty}a^i < K\frac{ab}{ab-1}+\frac{D}{1-a}.$$

b. If $\delta=1$, $\gamma=1$, $ab=1$, we obtain

$$\frac{K}{\log b}+\frac{D}{(1-a)\log b}\cdot\frac{1}{n+1}<C.$$

In both cases condition (33) is satisfied and thus the theorem is proved.

Theorem 11. *Given a function $\varphi(x)$ and any δ, where $0<\delta<1$, we can choose the coefficients a,b for a function $f(x)$ of type W in such a way that it will belong simultaneously to class $H(\delta,0)$ and to each of the classes $H^\infty(\delta,\gamma)$, where $\gamma<0$.*

Proof. Let us consider the particular case c of Theorem 9. We choose any δ and take $ab^\delta=1$. For $\gamma<0$ condition (39) is satisfied.

We note that we can always choose a,b in such a way that, with a chosen number $\theta(0<\theta<1)$, conditions (42), (40), (41) will be satisfied. Let us denote the expressions on the right side of (40) or (41) by A and B respectively. Conditions (42), (40) and (41) can be replaced by conditions (42) and by the conditions

$$b^{1-\delta}>\frac{1+A}{A}, \qquad a<\frac{B}{1+B}$$

whose realization, through a suitable choice of the coefficients a,b is always possible.

If we also take into account Theorem 10, Theorem 11 is completely proved.

It will be noted that for b we can also choose an integer, so that $f(x)$ will be periodic.

Remark. Theorems 8 and 9 permit the construction of a function $f(x)$ of type W, belonging simultaneously to classes $H(\delta,\gamma)$, $H^\infty(\delta,\gamma_1)$ only in the case of $\delta<1$ and $\gamma_1<\gamma=0$. For this reason, the construction of a function of type W for other values of the parameters δ,γ has still to be explained.

§ 6. Examples

A number of works induced by Weierstrass's example of a continuous non-differentiable function dealt in the first place with the construction of various examples of continuous functions for which in every point of the interval under consideration at least one of the derived numbers is different from the remaining ones or at least one of them is infinite.

In later examples, of both analytical and geometrical form, attention was paid to the question of obtaining the widest possible range of values of the coefficients a, b for which a function has the required kind of singularities [10]), with the retention of the simplest possible form of the function [11]), *e. g.* the form given by Weierstrass or the simplified form of Faber's function [12]). More general methods were also applied, and their efficiency was examined on classical examples, *i. e.* in a particular case. According to the generality of the method more or less sharp results [13]) are obtained in this way.

G. Faber has given an example of a function that has a singularity of a higher order than its non-differentiability, since it belongs, according to the classification which we have defined, to class $H^\infty(\delta,0)$ for every δ satisfying the inequality $0<\delta<1$. Further works aim at constructing examples of functions with a generalized singularity according to function $\omega(h)$, expressed by condition (3) [7]).

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[10]) Since the singularity required in this paper is expressed by condition (3) for every x, the method applied here limits *a priori* the range of the coefficients to the condition $ab>1$.

[11]) To the simplest examples of this type belongs the example given by van der Waerden [7].

[12]) Both functions are considered in examples 1 and 2. They are functions of type W.

[13]) W. Orlicz, on the basis of the general method which he applies, obtains, for example, for the coefficients of Weierstrass's function the conditions $ab>1+3\pi/2$, $a<1/13$ ([5], p. 35, Remarque).

The generalization of Hölder's condition permits the classification of functions with regard to their degree of continuity, the classification of functions according to the logarithmic-power scale having found application in the first place. The results of Orlicz's [5] work, in which he has given the necessary and sufficient condition of the existence of a function satisfying simultaneously conditions (2) and (3), make it actually possible to analyse this classification more exactly and, in consequence, to analyse some of the known examples.

The examples given below directly result from the application of the general method and concern exclusively the logarithmic-power scale. By imposing condition T on the function $\varphi(x)$ we obtain by the general method results even better than the classical results, obtained by the use of the individual method. A closer analysis of the function under consideration improves the results even more[14]). The examination itself is greatly simplified. The examined function is classified with regard to both its "degree of continuity" and its "degree of singularity" by indicating the classes $\mathrm{H}(\delta,\gamma)$ and $\mathrm{H}^\infty(\delta,\gamma_1)$ to which it simultaneously belongs. If this is so for every γ_1 satisfying $\gamma_1<\gamma$, then class $\mathrm{H}(\delta,\gamma)$ is the "narrowest" class in the logarithmic-power scale to which the examined function belongs. In this manner the results obtained become more complete. From this aspect we analyse some more of the later examples, obtaining generalized and sharpened results[15]).

On the ground of the method applied, the construction of a universal example, mentioned in the Introduction, for classification according to the logarithmic-power scale becomes possible[16]).

For the construction of each of the examples given below we can take integer values for b_n, *i. e.* the construction concerns in particular periodic functions with the required properties.

EXAMPLE 1. Taking $\varphi(x)=\cos x$ we choose the values of the coefficients a,b of a function $f(x)$ of type W in such a way that $f(x)$ belongs simultaneously to classes $\mathrm{H}(\delta,0)$ and $\mathrm{H}^\infty(\delta,\gamma)$, where $0<\delta<1$, $\gamma<0$.

In particular every such function will belong to class $\mathrm{H}^\infty(1,0)$ and thus its derived numbers will nowhere be all finite.

a. For the function $\cos x$ we obtain $K=1$ and on the basis of Lemma 1: $r=\pi/2$, $s=\pi$, $D=2$. Let us apply Theorem 9 (case c), substituting $\delta=1$, $\gamma=0$ in (39), (40), (41) and taking $\theta=2/3$. We obtain the inequalities

$$ab>1+\frac{3}{2}\pi, \qquad a<\frac{1}{13}$$

[14]) See Examples 1 and 2.

[15]) See Examples 3 and 4.

[16]) See Examples 5 and 6.

which constitute a sufficient condition for the coefficients a,b under which $f(x)$ belongs to $\mathrm{H}^\infty(1,0)$ and thus nowhere has a finite derivative[13]). The result obtained is weaker than that demanded at the beginning.

b. To obtain the result demanded at the beginning let us use Lemmata 2 and 3 and note that the function $\cos x$ satisfies condition T. Using the notation of Lemmata 2 and 3, we can take

$$h^*=\frac{\pi}{3}, \qquad q=\frac{3}{2\pi}, \qquad d=qh^*=\frac{1}{2}.$$

Let us apply Theorem 9 (case d) and take $\theta=2/3$. Considering (39), (22), (23) and Theorem 10, we obtain

$$ab>1+\pi, \qquad a<\frac{1}{13}, \qquad ab^\delta=1$$

as a sufficient condition under which the function $f(x)$ belongs simultaneously to classes $\mathrm{H}(\delta,0)$ and $\mathrm{H}^\infty(\delta,\gamma)$, where $\gamma<0$, $0<\delta<1$.

Now let us take for b an integer of the form $b=6n$. In this case $h^*=l/6$, and conditions (22) and (23) can be replaced by one condition (24). In view of this we obtain

$$ab>1+\frac{2}{3}\pi, \qquad ab^\delta=1\ ^{17)}$$

as a sufficient condition under which the function $f(x)$ has the properties demanded at the beginning.

c. The above results have been obtained by using general methods without an individual examination of the function $\cos x$. An individual examination enables us to obtain the most advantageous values for constant d,h^*, *i. e.* values for which the quotients d/h^*, d/D have the greatest possible values.

To determine the constant $h^*=|h_x|$, for which the inequality

$$|\cos(x+h_x)-\cos x|\geqslant d>0$$

is satisfied for every x (condition T), we convert this inequality into the following

$$\left|\sin\left(x+\frac{h_x}{2}\right)\right|\geqslant\frac{d}{2\left|\sin\frac{h_x}{2}\right|}=g.$$

It is easy to see that, in order that the above inequality be satisfied for every x, we must have

$$0<g\leqslant\frac{\sqrt{2}}{2}$$

[17]) The conditions given by Weierstrass are as follows: $0<a<1$, $ab>1+3\pi/4$, b an odd integer.

where, according to g, we can take for h^* one of the values satisfying the inequality

$$(43)\qquad 2\arcsin g \leqslant h^* \leqslant \pi - 2\arcsin g.$$

The greatest values for d/h^* are obtained with $g=\sqrt{2}/2$, $h^*=\pi/2$, *i. e.* with $d=1$. Then the relation d/D will also have the greatest possible value. Choosing $\theta=5/6$ we now obtain the following relations for the coefficients a,b

$$ab>1+\frac{3}{5}\pi,\qquad a<\frac{1}{13},\qquad ab^{\delta}=1.$$

Let us now suppose that b is an integer and a multiple of number k, with

$$(44)\qquad h^*=\frac{m}{k}l,$$

where m,k are natural numbers. In this case, in view of Theorem 5*, it is sufficient that the coefficients a,b satisfy conditions (24) and (42), *i. e.* that

$$(45)\qquad ab>1+\frac{h^*}{d}\quad\text{and}\quad ab^{\delta}=1.$$

In this case the function $f(x)$ will be periodic and will belong simultaneously to classes $\mathrm{H}(\delta,0)$ and $\mathrm{H}^{\infty}(\delta,\gamma)$, where $0<\delta<1$, $\gamma<0$.

From (43) it follows that the values which can be chosen for h^* must be smaller than π, and in order to obtain the smallest possible h^*/d, it is necessary to choose

$$g=\sin\frac{h^*}{2}\quad\text{if}\quad h^*\leqslant\frac{\pi}{2}\qquad\text{and}\qquad g=\sin\left(\frac{\pi}{2}-\frac{h^*}{2}\right)\quad\text{if}\quad h^*>\frac{\pi}{2}.$$

In both cases the smaller $|h^*-\pi/2|$ we take, the smaller value we shall obtain for h^*/d; it will be smallest when $h^*=\pi/2$.

It is easy to verify that for $k\geqslant 4$, if we choose in (44) the number m for a given k so as to obtain the smallest possible values for $|h^*-\pi/2|$, then the largest of those smallest possible values will be obtained for $k=6$ if $h^*\leqslant\pi/2$ and for $k=7$ if $h^*>\pi/2$. Listing the respective values we obtain

k	4	6	7
m	1	1	2
h^*	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{2\pi}{7}$
$\frac{h^*}{d}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{4\pi}{7\sin\frac{4\pi}{7}}<\frac{2\pi}{3}$

Thus if the coefficients a,b satisfy the conditions

$$(46)\qquad ab>1+\frac{2}{3}\pi,\qquad ab^{\delta}=1,$$

where b is an arbitrary integer not smaller than 4, then the function $f(x)$ is periodic and belongs simultaneously to classes $\mathrm{H}(\delta,0)$, $\mathrm{H}^{\infty}(\delta,\gamma)$, where $0<\delta<1$, $\gamma<0$.

In the case of $b=4n$, conditions (46) can be replaced by the following

$$ab>1+\frac{1}{2}\pi,\qquad ab^{\delta}=1\ ^{18}).$$

From conditions (45) follows the inequality

$$(47)\qquad b>\left(1+\frac{h^*}{d}\right)^{1/(1-\delta)}$$

and thus, if we increase δ, we must increase b so as to satisfy the above inequality. Thus, for example, for $\delta=1/2$ the smallest possible integer b satisfying (47) is 8. However, it is easy to calculate, considering (46), that it suffices to take $0<\delta\leqslant 1/3$ in order that $f(x)$ belong to class $\mathrm{H}(\delta,0)$ and to all classes $\mathrm{H}^{\infty}(\delta,\gamma)$ (where $\gamma<0$) for every integer $b\geqslant 4$ and a suitably chosen value of a.

It will be noted that, on the basis of the above method, we cannot construct a function of type W satisfying the conditions of our example if the value of the coefficient b is 3. In that case we must take

$$k=1,\qquad m=1,\qquad h^*=\frac{2\pi}{3},\qquad d=\frac{\sqrt{3}}{2},\qquad \frac{h^*}{d}=\frac{4\pi}{\sqrt{27}}>2;$$

thus condition (47) would not be satisfied for any δ, where $0<\delta<1$.

Example 2. Suppose that $f(x)$ is of type W and that

$$\varphi(x)=\min_p|x-p|,$$

where p assumes arbitrary integer values. We choose such values of the coefficients a,b of $f(x)$ that $f(x)$ belongs simultaneously to classes $\mathrm{H}(\delta,0)$ and $\mathrm{H}^{\infty}(\delta,\gamma)$, where $0<\delta<1$, $\gamma<0$.

We note that the function $\varphi(x)$ satisfies condition T and that $K=1$, $l=1$. When b is an integer and $b\geqslant 4$ we can take

$$h^*=\frac{1}{b},\qquad d=h^*,\qquad \frac{h^*}{d}=1.$$

[18]) It will be noted that taking for example $b=4$, $a=3/4$, we can, by applying the above method, show directly in a comparatively simple manner that the corresponding function $f(x)$ of type W has nowhere a finite derivative.

Let us consider Theorems 9 (case d) and 10, *i. e.* conditions (39), (24) and (42). We shall obtain conditions (45) for the coefficients a, b, and thus in the case of the example in question

$$ab>2, \qquad ab^{\delta}=1.$$

They are sufficient in order that $f(x)$ belong simultaneously to classes $\mathrm{H}(\delta,0)$, $\mathrm{H}^{\infty}(\delta,\gamma)$, where $0<\delta<1$, $\gamma<0$ and thus in particular that its derived numbers be nowhere all finite [19].

From these conditions it follows that $b>2^{1/(1-\delta)}$, and hence that with suitably small δ we must choose suitably large b. It is, however, sufficient to take $0<\delta<1/2$ in order that, for every integer $b\geqslant 4$ and a suitably chosen value of a, the function $f(x)$ belong simultaneously to classes $\mathrm{H}(\delta,0)$, $\mathrm{H}^{\infty}(\delta,\gamma)$, where $0<\delta<1$, $\gamma<0$.

It will be noted that on the basis of the above method we cannot construct a function of type W satisfying the conditions of our examples for $b=3$. In the latter case we have $h^*=1/3$, $d=1/6$, $h^*/d=2$, and thus conditions (45) cannot be simultaneously satisfied for any δ.

Example 3. G. Faber has given the following example of a function:

$$f(x)=\sum_{n=1}^{\infty}10^{-n}\varphi(2^{n!}x),$$

where $\varphi(x)=\min\limits_{p}|x-p|$ (p integer), and shown that it belongs to class $\mathrm{H}^{\infty}(\delta,0)$ for $\delta>0$ [7].

This result can be generalized for an arbitrary function $\varphi(x)$ satisfying condition T for which

$$\frac{d}{D}>\frac{1}{9}.$$

With this supposition $f(x)$ belongs even to $\mathrm{H}^{\infty}(0,\gamma)$ for every $\gamma<0$ (thus it does not belong to any of the classes of the logarithmic-power scale $\mathrm{H}(\delta,\gamma)$), as in this case conditions (34), (37) and (21) of Theorem 9 (case b) are satisfied for $\delta=0$, $\gamma<0$ (in (21) we take $\theta=0$).

In Faber's case $D=1/2$, and we can take $d=1/4$, so that $d/D>1/9$.

Example 4 [20]. We shall examine a function of type O

$$f(x)=\sum_{n=1}^{\infty}a^{n^2}\varphi(b^{n(n+a)}x)$$

in the following two cases:

a. $0<a<1$, $\quad ab>1$, $\quad a=0$.

[19]) The conditions given by Knopp ([4], p. 18) are as follows: $0<a<1$, $ab>4$, b even.

[20]) This example, in a slightly less general form, was examined by Orlicz ([5], p. 33, 34 and 38). The results obtained there have been extended here.

In this case the function $f(x)$, independently of the choice of $\varphi(x)$, belongs simultaneously to classes $\mathrm{H}(\delta,0)$ and $\mathrm{H}^{\infty}(\delta,\gamma)$ for every γ, where $\gamma<0$, and for δ satisfying $ab^{\delta}=1$ $(0<\delta<1)$.

b. $0<a<1$, $\quad ab=1$, $\quad a>0$.

In this case, if

$$b>\left(1+\frac{2sK}{D}\right)^{1/a} \tag{48}$$

(and thus for every b, provided we choose a suitably large a) the function $f(x)$ belongs simultaneously to classes $\mathrm{H}(\delta,0)$ and $\mathrm{H}^{\infty}(1,\gamma)$ for every δ and γ, where $0<\delta<1$, $\gamma\geqslant 0$.

Case a. We notice that conditions (33), (34), (37) and (38), in which we should take $a_n=a^{n^2}$, $b_n=b^{n^2}$, are satisfied.

Condition (33) is satisfied since

$$Kb_n^{\delta-1}\sum_{i=1}^{n}a_ib_i+Db_{n+1}^{\delta}\sum_{i=n+1}^{\infty}a_i<\frac{K}{1-b^{\delta-1}}+\frac{D}{1-a}.$$

The fulfilment of the other conditions is obvious.

Therefore the function $f(x)$ belongs simultaneously to both of the above-mentioned classes.

Case b. We notice that conditions (32), (35), (38) and (34) are satisfied; in the first of them we should take $\gamma=0$, in the last $\delta=1$ and in all of them $a_n=a^{n^2}$, $b_n=b^{n(n+a)}$.

Condition (35) is satisfied by (48) since

$$\lim_{n\to\infty}\frac{1}{a_nb_n}\sum_{i=1}^{n-1}a_ib_i=\frac{1}{b^a-1}<\frac{D}{2sK},$$

(in (35) we can take $\theta=1$). The fulfilment of the other conditions is obvious.

Therefore the function $f(x)$ belongs simultaneously to both of the above-mentioned classes.

Example 5. We give an example of a function $f(x)$ of type O, which, according to the values of the parameters δ,γ, belongs simultaneously to classes $\mathrm{H}(\delta,\gamma)$, $\mathrm{H}^{\infty}(\delta,\gamma_1)$ (where $\gamma_1<\gamma$) in the whole range of the logarithmic-power scale. The definition of the coefficients of the functions does not depend of the choice of the function $\varphi(x)$. The coefficients a_n, b_n of $f(x)$ are defined as follows:

$$a_n=A^{-\delta\cdot 2^{\lambda(n)}}\cdot 2^{\gamma\lambda(n)}\cdot\frac{1}{n^2}, \qquad b_n=A^{2^{\lambda(n)}},$$

where $A>1$, $\lambda(n)=n^2$. In the case of $0<\delta<1$ we can also take $\lambda(n)=n$.

In order to verify that $f(x)$ has the required properties, we apply Theorems 8 and 9 and examine conditions (32), (34), (37) and (38); in (34) γ should be replaced by γ_1.

Condition (32) is satisfied since

$$\sum_{n=1}^{\infty} a_n b_n^{\delta} (\log b_n)^{-\gamma} = \sum_{n=1}^{\infty} \frac{1}{n^2} (\log A)^{-\gamma} < \infty .$$

Condition (34) is satisfied since

$$a_n b_n^{\delta} (\log b_n)^{-\gamma_1} = 2^{\lambda(n)(\gamma-\gamma_1)} \cdot \frac{1}{n^2} (\log A)^{-\gamma_1} \to \infty .$$

Condition (37) is satisfied since

$$\frac{a_{n-1} b_{n-1}}{a_n b_n} = A^{-(1-\delta)(2^{\lambda(n)}-2^{\lambda(n-1)})} \cdot 2^{-\gamma(\lambda(n)-\lambda(n-1))} \left(\frac{n}{n-1}\right)^2 \to 0.$$

Condition (38) is satisfied since

$$\frac{a_{n+1}}{a_n} = A^{-\delta(2^{\lambda(n+1)}-2^{\lambda(n)})} \cdot 2^{\gamma(\lambda(n+1)-\lambda(n))} \left(\frac{n}{n+1}\right)^2 \to 0.$$

We notice that the supposition $\lambda(n) = n^2$ is used only to prove condition (37) for $\delta = 1$ and to prove condition (38) for $\delta = 0$.

If we take for A an integer, then $f(x)$ is periodic.

We note that for $\delta = 1$ we can choose γ ($\gamma > 0$) arbitrarily near zero, obtaining in that manner an example of a function of class $\mathrm{H}(1,\gamma)$ which is arbitrarily near the Lipschitz class, *i. e.* the class $\mathrm{H}(1,0)$, and in spite of this belongs to $\mathrm{H}^{\infty}(1,0)$, *i. e.* its derived numbers are nowhere all finite.

On the other hand, for $\delta = 0$ and $\gamma < 0$ we have an example of a function belonging to $\mathrm{H}(0,\gamma)$ and, simultaneously to $\mathrm{H}^{\infty}(0,\gamma_1)$, where $\gamma_1 < \gamma$. This function does not, therefore, belong in particular to any of the classes $\mathrm{H}(\delta,0)$ of the power scale.

Therefore this example is universal for the logarithmic-power scale.

EXAMPLE 6. Giving the coefficients a_n, b_n the form

$$a_n = A^{-\delta 2^{\lambda(n)}} \cdot 2^{\gamma\lambda(n)} \cdot \frac{1}{[\lambda(n)]^k}, \qquad b_n = A^{2^{\lambda(n)}}, \tag{49}$$

where $k > 0$, $A > 1$, $\lambda(n) = n!$, we also obtain an example of a function $f(x)$ of type O belonging, in the range of the logarithmic-power scale with an arbitrarily chosen $\varphi(x)$, simultaneously to classes $\mathrm{H}(\delta,\gamma)$ and $\mathrm{H}^{\infty}(\delta,\gamma_1)$, where $\gamma_1 < \gamma$, $0 \leqslant \delta \leqslant 1$. This is self-evident if we consider the conditions mentioned in example 5.

Moreover, the function $f(x)$ thus defined constitutes in the case of $\gamma = 0$, $\delta = 0$, an example of a function of type O which belongs to each of the classes $\mathrm{H}^{\infty}(0,\gamma_1)$, where $\gamma_1 < 0$ (in particular, it does not belong to any of the classes $\mathrm{H}(\delta,\gamma)$ of the logarithmic-power scale). Since in this case conditions (34) (in which γ should be replaced by $\gamma_1 < 0$), (37) and (38) are satisfied, as can be seen from the examination of the conditions in Example 5.

Thus the function defined above constitutes another, still wider, universal example of a function of type O, comprising also the case $\delta = 0$, $\gamma = 0$.

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